Stability of Networks

by

Ko Tung Yeung

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This is to certify that I have examined the above MPhil thesis and have found that it is complete and satisfactory in all respects, and that any and all revisions required by the thesis examination committee have been made.

Dr. Kwok-Yip Szeto, Thesis Supervisor

Prof. Ping Sheung, Head of Department

Department of Physics

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# TABLE OF CONTENTS

Title Page i
Authorization Page ii
Signature Page iii
Acknowledgements iv
Table of Contents v
List of Figures vii
List of Tables ix
Abstract x

Chapter 1 Introduction 1
  1.1 Introduction: Probability of Stability and the model 4
  1.2 Stability of Class Random (ER graph) 9
  1.3 Conclusion 14

Chapter 2 Probability of Stability of a Line, a Star and a Ring 15
  2.1 Introduction 16
  2.2 Stability of a Line and a Star 17
  2.3 Stability of a Ring 22
  2.4 Conclusion 28

Chapter 3 Hierarchical Structure of Rings 29
  3.1 Introduction 30
  3.2 Hierarchical Structure: 2 Layers 31
  3.3 Conclusion 45
LIST OF FIGURES

1.1 Example of a network with 4 nodes and 3 edges. 8
1.2a $P_{ER}$ vs $\sigma$ for a 6 nodes system with 5 edges. 11
1.2b $P_{ER}$ vs E for a 6 nodes system with $\sigma$ set to be 0.4 12
1.2c $P_{ER}$ vs N for networks with 15 edges, $\sigma$ set at 0.25 12
1.2d $P_{ER}$ vs $\sigma$ for a 20 nodes system with 50 Edges 13
2.1 All possible topologies for a 6 nodes system with 5 edges 20
2.2 Probability of Stability for T1 to T6 vs different value of $\sigma$ 21
2.3 All the possible topologies when one adds 1 more edge to the line of 6 nodes 25
2.4 Probability of stability for a 6 nodes system with 6 edges vs different value of $\sigma$ 26
2.5 Probability of Stability of a line, a star and a ring vs standard deviation $\sigma$ 26
2.6 Probability of stability for ring and ER (with 6 edges) vs different value of $\sigma$ 27
2.7 $D_{Ring}$ vs $\sigma$. It is very clear that the Ring is much more stable than the ER graph 27
3.1 An example of a ring of giant 31
3.2 D curves for different 2L 36
3.3 D values for 2L with 12 nodes a) 2L(4,3) b) 2L(3,4) c)
3.4 a) $P_{2L(4,3)}(15)-P_{2L(X)}$ curves with all $P_{2L(X)}$ belong to $2L(4,3)$. b) $P_{2L(4,3)}(15)-P_{2L(X)}$ curves for $2L(3,4)$.

4.1 Hierarchical structure: 3 layers

4.2 The D charts of $3L$, $2L(3,9)$ and $2L(9,3)$ with $N=27$

4.3 The D($3L;2L(3,9)$), D($3L;2L(9,3)$) and D($2L(3,9);2L(9,3)$) as a direct comparison between 3 classes

5.1 The student’s computer network

5.2 D chart for Halfling with $N=6$

5.3 D chart for Halfling with $N=10$

5.4 D chart for Halfling with $N=20$

6.1 D($3L(3,3,3); Halfling$) for $N=27$

6.2 Computer networks with 12 nodes and 16 edges

6.3 D($HH; HR$) for $3L(15,3,3)$ with 135 nodes
LIST OF TABLES

3.1 All possible values of m, E, C_{L1} and C_{L2} for N=12  40

4.1 All possible value of E for a 3L(3,3,3) network  53
Stability of Networks

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Department of Physics

The Hong Kong University of Science and Technology

Abstract

For a given number of nodes N and links L, we can arrange them in many different ways and thus different topology would be resulted and their network’s stability could be very different. Here we study some networks with special feature and we classify them into different class of topologies. We then compare the static stability between networks different class of topologies. Amazingly, we found that some class of topologies can be much more statically stable than the other class of topologies. We introduce the hierarchical structure, which is a class of topology that is about 10% more stable than the connected Erdos-Renyi (ER) graphs. In addition, our hierarchical structure has a short radius which is a good feature for building some artificial networks like computer networks.
Chapter 1 Introduction

There are many kinds of networks in the real world e.g. food web [1], the Internet, electrical circuits [2], traffic networks, social networks and etc. To study the general properties of these real world systems, we have the mathematical tools called graph theory. One of the properties is the stability of the network [3]. In 1971 & 1972, RM May introduced a model [4] [5] to study the probability of stability of ecosystem. His model assumed that the ecosystem was originally in an equilibrium state, i.e. the population of each species in the ecosystem would remain unchanged as time evolved. If for some reasons, such as earthquake, diseases, draught etc, the number of each species was disturbed in a small amount compared with the original population of that species, then the question is, as time goes by, would the network return to its original equilibrium state? If so, the network is considered as stable. Otherwise it is unstable.

Mathematically, RM May considered a system with N interacting species where their interactions can be written in the form of some non-linear set of first-order differential equations. By Taylor expansion around in the neighborhood of the equilibrium point, the stability of the possible equilibrium point for such a system can be characterized by the equation \( \frac{dx}{dt} = Ax \), where \( x \) is a \( N \times 1 \) column vector of the populations \( x_i \). \( A \) is the \( N \times N \) matrix that has element \( A_{ij} \) which specify the effect of species \( j \) on species \( i \) near the equilibrium point. The sign and magnitude of \( A_{ij} \) can be determined by a real food web, and \( A_{ij}=0 \) when species \( i \) and \( j \) has no interaction. But for general purpose, RM May simply assumed there are a portion of \( A_{ij} \) that equal to zero and used symbol \( C \) to express the probability that any pair of species will interact. In other words, \( C \) is measured as the percentage of non-zero
elements in the matrix. He also assumed that the non-zero elements $A_{ij}$ follows a normal distribution with zero mean and mean square value $\alpha$. RM May chose a zero mean because he believed the interactions are equally likely to be positive and negative. And for the meaning of $\alpha$, he suggested that it can be understood as the average interaction strength between the species.

Here, Wigner semicircle law comes in. For such a matrix $A$, the probability distribution function $P(\lambda)$ of the eigenvalues $\lambda$ of $A$ is

$$P(\lambda) = \frac{2\sqrt{R^2 - \lambda^2}}{\pi R^2}$$

for $-R < \lambda < R$. And for the range $\lambda < -R$ or $\lambda > R$, $P(\lambda) = 0$. And so $R$ would then be the maximum eigenvalue $\lambda_{\text{max}}$ of $A$. The meaning is that the value of $\lambda_{\text{max}}$ is limited and also we can find $\lambda_{\text{max}}$ analytically [6], [7]. In R.M. May’s model, the system in an equilibrium point, once disturbed, it will restore to its equilibrium point after a specific damping time, and RM May had rescaled this damping time for each species by setting the diagonal of $A$ to be -1. This -1 diagonal would then shift the center of the semicircle of $P(\lambda)$ to negative and therefore $\lambda_{\text{max}}$ has a chance to be negative. If $\lambda_{\text{max}}$ is positive then this system represented by $A$ is considered to be unstable, otherwise it is considered as stable. Since $A$ is a random matrix with diagonal being -1, RM May could have an unbounded number of ensembles of the same models because he could draw different random numbers for the random matrix $A$. Then, he calculated the probability $P(N, \alpha, C)$ that a specific matrix drawn from the ensembles correspond to a stable system. By applying Wigner semicircle law, he showed that for large $N$ such a matrix will be almost certainly stable ($P \rightarrow 1$) when $\alpha < (NC)^{-1/2}$. And certainly unstable ($P \rightarrow 0$) when $\alpha > (NC)^{-1/2}$. Thus there is a transition of $P$ from 1 to 0 at $\alpha = (NC)^{-1/2}$ and this transition would become very sharp when $N$ increases.
This model, raised by RM May, is accepted as a general mathematical model for the stability analysis. And this model can be applied to other networks besides ecosystem. However, RM May’s model considered the ecosystem as Erdos-Renyi graph (ER graph) [8] or equivalently the random graph. Therefore, all his conclusions only applied to ER graph [9]. Fortunately, his technique of analyzing the stability of a network can be applied to any type of network such as small world network, BA model or some real networks [10][11][12][13]. So we can raise a question, which kind of network would be the most stable? This question can be easily answered when one used the RM May’s technique to analysis different kinds of networks. Also, the results are very limited in applications because we are restricted to a few number of known networks (ER graph, BA, small world etc). Therefore, we can further ask the question: suppose we have a given number of nodes and edges, how can we construct a more stable network? This is a very interesting and important question and the results can have wide applications. Let’s consider the construction of an electricity supply systems, and one is given a fixed amount of fund and therefore one has a fixed number of electricity stations and cables (nodes and edges). There would be a lot of different ways to construct the network because there are a lot of different method to connect the electricity stations and cables. Of course the objective is wants to find a way to construct a stable electrical system to ensure that electricity would be always supplied to the city. We see that this is exactly the question we raised and demonstrate the importance of network architecture. In this thesis, we are going to find a way to construct a more stable network by the hierarchical method. First we study a small network, study how to connect the nodes and edges such that it is more stable. It turns out the most stable one is a ring and so we believe a hierarchical structure of rings should be stable.
1.1 Introduction: Probability of stability and the model

We can use a matrix, denoted by $A$, to represent a given network with $N$ nodes and $E$ edges. By naming the $N$ nodes from 1, 2, 3… to $N$, we can use the matrix element $A_{ij}=1$ to represent the nodes $i$ and nodes $j$ are connected by an edge. We use $A_{ij}=0$ to represent there is no edge connecting node $i$ and $j$ directly.

Normally, a real network usually has edges carrying weights that are not equal to 1. So we replace the non-zero elements $A_{ij}$ by the corresponding weights. And of course, the resulted matrix must be symmetric as there are no difference between $A_{ij}$ and $A_{ji}$. To find if this particular network is stable or not, we first need to replace the diagonal elements of $A$ by -1. And then the maximum eigenvalue $\lambda_{\text{max}}$ is found. If $\lambda_{\text{max}}$ is smaller than zero, then this network would be stable. Otherwise, this network is considered as unstable.

Since a given network can only be either stable or unstable with no uncertainty. However, as mentioned above, for a given number of nodes and edges there can be many methods to connect them. For example, if one connects the nodes and edges randomly, then a random graph or ER graph would be resulted. Some familiar methods are the small world network and BA model. For simplicity, let these different methods be called as different classes. So ER graph would be one class, BA being another class and small world being yet another class. Furthermore, for a single class, there can still be a lot of way to connect $N$ nodes and $E$ edges. For further reference, let us call each single way of connection as a single topology. For example the ER graph with $N$ nodes and $E$ edges, one can easily think of many different ways to connect them and thus many different topologies would be resulted. In addition, even for one topology, one can have different weights assigned to the edges of that
topology and thus many different configurations can be obtained. Since the weights can be any real numbers, one can have infinitely many different configurations for a particular topology. Figure 1.1 shows the situation clearly. In this thesis, all the weights are random numbers generated by computer following a normal distribution with zero mean and standard deviation $\sigma$. And thus different configurations of a single topology are generated by putting different sets of random numbers into the same topology. Because the magnitude of the weighting of an edge represent how strong is the interaction between the two connected nodes, the $\sigma$ is then interpreted as the interaction strength between the nodes. As with $\sigma$ becomes larger, the magnitude of weightings becomes larger. Note that the zero mean and the standard deviation $\sigma$ are made exact with no errors. This can be achieved by the following method: suppose we have $E$ edges and thus we need $E$ random numbers, we generate $E-2$ random number by computer and then we calculate the remaining 2 numbers such that the mean and $\sigma$ would be exactly the same as those that we want. Although two numbers were chosen, there would be no difference when $E$ is large enough.

From the above arguments, one can ask if a network is stable by finding out its maximum eigenvalue $\lambda_{\text{max}}$, but one can never ask whether a class is stable or not as each class contains infinite many different configurations in which some are stable while some are not. Therefore, we can only ask for the probability of stability for a class, denoted by $P_{\text{class}}$, which is the ratio of the number of stable configurations to the number of configurations. Unfortunately, these numbers are infinite, so practically if we were going to find the $P_{\text{class}}$, we can only generate a finite sample size of $N_t$ configurations which are all belong to the class under consideration. Then for each configuration we find the maximum eigenvalue $\lambda_{\text{max}}$ of the corresponding
matrix to see if they were stable or not. Suppose there are \( N_s \) configurations out of \( N_t \) are stable, we can obtain \( P_{\text{class}} \) as the following:

\[
P_{\text{class}} = \frac{N_s}{N_t}
\]  

(1.1)

And this technique of finding the probability of stability is the method introduced by RM May.

Recall that \( P_{\text{class}} \) is not a function of \( N_s \) and \( N_t \), but a function of \( N \), \( E \), \( \sigma \) and the class. To estimate the error of \( P_{\text{class}} \), first we must understand that \( P_{\text{class}} \) is actually the mean value of the probability of stability of the configurations. Notice that the probability of stability of a given configuration can either be 1 or 0 (stable or unstable). Now we have \( N_t \) configurations, and each configuration gives us a probability of stability of value 1 or 0. Therefore, the mean value of the probability of stability of these \( N_t \) configurations is just the sum of the probability of stability of all configurations divided by \( N_t \). However, the sum of the probability of stability of all configurations is just \( N_s \) and thus the mean value is \( P_{\text{class}} \) in equation (1.1). Now the error of a mean value is given by:

\[
\Delta P_{\text{class}}^2 = \frac{\sigma_p^2}{N_t}
\]  

(1.2)

where

\[
\sigma_p^2 = \frac{1}{N_t - 1} \sum (P_i - P_{\text{class}})^2
\]  

(1.3)

and \( P_i \) stands for the probability of stability of configuration \( i \) (either 1 or 0).
Plugging (3) into (2), one can easily obtain:

\[
\Delta P_{\text{class}} = \sqrt{\frac{P_{\text{class}} (1 - P_{\text{class}})}{N_t - 1}}
\]  

(1.4)

which is the estimated error for \( P_{\text{class}} \).

Finally, one important point must be made clear: only those configurations which are connected network would be counted toward \( N_t \). Or simply speaking, we are only interested in connected networks and reject those disconnected networks. The reason is simple: a disconnected network can be considered as several independent connected networks, so it would be meaningless to consider them as a whole.
Figure 1.1. Example of a network with 4 nodes and 3 edges. The black dots represent nodes and the lines represent edges. The decimal numbers stand for the weights for the corresponding edges. In this example, we have 2 different ways to connect 4 nodes with 3 edges and thus 2 different topologies are obtained. Furthermore, with different weights of edges assigned, we can have many different configurations for each topology. As the weights are real number with no restriction, actually there can be infinitely many different configurations.
1.2 Stability of Class Random (ER graph)

An ER graph would sometimes be called Class random or the random graph in this thesis. And its corresponding adjacent matrix is the famous random matrix. A random matrix represents a network with its nodes randomly connected by edges. The matrix is constructed by making an $N \times N$ dimensional matrix with all elements be 0. Suppose we have $E$ edges in our network, we randomly pick $E$ elements from those elements $A_{ij}$ with $j > i$ and set them to be 1. Due to the symmetry of $A$, we set $A_{ji} = A_{ij}$ and thus all together we have $2E$ ones in our matrix $A$.

However, the resulted graph represented by this matrix may be disconnected and thus the graph may have 2 or more components. As these components do not interact with each other, we do not consider them as a graph with dimension $N$. So if the resulted matrix is disconnected, then we would not study them as our subject. We study only those which are connected graphs.

The random matrix is an important class of topology as its stability always serves as a reference of other classes of topologies. Consider a man who wants to construct a network and he possesses no knowledge of the network architecture, then he would just arrange the nodes and edges randomly and thus the random graph. So a class of topology is considered as “stable” if it is more stable than the random graph with the same number of nodes and edges.

In section 1.1, it was mentioned that $P_{\text{class}}$ is a function of $N$, $E$, $\sigma$ and class. To see this is the fact, we use ER graph as an example here.

Figure 1.2a shows the probability of stability of a 6 nodes Class random with 5 edges vs the different values of $\sigma$. As one can see there is a region of $\sigma$ where $P_{\text{ER}}$ is always 1 and another region of $\sigma$ where $P_{\text{ER}}$ is always 0. There is a transition
region where $P_{ER}$ drops from 1 to 0. This is expected as RM May has already pointed out in his paper.

Figure 1.2b shows the $P_{ER}$ of a 6 nodes Class random with different number of edges. The $\sigma$ was chosen to be 0.4. It shows a general relation between $P_{class}$ and $E$: $P_{class}$ decreases as $E$ increases as predicted by RM May. Notice that $\sigma$ must be chosen appropriately in order to show such a relation of $P_{class}$ and $E$. For example, if $\sigma$ was chosen to be a value greater than 0.63, from Figure 1.2a, one can see $P_{ER}$ would be zero for a 6 nodes system with 5 edges. And no doubt, with an $E$ larger than 5, the $P_{ER}$ would also be zero. Thus, suppose we use any value of $\sigma$ greater than 0.63 to plot Figure 1.2b, the resulted graph would be a straight line $P_{ER}=0$ for all values of $E$ and this cannot show the correct relation between $P_{class}$ and $E$. A similar case would occur when $\sigma$ is too small. For example if we use $\sigma=0.01$ to plot Figure 1.2b, one should expect a straight line $P_{ER}=1$ for a reasonable large $E$. But this time, when $E$ is very large, the $P_{ER}$ can still decrease as the effect of large $E$ eventually overcome the effect of small $\sigma$. (For a system of 6 nodes however, the maximum value of $E$ is only 15.) Thus, only by choosing the $\sigma$ near the transition region where $P_{ER}$ drops from 1 to 0 can show the relation of $P_{class}$ and $E$ correctly.

Figure 1.2c is more interesting, which shows $P_{ER}$ vs the number of nodes $N$. One might expect a decreases in $P_{ER}$ as $N$ increases. However the curve in Figure 1.2c is increasing rather than decreasing. The reason is that when there are 6 nodes with 15 edges, the Class random is not 100% stable because it is a fully connected network which is known to be unstable. However when the number of nodes $N$ is increased to 16, the network is almost empty with edges and we know that generally such kind of networks is usually stable. As a result, the curve is increasing in Figure 2c. The stability gained is all due to the changed of the topology of the network, which would
be discussed in detail in the next chapter. Note again $\sigma$ is chosen to be 0.25 in Figure 1.2c as the same reason as in Figure 1.2b. If $\sigma$ is not well chosen, one may not be able to obtain the trend shown in Figure 1.2c.

Finally let us see what will the shape of Figure 1.2a be changed when $N$ and $E$ become large. Figure 1.2d shows the $P_{ER}$ vs $\sigma$ for a 20 nodes system with 50 edges. One can see the transition region now become narrower and shifted to the left. This result is important as it has the following implication: when $N$ becomes a very large number, the transition region will become very narrow and we can consider it is a sudden drop from $P_{ER}$ from 1 to 0. Thus the curve shown in Figure 1.2a would become a step function with a sharp critical value of $\sigma$, denoted by $\sigma_c$, where the transition occurs.

Figure 1.2a. $P_{ER}$ vs $\sigma$ for a 6 nodes system with 5 edges with sample size $N_t=100000$. The error bars were shown but too small to see. When $\sigma$ is smaller than 0.45, $P_{ER}\rightarrow 1$. When $\sigma$ is greater than 0.63, $P_{ER}\rightarrow 0$. Between these two values of $\sigma$, there is a transition region of $P_{ER}$.
Figure 1.2b. $P_{ER}$ vs $E$ for a 6 nodes system with $\sigma$ set to be 0.4, $N_t=100000$. As $E$ increases, $P_{ER}$ decreases.

Figure 1.2c. $P_{ER}$ vs $N$ for networks with 15 edges, $\sigma$ set at 0.25 and $N_t=100000$. The $P_{ER}$ increases as $N$ increases in this particular case.
Figure 1.2d. $P_{ER}$ vs $\sigma$ for a 20 nodes system with 50 Edges. The curve looks like that shown in Figure 1.2a expect the transition region now shift to the left and become smaller.
1.3 Conclusion

As we can see, $P_{\text{class}}$ is a function of $N$, $E$, $\sigma$. However, $\sigma$ must be well chosen in order to see the effects of $N$ and $E$ on $P_{\text{class}}$. Generally speaking, one should avoid testing the network in the region where $P_{\text{class}}$ is either 1 or 0. This is especially important when we compare the probability of stability of two different classes. If we choose a very large value of $\sigma$, then both classes can fall in the region where the $P_{\text{class}}$ for both of them are 0 and we would conclude that the two classes have no difference in term of stability while actually they do have a different near the transition region. When $N$ becomes very large, although the transition region disappear, different classes would lead to a different $\sigma_c$. Therefore, a class with a larger $\sigma_c$ would mean it is a more stable class.

The interesting fact that the $P_{\text{class}}$ increases as $N$ increases would be investigate in details in the following chapter, which is on the class of the network.
Chapter 2 Probability of Stability of a Line, a Star and a Ring

From chapter 1, we have seen $P_{\text{class}}$ is a function of $N$, $E$, $\sigma$ and the class. But we have not yet tested the effect of the class on $P_{\text{class}}$. While the dependence of $P_{\text{class}}$ on $N$, $E$ and $\sigma$ can be very straightforward as shown in Figure 1.2a, 1.2b and 1.2c, the relation between $P_{\text{class}}$ and the class can be very complicated and even unexpected. Amazingly, we will find that with a fixed $N$ and $E$, one can obtain a much more stable network by appropriately constructing the network, or in other words: choosing a good class.

This chapter contains the most fundamental elements of this thesis as all of the materials in the following chapters were based on the discovery in this chapter. As I have mentioned in the very beginning in chapter 1, we will first find out some very stable classes for a small network and then build up a much larger one by using the hierarchical structure of such small networks. The reason is simple: we believe that the stability of a hierarchical structure would be similar to those smaller sub-structures, although this is not necessarily true. In chapter 2, we show the way to construct a stable small network.
2.1 Introduction

In this chapter, we shall focus on a 6 node system and try to find out the most stable topology. We study 3 most basic topologies here: a line, a star and a ring.

In this chapter we shall show that by using the appropriate topology, we can construct a network with more edges being more stable than a network with fewer edges, which is amazing because usually one would expect the more edges the network has, the more unstable the network is.
2.2 Stability of a Line and a Star

Given 6 nodes, we want to find the most stable topology regardless of the number of edges. From Figure 1.2b, we know that when as $E$ increases, $P_{\text{class}}$ decreases. Thus one will guess the most stable topology of a 6 nodes system should have the minimum possible number of $E$. And for a 6 nodes connected network, the minimum possible $E$ would be 5. So we shall only focus on the 6 nodes with 5 edges. Figure 2.1 shows all the possible topologies for such a network and I name them as $T_1$, $T_2$, … to $T_6$ as shown on the right in Figure 2.1.

Just like the case of a class, we can ask for the probability of stability of a topology. The only difference is that for a class, different topologies and configurations of that class contribute to $P_{\text{class}}$. While for a topology, $P_{\text{topology}}$ is only contributed by different configurations of that topology.

Figure 2.2 shows the $P_{\text{topology}}$ for the six topologies vs different values of $\sigma$. One can immediately see that $T_1$ is the most stable topology in the transition area. On the other hand, the most unstable one is $T_6$. Note that the curve for $T_6$ is a step function, where the $P_{T_6}$ drop suddenly from 1 to 0 at $\sigma_c = 0.45$.

As $T_1$ and $T_6$ being the most stable and most unstable topology respectively, I am going to give them names. From the shape of $T_1$ and $T_6$, let us call $T_1$ as a line and $T_6$ as a star.

A line has important implications. First, it is the most stable topology among the six as shown in Figure 2.1. Second, from the knowledge of “more edges the more unstable the network is”, the most stable topology for 6 nodes system must be the line. If one wants to add more edges, then one must sacrifices the stability for the additional edges. Third, a line can be constructed with any value of $N$. Thus for any
fixed value of N, the most stable topology must then be the line (one can easily obtain this result when they compare the probability of stability between the line and any other topologies for that value of N).

We reach a very important conclusion:
The line is the most stable topology.

So the aim of this thesis is basically achieved: we have found the most stable topology.

What about the star? Is it the most unstable topology in the world? No, the star is not the most unstable topology. While the star indeed would be the most unstable topology for any N with E=N-1, it is absolutely not the most unstable topology for any N. (where E=N-1 is the minimum possible E that can connect all the nodes together.) Actually the most unstable topology for any value of N is the fully connected network, i.e. a network which is saturated with edges and it is impossible to add any more edges to the network.

This result can be frustrating to one interested in computer networking architecture. We all know that in computer networks, the most unstable topology is the line! And a star, although not very stable at all, still much more stable than a line! So why would we have such a ridiculous conclusion? Actually the conclusion made here is not contradicting the knowledge of computer networking architecture.

In computer networking architecture, a line is unstable because when ever there is a single edge broken, then the whole network will become disconnected and thus the network is ruined. A star, on the other hand, is more stable because when one edge is broken, the other computers can still communicate. However, the probability of
stability given in this thesis is asking about how easily a network would return to its equilibrium state after a small perturbation. Especially, the edges in our model would never be broken. Or more precisely, the computer networking problem is a dynamic one [14] while our model is a static one. Thus the meaning of “stable” is different in the computer networking architecture and the model given here. Yet, when we consider the spreading of computer virus, then our model would be applicable as virus spreading can be a static problem. Imagine, a virus is hardly spread in a line but easily spread in a start network. Therefore, one must be aware of the meaning of stability in the model.

The problem of computer networks also raise a problem: the line is not practical in the real world system. Who wants to build a network in a line form? We want a more realistic topology that is more applicable in the real world. As we know from computer networking architecture, a ring network is better than a line and a star. In addition, a ring is simple. Thus we are going to study the ring in the next section.
Figure 2.1. All possible topologies for a 6 nodes system with 5 edges. They were named as T1, T2, … to T6 as shown on the right.
Figure 2.2. Probability of Stability for T1 to T6 vs different value of $\sigma$. Obviously the most stable topology is T1, the line.
2.3 Stability of a Ring

From the data shown in 2.2, we now know that the most stable topology for a 6 nodes system with 5 edges is the line. However, in real world, we seldom see the line exist alone. Therefore, to make things more realistic, we are going to make a further step and find the most stable topology if we added one edge to those shown in Figure 2.3. (Possibly be not the most stable, but at least more stable.) Naturally, one would like to add the edge to an originally stable topology such that the resulted topology should be more stable than that of adding the edge to an originally unstable one. Simply speaking, we should add the edge to the line rather than the other topologies shown in Figure 2.1 such that we are more confident that the resulted topology is more stable. Figure 2.3 shows all the possible way to add one more edge to the line of 6 nodes. I give them names from T7 to T12 as shown on the right of the topologies.

Again, we should ask for their probability of stability and they are shown in Figure 2.4. This time, it is not so obvious that which topology is more stable. One can see when $\sigma$ is smaller than 0.51, T7 is the most stable topology. However, when $\sigma$ is greater than 0.51, T10 becomes the most stable topology. But overall, we can see that in the transition area, T7 is the most stable one. In addition, T7 is just the topology of a Ring, which is a common topology in the real world system while T10 isn’t. Therefore, we would consider the ring as the most stable topology and ignore T10.

Figure 2.5 shows the probability of stability for the line, star and ring. Amazingly, the ring, in most region of the transition area, is more stable than the star even though the ring has one more edge than the star! So by choosing an appropriate topology, one could indeed build a more stable network with more edges. And now we can explain the phenomenon as shown in Figure 1.2c where $P_{ER}$ increases as $N$ increases.
With fixed number of edges $E=15$, a network with 6 nodes must be a fully connected network, which is the most unstable topology as has mentioned. In contrast, for a network with $N=16$ with $E=15$, the network is actually a tree graph (like those shown in Figure 2.1), which is more stable than a fully connected network even it has a larger $N$. So the phenomenon in Figure 1.2c is all because the change in topology when $N$ becomes larger.

One thing must be made clear. For a given $N$ and $E$, let’s define the network $A$ in the form of the most stable topology that the $N$ and $E$ allowed. Next, let’s consider another network $B$ also has $N$ nodes but with a larger $E$, say $E+1$, and it is also in the form of the most stable topology that $N$ and $E+1$ allowed. Then $B$ can never be more stable than $A$. This agrees with the result of RM May on ER network, where the network with higher connectivity is less stable. Here connectivity increases with $E$. Only when $A$ is not in the form of most stable topology allowed by $N$ and $E$, $B$ can have a chance to be more stable than $A$. Therefore, the line must be the most stable topology as it is the most stable topology with minimum possible number of edges $E$ allowed by a given $N$. As an example, Figure 2.5 shows that the line is more stable than the ring.

Finally, we want to compare if the ring is better than a general network. By means of a general network, we refer to connected ER graphs. Figure 2.6 shows this is indeed the truth. To see more clearly how better the ring is over ER, we define a value $D$:

$$D_{\text{Class}}(N,E,\sigma) = P_{\text{Class}}(N,E,\sigma) - P_{\text{ER}}(N,E,\sigma) \quad (2.1)$$

So the $D$ value is just the difference between a specific class and the ER graph.
(not necessarily a class, can also be a topology.) And it is understood that the N, E and \( \sigma \) are the same in (2.1). Figure 2.7 shows the \( D_{\text{Ring}} \) curve. One can see that the ring is much more stable than the ER graph. The peak is about 0.22 which means the ring can be 22% more stable than ER graphs. Which is indeed amazing because with the same resources (nodes and edges), one can build a much more stable network when he choose the right topology. Note that there is a small region where \( D_{\text{Ring}} \) is negative. And we shall usually see similar negative part in other topologies. But since this negative region is small and the magnitude is very small (which is about -0.01, which is 1% and indeed very small compared to 22%), we can simply ignore it and say the Ring is always being more stable than connected ER graph.
Figure 2.3. All the possible topologies when one adds 1 more edge to the line of 6 nodes. The names were given on the right of the corresponding topology.
Figure 2.4. Probability of stability for a 6 nodes system with 6 edges vs different value of $\sigma$. In most of the transition region, T7 is the most stable topology.

Figure 2.5. Probability of Stability of a line, a star and a ring vs standard deviation $\sigma$. Even though a ring has one more edge than a star, yet the ring is more stable than a star. And always, the line is the most stable topology.
Figure 2.6. Probability of stability for ring and ER (with 6 edges) vs different value of $\sigma$. As expected, the ring is more stable than the ER graphs.

Figure 2.7. $D_{\text{Ring}}$ vs $\sigma$. It is very clear that the Ring is much more stable than the ER graph. The peak is about 0.22 meaning that the ring can be 22% more stable than the ER graph which is amazing. There is a small region of the curve where it is negative, but since the magnitude is small (compared to 22%), we just ignore it.
2.4 Conclusion

As we have seen, the most stable topology is the line. However, a line is not realistic and also it is very unstable in the sense of dynamic problem (e.g. computer networking problem), we cannot accept such a topology.

In contrast, the Ring, being a simple and stable topology in both static (our model) and dynamic problem, it would undoubtedly be the best choice for building our hierarchical structure in the later chapters.
Chapter 3 Hierarchical Structure of Rings

3.1 Introduction

As we have seen, a ring is a very stable topology in both dynamic and static perspectives. However, a ring has a major defect: it cannot be a good topology for a large network. While one can accept building a ring with 6 nodes, one will never accept building a ring, say, with 10,000 nodes. There are many reasons for rejecting such a large ring network. First, for a large network, especially when we consider real world system, there can be many different topologies and thus a ring would be too simple and unrealistic. Second, the radius of a ring would be very large when the number of nodes $N$ is large. The communication between two distant computers would be very inefficient when there are 10,000 computers connected as a ring network. Third, in the view of dynamic problems, a ring is only stable when the number of nodes $N$ is small. When 2 non-adjacent edges are broken, then a ring would be broken into 2 small networks which is undesirable. For a ring with 6 nodes, keeping all the 6 edges functional is easy and thus the ring would not be easily broken. But we would immediately see that keeping 10,000 edges functional for a ring with 10,000 nodes is not an easy task. One can think of many reasons why we should reject the ring. To solve these problems, we propose the method of
hierarchical structure [15][16].

As mentioned before, we believe a hierarchical structure of some stable sub-structures should also be stable. From chapter 2, we have found that the ring is a good choice as our sub-structure. In this chapter, we are going to build a much larger network by connecting many rings together. Then we compare the results with ER graph and the results shows such a hierarchical structure is actually more stable as expected.
3.2 Hierarchical Structure: 2 layers

Before we discuss hierarchical structure, we first introduce definitions and terminologies. Suppose we have \( N \) nodes, we can divide them into \( q \) sub-groups, each sub-group would then have \( p = \frac{N}{q} \) nodes (I choose an \( N \) or \( q \) by making \( \frac{N}{q} \) an integer). We call these sub-groups as giant nodes. For each giant node, I connect the \( \frac{N}{q} \) nodes as a ring network with \( \frac{N}{q} \) edges and so each giant node must be a connected graph. All the \( q \) giant nodes are connected in the manner of a ring again to obtain a ring of giant nodes, see figure 3.1.

![Figure 3.1](image_url)  

Figure 3.1. An example of a ring of giant. Each small loop is a giant node with \( \frac{N}{q} \) nodes. The dotted line represents the edges connecting the giant nodes.

In figure 3.1, we can have two kinds of edges. First, the edges within the giants, with both ends connecting to nodes belong to a single giant. For instance, we call
them the inner edges. Second, the edges which connect the giants into a ring of giant, with one of its end connecting to a giant while the other end connecting to another giant. We call these edges as outer edges.

Now we are going to define the 1st layer of the network as all inner edges and nodes. And we define the 2nd layer of the network as all the outer edges and giants. In other words, 1st layer refer to anything belong to the giants; 2nd layer refer to anything belong to the ring of giants except those belong to the 1st layer. To specify such a topology, we can use the following definition:

\[ L_2(p, q) \]  

where \( L_2 \) stands for there are two layers, \( p \) is the number of nodes within each giant and \( q \) is the number of giants. (Note: \( p = N/q \))

There are \( p \) nodes in each giant and we need at least \( p \) edges to connect them into a giant. And since there are \( q \) giants, there would be \( p \times q \) edges within all the giants. Finally we need at least \( q \) edges to connect these giants into a ring of giant, thus the minimum possible number of edges for constructing a \( L_2(p, q) \) would be:

\[ pq + q \]  

Basically, a ring of a giant shown in figure 3.1 would have at least \( (p+1)q \) edges.
For network with N nodes and E edges where E is larger than (p+1)q, we add the extra \((E - (p+1)q)\) edges to the 1\(^{st}\) layer (i.e. adding them as inner edges) rather than to the 2\(^{nd}\) layer (i.e. adding them as outer edges). Specifically, we randomly choose a giant and randomly choose 2 of its nodes and connect them together. In this way, the ring structure of that particular giant would be destroyed but the global structure shown in figure 3.1 can still be maintained (i.e. the ring of giant would be unaffected). The extra \(E - (p+1)q\) edges are added to the 2\(^{nd}\) layer only when there are no more edges can be added to the 1\(^{st}\) layer. In this case, all the giants are fully connected. Or more precisely, when

\[
E \leq \frac{pq}{2} (p - 1) + q
\]  

(3.3)

then no edges would be added to the 2\(^{nd}\) layer. The right side of (3.3) is obtained as the following: we have p nodes in a giant, so a giant can have a maximum of \(p(p-1)/2\) edges. We have q giants and thus \(pq(p-1)/2\) edges. Finally we add q edges to connect the giants together, and therefore equation (3.3) is obtained.

And finally one more rule for adding extra edges. The number of edges belong to the 2\(^{nd}\) layer (outer edges) must be less than the number of edges belong to the 1\(^{st}\) layer (inner edges). Otherwise, our hierarchical structure would be destroyed.
because we cannot distinguish the giants from the whole network. Mathematically, the allowed maximum number of edges belong to 2\textsuperscript{nd} layer would be given by:

Number of outer edges \leq \frac{pq}{2} (p - 1) - 1 \quad (3.4)

where the term pq(p-1)/2 is the number of edges of all giants which are fully connected. And the minus 1 would simply means these are less outer edges than inner edges. Meanwhile, there are pq(p-1)/2 inner edges. By adding the number of inner edges and outer edges, we then obtain the maximum number of edges allowed in our hierarchical structure:

\[ E \leq pq(p - 1) - 1 \quad (3.5) \]

A fixed value of N with different values of q can form a lot of different ring of giants. The larger the q, the smaller the giants are. And we can compare the stability between different q.

Figure 3.2a shows the $D_{L2(6,3)}$ curve with E=21. (For a fixed value of SD, $D_{2L(6,3)} = P_{L2(6,3)} - P_{ER}$, where $P_{L2(6,3)}$ is the probability of $L_2(6,3)$ and $P_{ER}$ is that of ER graph. ) E=21 is chosen because this is the value of (p+1)q, the minimum number
of edges possible for $L_2(6,3)$. We can also divide the 18 nodes into 6 giants ($p=3$, $q=6$), then we need at least 24 edges because $(p+1)q = 24$ in this case. Figure 3.2b shows the D curves for both $L_2(6,3)$ and $L_2(3,6)$ with $E=24$. Obviously, the stability of a network with 18 nodes is higher when they are divided into 6 giants ($L_2(3,6)$ is about 4% more stable than $L_2(6,3)$). One can find that the D curves of $L_2(6,3)$ in Fig 3.2a and 3.2b are almost the same, even the values of $E$ are different for them. This is no surprise because when $E$ increased, both $P_{L_2(6,3)}$ and $P_{ER}$ decrease, while the difference between them $D_{L_2(6,3)} = P_{L_2(6,3)} - P_{ER}$ remains the same.

Figure 3.2c shows the $D_{L_2(6,3)}$ curve with $E=48$. 48 is the maximum value of $E$ satisfy inequalities (3.3). i.e. all the giants are fully connected, but still there are no edges added to the 2$^{nd}$ layer. Because a fully connected network is usually very unstable, so we would expect a collection of multiples fully connected networks to be very unstable too. But surprisingly, Fig 3.2c is positive which means that $L_2(6,3)$ is more stable than the random graph even when there are 3 fully connected giants. Moreover, when comparing to Fig. 3.2a, the peak is shifted and higher in 3.2c. In addition, the width of the peak is smaller. Note that for $E=21$, $E=24$ and $E=48$, there are no edges added to the 2$^{nd}$ layer in the $L_2(6,3)$ and $L_2(3,6)$ for Fig 3.2a, 3.2b and 3.2c.
Figure 3.2. D curves for different L. Fig. 3.2a shows the D for L_2(6,3) with E=21. Fig. 3.2b shows the D of L_2(6,3) and L_2(3,6) with E=24. Fig. 3.2c shows the D curve of L_2(6,3) with E=48.
As usual, we are interested in the connectivity $C$ of the network. In addition, since we have giants as our sub-structures and a ring of giants as our global structure, we may also want to know the connectivity of them. For the connectivity of the sub-structures, let us define $C_L^1$ as the total number of edges within all giants divided by the maximum possible number of edges in all giants, or mathematically:

$$C_L^1 = \frac{2E_L^1}{pq(p-1)} \quad (3.6)$$

where $E_L^1$ is the total number of edges within all giants, or in other words the number of edges belong to the $1^{st}$ layer. We call $C_L^1$ as the connectivity of the $1^{st}$ layer.

Similarly, we define $C_L^2$ as the connectivity of the $2^{nd}$ layer. It is the number of edges belong to the $2^{nd}$ layer $E_L^2$, divided by edge capacity of the $2^{nd}$ layer (the edge capacity is $q(q-1)p^2/2$). Thus, $C_L^2$ is given by:

$$C_L^2 = \frac{2E_L^2}{q(q-1)p^2} \quad (3.7)$$

And the connectivity $C$ of the whole network is given by:

$$C = \frac{2E}{N(N-1)} \quad (3.8)$$
To investigate the effect of the $C_{L1}$ and $C_{L2}$ on the probability of stability, the case of $N=12$ was studied, which is the smallest possible $N$ that allows different value of $q$, $E$, $C_{L1}$ and $C_{L2}$. Table 3.1 shows all the allowed values for $N=12$ and one can see when $q=3$, no extra edges were added to the 2nd layer until $E=22$ and the same case for $q=4$ is until $E=16$. The $D$ values of the $L_2(4,3)$ and $L_2(3,4)$ are shown in figure 3.3.

From Fig. 3.3a, the graph of $D$ curves for $L_2(4,3)$, one can see the $D$ values decreases when $E$ increases from 15 to 21. Surprisingly, $E=21$ is the point where all the giants are fully connected and still no extra outer edges were added and it also posses the highest peak among all the curves in Fig 3.3a. From the trend we see that when the giants have more and more edges in 1st layer, the larger and larger $D$ values would be obtained! The further increase of $E$ beyond 21 would then lead to the decrease of the peak as expected because the global structure is now disturbed.

Fig. 3.3b shows the $D$ curves of $L_2(3,4)$. By noting $E=16$ is the only point where no extra outer edges exist, further increasing $E$ would lead to the decrease of the peaks. This is exactly the same as what was happened for $L_2(4,3)$, and so Fig. 3.3a and 3.3b have the same pattern.

To compare the $L_2(4,3)$ and $L_2(3,4)$ directly, I define a value:
\[ D(A; B) = P_A - P_B \]  \hspace{1cm} (3.9)

where \( P_A \) and \( P_B \) are the probability of stability for class A and B respectively.

Fig. 3.3c shows the value of \( D(\text{L}_2(4,3) ; \text{L}_2(3,4) ) \). Positive value means \( \text{L}_2(4,3) \) is more stable and negative value means \( \text{L}_2(3,4) \) is more stable. The curve with \( E=18 \) is the border, which is very close to zero. \( \text{L}_2(3,4) \) is better with smaller \( E \), while \( \text{L}_2(4,3) \) is better with larger \( E \). Again, the highest peak occurs when \( E=21 \). But further increase of \( E \) would then decrease the \( D \) values.

Although \( E=21 \) has the highest peak in Fig. 3.3a, it does not mean \( E=21 \) has the highest \( P \). Normally, the highest \( P \) belong to the minimum \( E \) and in this case it would be \( E=15 \). To see the effect on \( P \) as \( E \) increases, I plot the curves \( P_{\text{L}_2(4,3)}(15)-P_{\text{L}_2(X)} \) in Figure 3.4 where \( P_{\text{L}_2(4,3)}(15) \) is the \( P \) curves of \( \text{L}_2(4,3) \) with \( E=15 \) and \( P_{\text{L}_2(X)} \) is another \( P \) curve of either \( \text{L}_2(4,3) \) or \( \text{L}_2(3,4) \) with \( E=X \). Fig 3.4a shows those \( P_{\text{L}_2(4,3)}(15)-P_{\text{L}_2(X)} \) with all \( P_{\text{L}_2(X)} \) belong to the class \( \text{L}_2(4,3) \) while Fig. 3.4b with all \( P_{\text{L}_2(X)} \) belong to \( \text{L}_2(3,4) \). As expected, all the curves should be positive. Notice that there is a small portion of \( P_{\text{L}_2(4,3)}(15)-P_{\text{L}_2(3,4)}(16) \) in Fig. 3.4b with negative values. This means even with a larger \( E \), the \( \text{L}_2(3,4) \) is more stable than the \( \text{L}_2(4,3) \) in that region.
Table 3.1. All possible values of m, E, C_{l_1} and C_{l_2} for N=12. The dashed line in the table separate the region where extra edges were added to the 2nd layer or not. C is the connectivity of the whole network. Peak is the peak value of D and $\sigma_p$ is the corresponding SD. A is the area of the curve under the D curve (the positive area – negative area).

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<td>0.3485</td>
<td>0.0808</td>
<td>0.29</td>
<td>0.0059</td>
</tr>
</tbody>
</table>
Fig. 3.3a

Fig. 3.3b
Figure 3.3c

Figure 3.3. D values for $L_2$ with 12 nodes. Fig. 3.3a is the D values of $L_2(4,3)$. Fig. 3.3b is the D values of $L_2(3,4)$. For easy reference, the topology of the network for Fig. 3.3a and 3.3b are shown at the bottom of the graphs. Fig. 3.3c is $D(L_2(4,3);L_2(3,4))$. The legend shows the value of $E$ of the corresponding curve.
Figure 3.4. Fig. 3.4a is the $P_{L^2(4,3)}(15)-P_{L^2}(X)$ curves with all $P_{L^2}(X)$ belong to $L_2(4,3)$. Fig. 3.4b is the $P_{L^2(4,3)}(15)-P_{L^2}(X)$ curves for $L_2(3,4)$. 
3.3 Conclusion

Obviously, hierarchical structure of rings is a stable network compared with the connected ER graph. This is not surprising because the rings are meant to be stable and so a collection of rings is also stable as expected. However, for a given value of N, there can be several different choices of p and q. The problem of choosing the p and q has no easy solution. Yet, we can have a clue from Fig. 3.3c, where the highest peak occurs when E=21. E=21 is a special value at which all the giants of L₂(4,3) are fully connected meanwhile no additional edges were added to the 2\textsuperscript{nd} layer. Thus, when a fixed value of N and E are given, one may first find out the value of p and q at which all the giants are fully connected while no extra edges are added to the 2\textsuperscript{nd} layer. Unfortunately, in general the given E would not allow us to make such a topology. In this case, it would be very difficult to choose the values of p and q as we shall see in the next chapter.
Chapter 4 Hierarchical Structure of Rings: 3 layers

4.1 Introduction

As a single ring is stable, a collection of them should also be stable. Yet, there are many ways of connecting these rings. Again, we borrow the idea from a ring where a number of nodes are connected in a loop to form the ring. Therefore we connect multiples rings in a loop to form the \( L_2(p, q) \). In other words, when each node in a ring is expanded into a ring, then the resulted network is actually a \( L_2(p, q) \) where we have 2 layers of rings. Following this idea, one can continuously expand each node into a ring, than one can easily obtain as many layers of rings as needed. In this chapter, we shall study 3 layers of rings.

When nodes connected in the manner of loop, we call it a ring. When each of these single nodes is expanded into a giant, then the ring would become a ring of giants as shown in Figure 3.1. However, we can expand each single node in the ring of giants into a giant again and we would obtain a hierarchical structure with 3 layers. An example is shown in Figure 4.1.
Figure 4.1. On the left is the ring of nodes. Each black dot represents a node. After each single node is expanded into a giant, then a ring of giants would be obtained as shown in the middle. Yet we can expand each node into a giant again and we obtain the graph on the right, where the originally black dots now become small loops which represent giants.

Figure 4.1 only shows a special case. In general, we can have $p$ nodes to form a giant, $q$ giants to form a ring and finally $r$ rings to form the complete graph. So we have 3 degree of freedom. Following our definition in the previous section, I call the giants as the 1st layer, a ring of giants as 2nd layer and the collection of such rings of giants as the 3rd layer. So I call this class as $L_3(p,q,r)$, where $L_3$ stands for “3 layers”.

The rule to connect two giants is the same as before: randomly pick a node form each of the 2 giants and connect them with an edge. Similarly, we connect two ring of giants by randomly picking a node from each of the 2 rings and connecting them together. Thus, build a class $L_3(p, q, r)$, we need a minimum number of edges $E$ given by:

$$E \geq (pq + q)r + r = [(p + 1)q + 1]r \quad (4.1)$$
When $E$ is more than this value, then the edges are added to the 1\textsuperscript{st} layer. When 1\textsuperscript{st} layer is fully filled up, then we add the edges to the 2\textsuperscript{nd} layer. After 2\textsuperscript{nd} layer is fully filled, then we add edges to the 3\textsuperscript{rd} layer. Again, the number of edges in the 1\textsuperscript{st} layer (giants) must be more than those in 2\textsuperscript{nd} layers. Because of this, one should notice that the 2\textsuperscript{nd} layer will never be fully filled with edges and thus we can never have any extra edges added to the 3\textsuperscript{rd} layer. As a result, there are always $r$ edges exist in the 3\textsuperscript{rd} layer.

Again, we are interested in the connectivity of each layer. Define $C_{L1}$, $C_{L2}$, $C_{L3}$ as the connectivity of the 1\textsuperscript{st}, 2\textsuperscript{nd} and 3\textsuperscript{rd} layer, which is portion of filled edges exist in the 1\textsuperscript{st} layer, 2\textsuperscript{nd} and 3\textsuperscript{rd} layer respectively. Or mathematically, the total number of edges in that layer divided by the maximum possible number of edges in that layer:

\[
C_{L1} = \frac{E_{L1}}{pqr(p-1)/2} \tag{4.2}
\]

\[
C_{L2} = \frac{E_{L2}}{p^2qr(q-1)/2} \tag{4.3}
\]

\[
C_{L3} = \frac{E_{L3}}{p^2q^2r(r-1)/2} \tag{4.4}
\]

where $E_{L1}$, $E_{L2}$ and $E_{L3}$ are the number of edges belong to the 1\textsuperscript{st} layer, 2\textsuperscript{nd} layer and
3\textsuperscript{rd} layer respectively. And of course, the sum of the three denominators in equation (4.2), (4.3), (4.4) is equal to \( N(N-1)/2 \), the maximum possible number of edges of a N nodes network. However, as mentioned before, there are always \( r \) edges in the 3\textsuperscript{rd} layer and so (4.4) is reduced to:

\[
C_{L3} = \frac{1}{p^2q^2(r-1)/2}
\] (4.5)

which is actually a constant.
4.2 Stability of Hierarchical Structure of Rings with 3 Layers

The smallest value of \( p, q \) and \( r \) are 3 and thus the smallest possible \( N \) for \( L_3 \) would be \( L_3(3,3,3) \) with 27 nodes. On the other hand, we can also build \( L_2 \) graph with 27 nodes. We have 2 possibilities here, \( L_2(9,3) \) and \( L_2(3,9) \). Figure 4.2 shows the D value of \( L_3(3,3,3), L_2(9,3) \) and \( L_2(3,9) \) with \( N=27 \). And Table 4.1 shows the values of \( C_{L1}, C_{L2}, C_{L3} \) for the \( L_3(3,3,3) \).

All 3 graphs have the same \( E \) starting from 39 to 53, which is the range of all allowed values of \( E \) of \( L_3(3,3,3) \). However, \( L_3(3,3,3) \) has all its giants fully connected at \( E=39 \), so all the extra edges were added to the 2\(^{nd} \) layer. And as said before, no edges can added to the 3\(^{nd} \) layer.

In Fig. 4.2a, we observe that the peak values do not vary much with the increasing \( E \). On the other hand, the \( L_2(3,9) \) does not share the same property. Although the peak of \( L_2(3,9) \) at \( E=39 \) is the same as that of \( L_3(3,3,3) \), one can see the peaks are decreasing with the increasing \( E \).

Since \( L_2(3,9) \) has 3 nodes as a single giant, so all the extra edges can only be added as outer edges. By destroying the global structure, the network can only decrease its stability and which is just the same case as shown in Fig. 3.3b.

For the \( L_2(9,3) \) (Fig. 4.2c), the peak is increasing slightly as \( E \) increases. We have
also observed similar pattern in Fig. 3.3a with E<21. All extra edges were added as inner edges and since the global structure is preserved, the peak is then slightly increased. However, the highest peak of \( L_2(9,3) \) is only about 0.12 which is smaller than all the peaks of \( L_3(3,3,3) \). Obviously, \( L_3(3,3,3) \) is the best in both stability and robustness to E. From these examples, preserving the global structure is very important to obtain a stable network [17]. Figure 4.3 shows the \( D(L_3; L_2(3,9)) \), \( D(L_3; L_2(9,3)) \) and \( D(L_2(3,9); L_2(9,3)) \) as the direct comparison between the P of each class.

Now it is very obvious that \( L_3 \) is much better than both \( L_2 \) for \( N=27 \).

As a side note, Fig. 4.3c shows the difficulty to choose the values of p and q for \( L_2(p, q) \). For \( E \leq 46 \), \( L_2(3,9) \) is more stable. But when \( E > 46 \), \( L_2(9,3) \) would become more stable. Thus \( E=46 \) is then a critical value. However, this value cannot be found unless we plot the curves as shown in Fig. 4.3c. Therefore there is no simple answer for choosing the values of p and q for \( L_2 \). In this case we expect this kind of difficulty is a general feature for hierarchical structure.
Fig. 4.2a $L_3(3,3,3)$

Fig. 4.2b $L_3(3,9)$
Figure 4.2. The D charts of $L_3$, $L_2(3,9)$ and $L_2(9,3)$ with $N=27$. The E starts from 39 to 53, which is the range of all allowed values of $L_3(3,3,3)$. For easy reference, the topologies of the corresponding networks are shown at the bottom of the graphs.
Table 4.1. All possible value of E for a L₃(3,3,3) network.

<table>
<thead>
<tr>
<th>E</th>
<th>E₁₀</th>
<th>E₁₁</th>
<th>E₁₂</th>
<th>C₁₂</th>
<th>C₁₂</th>
<th>C₁₃</th>
</tr>
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<tbody>
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</tr>
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<td></td>
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<tr>
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</table>
Figure 4.3. The $D(L_3; L_2(3,9))$, $D(L_3; L_2(9,3))$ and $D(L_2(3,9); L_2(9,3))$ as a direct comparison between 3 classes.
4.3 Choosing between $L_3$ and $L_2$

The success of the $L_3(p, q, r)$ is no coincidence. In fact, $L_3$ is well designed. Consider, for a $L_2$ class of topologies, the 2$^{\text{nd}}$ layer is the global structure of the network. But for a $L_3$ class of topologies, the 2$^{\text{nd}}$ layer now becomes a local structure and the 3$^{\text{rd}}$ layer is now the global structure. As persevering the global structure is very important for stability, $L_3$ is then more stable because no edges are allowed to be added into the 3$^{\text{rd}}$ layer in our example.

However, this does not mean $L_3$ is always being more stable than $L_2$. To illustrate this point, we should go back to Fig. 4.3c. For all values of $E$ in Fig. 4.3c, extra edges were filled into the 2$^{\text{nd}}$ layer of $L_2(3,9)$ while the extra edges were filled into the 1$^{\text{st}}$ layer of $L_2(9,3)$. In other words, we are actually destroying the global structure of $L_2(3,9)$ while preserving the global structure of $L_2(9,3)$. And more important is, $L_2(3,9)$ is still being more stable for some values of $E$. Thus, preserving the global structure inappropriately can sometimes lower the stability. As a result, we cannot guarantee that $L_3$ is always more stable than $L_2$ because preserving the global structure does not necessarily increases the stability. Therefore, choosing between $L_3$ and $L_2$ is not as simple as one might have thought.

To determine whether $L_3$ or $L_2$ should be chosen for a given $N$ and $E$, we have to
look at the details of $L_2(3,9)$, $L_2(9,3)$ and $L_3(3,3,3)$. When we compare $L_2(3,9)$ to $L_3(3,3,3)$, both $L_2(3,9)$ and $L_3(3,3,3)$ have 9 giants and each giant has 3 nodes, while the only difference is that the 9 giants are connected in different ways. But when we compare $L_2(3,9)$ or $L_3(3,3,3)$ to $L_2(9,3)$, we immediately found that the size of each giant for $L_2(9,3)$ is 9, which is different from that of $L_2(3,9)$ and $L_3(3,3,3)$. Since the giant sizes for $L_3(3,3,3)$ and $L_2(9,3)$ are different, we can hardly tell which is more stable unless we plot the $D(L_3(3,3,3) ; L_2(9,3))$ as have done in Fig. 4.3b. And in fact, there are some negative parts in Fig. 4.3b which stands for $L_2(9,3)$ is being more stable in that region. But for $L_3(3,3,3)$ and $L_2(3,9)$, we can predict $L_3$ is more stable even we do not carry out any computation because both of them have the same size of giants and we know that $L_3$ is a better class of topology. In general, $L_3(p, q, r)$ is more stable than $L_2(p, qr)$ and this is the situation where the global structure is preserved appropriately.
4.4 Conclusion

By choosing appropriate size of giants, $L_3$ is a more stable topology compare to $L_2$ because the global structure can be preserved in $L_3$. One can have $4L$, $5L$, … but we would not bother about them because we can predict the stability of more layers would not be much better than that of $L_3$. As we have seen, when we jump from ER graphs to $L_2$, we gained about 12% more stability. And when we jump from $L_2$ to $L_3$, we gained an additional 4% stability. From the trend, if we jump from $L_3$ to $L_4$, the stability we shall gain will not be more than 2%.

Basically, the purpose of this thesis is achieved as we have found a class of topologies which is stable (both statically and dynamically), easy to build and somewhat realistic. Yet, there is still a simple way to improve the stability. In this chapter, we fill the edges into the giants by randomly choosing 2 nodes within a giant and connect them together. Knowing that random connection between nodes cannot give a stable network (i.e. ER graphs), one would expect there is a better way to fill the edges into the giants rather than the random connection method. And this is what we are going to explore in the next chapter.
Chapter 5 Filling a Ring by Shorten its Radius

5.1 Introduction: The Purpose of Shorten the Radius of a Ring

In chapter 3 and 4, the rule for adding extra edges to the hierarchical structure is to first add the edges to the 1st layer. After the 1st layer is filled up with edges, then we add the edges to the 2nd layer. When we add edges to the 1st layer, we added the edges randomly into a giant node. In other words, we randomly choose 2 nodes in a ring and connect them together. However, knowing that ER graph is unstable when compared to many other topologies, we would not expect adding edges to a ring randomly could give us a stable network. Instead, we should expect there are some methods better than adding edges randomly to the ring. In this chapter, we will show that when the ring is large enough, filling a ring by shortening its radius would give a much more stable network than randomly adding edges to the ring. Of course, the ultimate goal is to use this method to add edges to the giants and obtain an even more stable hierarchical structure.

Suppose we have a ring of N nodes and N edges and now we are going to add some extra edges to this ring. As one can see there can be numerous ways for doing so. One way, as we were using before, randomly choose 2 nodes of the ring and connect them together. However, it seems that this method of adding edges is not
wise at all. Consider a group of students who own a ring computer network. One day
their teacher gives a new cable to the students. How should the students add this new
cable to their already existing ring computer network? Immediately we see that
adding this new cable to the network randomly is not efficient because one might
obtain something as shown in figure 5.1a.

A better way to add this new cable can be the one as shown in figure 5.1b. There
are two reasons to do so.

First, information can be transmitted much faster in 5.1b. Let’s say node 1 wants to
send a message M to node 7. Without the new cable, a possible path for this message
M is 1-2-3-4-5-6-7 (i.e. M travels from node 1 to 2, then from 2 to 3 and etc…). With
the new cable added, the new path is 1-0-6-7 which is much shorter than the original
path. Thus time is saved and M reaches its destination faster. On the other hand, 5.1a
does not have such benefits. The shortest path for M to travel in 5.1a is 1-0-11-9-8-1
which is obviously slower.

Second, 5.1b is dynamically stable [18]. Suppose two edges are broken, say, edge
0-1 (the edge between node 0 and node 1) and 6-7 are broken (the edge between
node 6 and 7). Then the network 5.1a would be broken into 2 small networks. On the
other hand, network 5.1b can still retain as a connected network and would not be
broken into 2. In general, 5.1a would not be broken into 2 small networks only when
the following is true: one broken edge must be either 9-10 or 10-11 while the other broken edge must not be 9-10 or 10-11. For 5.1b, it would not be broken into 2 small networks when: one broken edge must be on the right hand side of the new cable while the other broken edge must on the left hand side of the new cable. From this simple argument, one can see that the probability of 5.1a to be broken into 2 small networks is much greater than that of 5.1b. Thus 5.1b is more dynamically stable than 5.1a.

The reason for 5.1b better than 5.1a is simple: the radius of 5.1b is shorter than that of 5.1a [19]. (The eccentricity $\epsilon(v)$ of a graph vertex $v$ in a connected graph $G$ is the maximum graph distance between $v$ and any other vertex $u$ of $G$. The minimum graph eccentricity is called the graph radius.) Thus, when we want to add edges to a ring, we should add it in a way such that the radius can be reduced as much as possible. (It is easy to see that when the radius is reduced more, the faster the information can be transmitted and the more dynamically stable network would be obtained.)
So, we have the following algorithm to add extra edges to a ring:

1. Find the maximum distance of the network.

2. Find the pair of nodes corresponding to this maximum distance. If there are more than one pair, then randomly choose one pair and connect them together.

3. Repeat step 1 and 2 until all extra edges are added.

However, following this algorithm could still give us different topologies with different radius. Since step 2 involves randomly choosing a pair of nodes, choosing different pair of nodes would have different evolution of the topology and thus different radius. As a result, the topology obtained by this algorithm does not necessary gives us the shortest possible radius.
For future reference, I would call this class of topologies obtained by this algorithm as Halfling. This term appears in many fictions, which is a race of people with height being half of a normal human. Or simply speaking, they are very short. Thus it would be a good name for our class of topologies which has short radius. In addition, it is a good contrast between the term Halfling and the term giant.
5.2 Stability of Halfling

From the previous section, we have seen Halfling has two advantages over structures obtained by adding edges randomly to the ring: dynamically more stable and faster information transmission. However, we still need to investigate the static stability and which is our main concern. Figure 5.2a shows the $D_{Halfling}$ for 6 nodes with $E = 7, 8$ and $9$. And we can see that the result is not nice because there are negative portions which mean Halfling is less stable than the ER graph in those portions. Figure 5.2b shows the $D$ values for a 6 nodes ring network with extra edges added randomly (for simplicity, let’s call this kind of topologies as ring random or RR in short). This time, the $D$ values are all positive. This is not surprising because a ring is a stable topology when compared to ER graphs with the same $N$ and $E$. And when we add more edges randomly to both ER graphs and the ring, one should expect the one which was built based on the ring should be more stable. Be careful, although Halfling is also based on a ring, it was built with the rule to shorten the radius and so Halfling does not necessarily be more stable than ER graph. And figure 5.2c shows $D(Halfling; RR)$ and again there are some negative portions which is undesirable.

From figure 5.2c, we know that when we are going to build a $L_3(6,3,3)$, we should
just add the extra edges randomly to the giant nodes (i.e. the RR method) because it seems that we would not gain much if we made the giant nodes to become Halflings. Nevertheless, even if the giants were made into Halflings, we would not lose much static stability because figure 5.2c also shows a large part of positive area. And by doing so, one could trade off some static stability for some dynamic stability and information conduction speed.

As a conclusion, for a 6 nodes ring (or a giant), it has both advantages and disadvantages to make the ring become Halfling. One should consider their purpose of network and choose an appropriate class of topologies accordingly.
Figure 5.2. 5.2a shows the $D_{\text{Halfling}}$ with $N=6$. 5.2b shows the $D$ values for a ring with $N=6$ with extra edges added randomly. 5.2c shows $D(\text{Halfling}; \text{RR})$ with $N=6$. The legends show the number of edges.
What would the $D(\text{Halfing}; \text{RR})$ become when $N$ became larger? Figure 5.3 shows the graphs for Halfling with $N=10$. In figure 5.3a and 5.3b, we see that the negative portion now become smaller as we compare them with that for $N=6$. It seems that when $N$ become larger, the stability of Halfling become better. And the graph of $D_{\text{RR}}$ is not shown here because it does not concern us with our problem.

![Fig 5.3a](image_url)

Figure 5.3a shows $D_{\text{Halfling}}$ with $N=10$. 5.3b shows $D(\text{Halfling}; \text{RR})$ with $N=10$. The legends show the number of edges $E$. 

![Fig 5.3b](image_url)
Figure 5.4a shows the $D_{\text{Halfling}}$ with $N=20$. Now the negative portion has disappeared which means Halfling is always more stable than the ER graph. Figure 5.4b shows $D_{\text{RR}}$ with $N=20$. And figure 5.4c shows $D(\text{Halfling}; \text{RR})$, where the negative portion almost disappeared. And now the Halfling is more stable than RR which is a desirable result.

Now we have an important result: when $N$ is large enough, Halfling is more stable than the ER graph and RR (statically). In addition, Halfling is also dynamical stable and has high information conduction speed. As a result, we should always build a Halfling instead of ER graph or RR when $N$ is large.
Figure 5.4. Fig 5.4a shows $D_{\text{Halfling}}$ with $N=20$. Fig 5.4b shows $D_{\text{RR}}$ with $N=20$. Fig 5.4c shows $D(\text{Halfling}; \text{RR})$ with $N=20$. The legend shows the number of edges $E$. 
5.3 Conclusion

By shortening the radius of a ring, we can obtain a more dynamically stable and high information conduction speed network. And when \( N \) is sufficiently large, this method (Halfling) would also give a statically stable network. As a result, we should not add extra edges to the ring randomly. Instead, we should make them Halfling. The next step is to modify the 3 layers hierarchical structure such that the giants are made to be Halflings and hopefully the resulted network is even more stable than before.
6. Hierarchical Structure of Halflings

In chapter 1, I have pointed out the goal of this thesis is to find a class of topology, such that the associated network is generally more stable than a connected ER graph. Since such a network has the same number of nodes and edges as the ER graph, a network engineer can then follow our guideline to build a network without spending more resources. In chapter 2, we have seen that the ring is a very stable topology and so we wanted to build a hierarchical structure of rings as we believe a collection of stable networks should also gives us a stable network. And this was done in chapter 3 and 4, by arranging the giant nodes (rings) into a ring structure to obtain the two layers hierarchical structure. Then again we arranged multiple rings of giants to form the three layers hierarchical structure. And we have seen that these hierarchical structures are more stable compared with ER graph. But we are not yet satisfied with these results. As we know that an ER graph is generally not stable, we expect that filling edges into a giant randomly cannot be a good idea. Therefore, in chapter 5, I introduced the Halfling which is going to replace the old method of filling edges. It is evident that, when N is large enough, the Halfling is more stable than a giant filled with edges randomly. In addition, Halflings have two more advantages that the old method does not posses: it is more dynamically stable and has a higher information
transmission speed. Now we come to this final chapter, I will show by making the giants into Halflings, the hierarchical structure will become even more stable. In this method, we gain the two advantages of Halflings without spending more resources. The hierarchical structure of Halflings would then be our final answer to the goal of this thesis.
6.1 Introduction: Why the Hierarchical Structure of Halflings?

Why the hierarchical structure of Halflings? One may think the answer is simple: it is stable. However, the true answer is not as obvious as that. If one has followed chapter 5 carefully, one should have noticed that a piece of very important information is missing: how is the stability of Halflings compared to the hierarchical structures? Why didn’t I compared the stability of $L_3(3,3,3)$ with Halflings? Does it implicitly imply $L_3(3,3,3)$ is more stable? No. In fact, with $N=27$, Halfling is more stable than $L_3(3,3,3)$. And please note that $L_3(3,3,3)$ only has 3 nodes as a giant so it does not matter with the problem of filling edges into the giants randomly or not. In other words, for the case of $N=27$, $L_3(3,3,3)$ has no difference between the hierarchical structure of rings or the hierarchical structure of Halflings. Therefore, for $N=27$, Halfling is better than the hierarchical structure (see figure 6.1).
Figure 6.1 D(L₃(3,3,3); Halfling) for N=27. The legend shows the number of edges E.

From figure 6.1, one can see the negative portion is larger than the positive portion. This implies that the Halfling is more stable in general. And this lead us back to the question, why should one choose a hierarchical structure? As one can see with the same values of N and E, in general, Halfling is more stable both statically and dynamically stable and has a higher information conduction speed when compared with the hierarchical structure of Halflings or rings. It seems there are no reasons for one to pick the hierarchical structure. However, hierarchical structure has it own merits that simple Halfling does not have.

First, hierarchical structure is natural. There are many hierarchical structures in the real world, e.g. the ecosystem, social network, the Internet… etc [20].
Second, hierarchical structure is essential. For example let us consider the structure of a large company. A number of employees form a group as a department, like the computer department, accounting department or man and power department… etc. Then many of such departments form the whole company. Without such a hierarchical structure, one cannot manages so many employees in an efficient way.

Third, building a hierarchical structure can sometimes save us resources. This may sound contradicting. When one has a number of N and E (resources), one can build any network he wants. And by building Halfling one can obtain a stable network while one does not have to spend more N and E (resources). However, we have ignored an important fact: the cost for each edge can be different. An example is given in figure 6.2. In figure 6.2, the Halfling requires 8 edges which are crossing the countries’ borders (e.g. from the USA to England). Meanwhile, the 2L(3,4) only requires 4 such edges. In this case, we cannot build a Halfling simply because the cost to build a country border crossing cable is very expensive. Especially when the values of N and E are very large, this same problem can become much more serious. Thus, hierarchical structure can indeed save a lot of resources in this example. Notice that the weighting of an edge is not the geometric length of that edge. The weight of an edge is the strength of interaction between 2 nodes connected by that edge. So
while the weight of an edge can be very heavy, the geometric length of the edge can be as short as 1 centimeter. Thus, unless all nodes are only spanning a small area, one must consider building the hierarchical structure instead of a Halfling even when the Halfling is more stable.

Figure 6.2 Computer networks with 12 nodes and 16 edges. A node represents a computer and a edge represents a cable. There are 3 computers belong to each country, namely England, Germany, the USA and Australia. The left one is a Halfling and the right one is 2L(3,4).
6.2 Stability of Hierarchical Structure of Halflings

From the previous chapter, it is known that the Halfling is more stable only when N is large enough. This time we would choose L₃(15,3,3) such that 15 nodes as a giant which can be made into a Halfling. Figure 6.3 shows the D(HH; HR) where HH stands for the hierarchical structure of Halflings and HR stands for the hierarchical structure of rings. One can see when E=150, D(HH; HR) is very close to zero. This is correct because each giant node has 15.3 edges on average when E=150. So there should be no difference between HH and HR. When E=250, 350 and 450, the curves are all positive. Notice that when E=550, the curve is very close to zero. And the further increases of E would give a negative curve. When E=850, each giant was almost fully connected (93.1 edges per giant on average, and the capacity of each giant is 105 edges) and thus the difference between HH and HR is small. The most interesting point is E=550, which is then a critical value. The nature of this critical value is yet an unknown. Nevertheless, the positive area is still much larger than the negative area. So we can roughly say that HH is better than HR. In addition, since HH has the dynamic stability and a faster information conduction speed, we would prefer HH rather than HR.
Figure 6.3 D(HH; HR) for $L_3(15,3,3)$ with 135 nodes. The legend shows the number of edges.
6.3 Conclusion

This comes to the end. Finally we have found the hierarchical structure of Halfling excels in many aspects. It is more stable, both statically and dynamically, and it conducts information faster. Sometimes it may not be more stable than the hierarchical of rings, but it posses more advantages that hierarchical of rings does not. When situation is available, one can also build a Halfling rather than the hierarchical structure of Halflings because a single Halfling is even more stable. Although we cannot prove all these classes of topologies are the best, but at least we provided a guideline for building a stable network. Therefore, our main purpose is achieved.
7. Conclusion

The problem originated from R.M. May’s works of stability of random networks. He pointed out the probability of stability $P$ of large random networks would drop from 1 to 0 at a critical value of $\sigma$. As the number of nodes $N$ and edges $E$ increase, this critical value of $\sigma$ can become very small such that $P \to 0$ all the time. From his conclusion, it seems that all kind of networks must be unstable if the networks were large. However, he did not give any conclusion about $P$ when the network is not very large. It is then worthwhile to study how to construct a stable network that is small or medium in size.

For given $N$ and $E$, one can have many different methods to connect them and yielding different topologies. Some topologies are more stable than the others. Our purpose is then to find the stable topologies.

In general, a hierarchical structure is stable if its sub-structures are stable. Therefore, we would like to find out the most stable topology for a small network. It turns out to be the line. Unfortunately, we cannot use a line to be the sub-structure because a line is dynamically unstable. We then modified the line by adding one edge to make a ring. Since the ring is stable both dynamically and statically, we use the ring as our sub-structure of the hierarchical structure. We introduced the hierarchical structure of rings with 2 layers and 3 layers and they are stable. When the giant sizes
of the 2 layers and 3 layers are equal, then the 3 layers would be more stable than the 2 layers. But when the giant sizes are different, then no general conclusion can be made. Normally a hierarchical structure of rings with more layers should be more stable. However, from the trend of the data we obtained here, it seems that further increasing the number of layers would not further increases the stability much. Therefore we would not bother with so many layers and conclude that 3 layers are enough. (For a 27 nodes network, 3 layers are about 14% more stable than a random network.)

We want to further improve the stability of the hierarchical structure, and we observed that adding edges randomly to a network normally destroy the stability. As a result, we introduced a new method of adding edges, called Halfling, which add edges to the rings by shorting the radius of the ring. It turns out that this method would give us an even more stable hierarchical structure when the giant size is larger than 15 (hierarchical structure of Halflings about 3% more stable than a hierarchical structure rings). When the giant size is small, hierarchical structure of rings would be more stable.

In conclusion, the hierarchical structure of rings is our recommendation for building a stable network.
Bibliography


