Bandwidth Extension Algorithm
for Multiple Deterministic Systems

by

Xu Didi

A Thesis Submitted to
The Hong Kong University of Science and Technology
in Partial Fulfillment of the Requirements for
the Degree of Master of Philosophy
in the Department of Mechanical Engineering

August 2006, Hong Kong
Authorization

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by

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This is to certify that I have examined the above MPhil thesis
and have found that it is complete and satisfactory in all respects,
and that any and all revisions required by
the thesis examination committee have been made.

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Contents

Title Page ......................................................... i
Authorization Page ........................................... ii
Signature Page ................................................ iii
Acknowledgements ........................................... iv
Table of Contents ........................................... v
List of Figures ................................................ viii
List of Tables ................................................ xi
Abstract ....................................................... xii

CHAPTER 1 ........................................................................................................... - 1 -

INTRODUCTION ................................................................................................. - 1 -

1.1 Background ............................................................................................... - 1 -

1.2 Literature review ...................................................................................... - 4 -

1.3 Motivation ................................................................................................ - 5 -

1.4 Outline of the Thesis ................................................................................ - 7 -

CHAPTER 2 ......................................................................................................... - 9 -

PRELIMINARIES AND PRINCIPLES ......................................................... - 9 -

2.1 Mathematical Preliminaries ....................................................................... - 9 -

2.1.1 Signal & System .................................................................................... - 9 -

2.1.2 Laplace transform & Fourier transform ............................................... - 12 -

2.1.3 Transfer function & Frequency response ........................................... - 16 -

2.1.4 The magnitude-phase representation of the Fourier transform .......... - 19 -
2.1.5 Discrete Fourier transform & Fast Fourier transform................................. 22
2.2 Applicable Conditions and BWE Principle.................................................. 25
  2.2.1 Extension to nonlinear systems ......................................................... 25
  2.2.2 Basic principles .................................................................. 30

CHAPTER 3 ............................................................................................... 35

BWE ALGORITHM FOR LOW-PASS FILTER SYSTEM......... 35

3.1 Introduction ....................................................................................... 35
3.2 Theoretical Part ................................................................................. 36
  3.2.1 Algorithm for bandwidth extension ............................................... 36
  3.2.2 Algorithm for signal transmission ................................................. 39
3.3 Hardware Description ...................................................................... 41
3.4 Evaluation Terms ............................................................................ 42
3.5 Experimental Results and Analysis .................................................. 43
  3.5.1 Frequency response .................................................................. 43
  3.5.2 Application I----Recovery of wideband signal ......................... 43
  3.5.3 Application II----Signal transmission ......................................... 47
3.6 Discussion & Comments .................................................................. 49

CHAPTER 4 ............................................................................................... 52

BWE ALGORITHM FOR MICROPHONE SYSTEM................. 52

4.1 Introduction ....................................................................................... 52
4.2 Theoretical Part ................................................................................. 57
4.3 Hardware Description ...................................................................... 58
4.4 Test Environment ............................................................................. 61
4.5 Evaluation Terms ............................................................................ 64
  4.5.1 Sub-band log spectral distortion measure ......................... 64
  4.5.2 Ease of listening .................................................................. 66
4.6 Experimental Results and Analysis .................................................. 66
  4.6.1 Transfer function ................................................................. 66
  4.6.2 Experiment results-1 ............................................................. 71
  4.6.3 Experiment results-2 ............................................................. 74
4.6.4 Experiment results-3 ................................................................. - 78 -

4.7 Discussion & Comments ............................................................. - 81 -

CHAPTER 5 ......................................................................................... - 83 -

BWE ALGORITHM FOR IMAGING SYSTEM ......................................... - 83 -

5.1 Introduction .................................................................................. - 83 -

5.2 Multidimensional Fourier Transform ............................................. - 84 -

5.3 Simulation Results .......................................................................... - 86 -

CHAPTER 6 ......................................................................................... - 92 -

CONCLUSIONS .................................................................................. - 92 -

6.1 Future Work .................................................................................. - 92 -

6.2 Contributions ................................................................................ - 93 -

BIBLIOGRAPHY ................................................................................. - 94 -
## List of Figures

<table>
<thead>
<tr>
<th>Figure</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>2.1</td>
<td>10</td>
</tr>
<tr>
<td>2.2</td>
<td>32</td>
</tr>
<tr>
<td>3.1</td>
<td>36</td>
</tr>
<tr>
<td>3.2</td>
<td>37</td>
</tr>
<tr>
<td>3.3</td>
<td>39</td>
</tr>
<tr>
<td>3.4</td>
<td>40</td>
</tr>
<tr>
<td>3.5</td>
<td>41</td>
</tr>
<tr>
<td>3.6</td>
<td>44</td>
</tr>
<tr>
<td>3.7</td>
<td>45</td>
</tr>
<tr>
<td>3.8</td>
<td>46</td>
</tr>
<tr>
<td>3.9</td>
<td>47</td>
</tr>
<tr>
<td>3.10</td>
<td>48</td>
</tr>
<tr>
<td>3.11</td>
<td>49</td>
</tr>
<tr>
<td>4.1</td>
<td>53</td>
</tr>
<tr>
<td>4.2</td>
<td>56</td>
</tr>
<tr>
<td>4.3</td>
<td>57</td>
</tr>
<tr>
<td>4.4</td>
<td>60</td>
</tr>
<tr>
<td>4.5</td>
<td>60</td>
</tr>
<tr>
<td>4.6</td>
<td>61</td>
</tr>
<tr>
<td>4.7</td>
<td>62</td>
</tr>
<tr>
<td>4.8</td>
<td>63</td>
</tr>
<tr>
<td>4.9.1</td>
<td>Magnitude characteristic of impulse signal</td>
</tr>
<tr>
<td>4.9.2</td>
<td>Phase characteristic of impulse signal</td>
</tr>
<tr>
<td>4.10</td>
<td>Repetitive impulse signals in experiments</td>
</tr>
<tr>
<td>4.11.1</td>
<td>Magnitude characteristic of transfer function</td>
</tr>
<tr>
<td>4.11.2</td>
<td>Phase characteristic of transfer function</td>
</tr>
<tr>
<td>4.12.1</td>
<td>Magnitude properties of complex input sound</td>
</tr>
<tr>
<td>4.12.2</td>
<td>Phase properties of complex input sound</td>
</tr>
<tr>
<td>4.13.1</td>
<td>Magnitude properties of output sound</td>
</tr>
<tr>
<td>4.13.2</td>
<td>Phase properties of output sound</td>
</tr>
<tr>
<td>4.14.1</td>
<td>Magnitude properties of modified output</td>
</tr>
<tr>
<td>4.14.2</td>
<td>Phase properties of modified output</td>
</tr>
<tr>
<td>4.15.1</td>
<td>Magnitude properties of complex input sound</td>
</tr>
<tr>
<td>4.15.2</td>
<td>Phase properties of complex input sound</td>
</tr>
<tr>
<td>4.16.1</td>
<td>Magnitude properties of output sound</td>
</tr>
<tr>
<td>4.16.2</td>
<td>Phase properties of output sound</td>
</tr>
<tr>
<td>4.17.1</td>
<td>Magnitude properties of modified output</td>
</tr>
<tr>
<td>4.17.2</td>
<td>Phase properties of modified output</td>
</tr>
<tr>
<td>4.18.1</td>
<td>Magnitude properties of English speech by female</td>
</tr>
<tr>
<td>4.18.2</td>
<td>Phase properties of English speech by female</td>
</tr>
<tr>
<td>4.19.1</td>
<td>Magnitude properties of output speech</td>
</tr>
<tr>
<td>4.19.2</td>
<td>Phase properties of output speech</td>
</tr>
<tr>
<td>4.20.1</td>
<td>Magnitude properties of modified speech output</td>
</tr>
<tr>
<td>4.20.2</td>
<td>Phase properties of modified speech output</td>
</tr>
<tr>
<td>5.1</td>
<td>The process of compensation for object imaging system</td>
</tr>
<tr>
<td>5.2</td>
<td>The facet of a black background image</td>
</tr>
<tr>
<td>5.3.1</td>
<td>Magnitude properties of facet</td>
</tr>
<tr>
<td>5.3.2</td>
<td>Phase properties of facet</td>
</tr>
<tr>
<td>Section</td>
<td>Title</td>
</tr>
<tr>
<td>---------</td>
<td>----------------------------------------------------</td>
</tr>
<tr>
<td>5.4</td>
<td>Grid created by sampling by 2</td>
</tr>
<tr>
<td>5.5.1</td>
<td>Magnitude properties of grid</td>
</tr>
<tr>
<td>5.5.2</td>
<td>Phase properties of grid</td>
</tr>
<tr>
<td>5.6</td>
<td>2D image from standard system</td>
</tr>
<tr>
<td>5.7</td>
<td>2D image from practical system</td>
</tr>
<tr>
<td>5.8</td>
<td>Modified output image after simulation</td>
</tr>
</tbody>
</table>
List of Tables

<table>
<thead>
<tr>
<th>Table</th>
<th>Description</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>3.1</td>
<td>SNR results of 2 applications</td>
<td>50</td>
</tr>
<tr>
<td>4.1</td>
<td>RMS LSD results of 3 experiments</td>
<td>81</td>
</tr>
<tr>
<td>4.2</td>
<td>Ease of listening results of 3 experiments</td>
<td>82</td>
</tr>
</tbody>
</table>
Bandwidth Extension Algorithm for Multiple Deterministic Systems

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Abstract

Bandwidth extension (BWE) is a field that has attracted increasing attention in recent years because bandwidth reduction can happen, for example, during recording, transmission (including storage), or reproduction. One way to extend the bandwidth of output is changing the transmission systems physically, which is hard to implement in practice due to economical and historical reasons. Another traditional method is to derive the missing frequency components only from the available bandwidth-limited signal, without referring to the transmission process.

This thesis is concerned with a new BWE algorithm based on the transfer function of multiple deterministic systems, using the tools of Fourier transform. The traditional definition of transfer function in linear time-invariant systems is extended further into nonlinear systems, and the basic hardware requirement is the availability of changing the digital energy to the actual force.

Experiments and simulations of low-pass filter system, audio system, or imaging system are operated to explore the bandwidth extension properties of this algorithm. In general, it is discovered and proved that this BWE method is effective to recover high-band output signal for different systems, assessed by various evaluation terms.
Chapter 1

Introduction

This thesis is concerned with a bandwidth extension algorithm in multiple deterministic systems, with experimental verification in low-pass filter system, audio system, and imaging system. The transfer function or the frequency response is determined, either for linear systems or nonlinear systems respectively, and used to eliminate the bandwidth limitation of output signals.

1.1 Background

Bandwidth will always be a limited resource. The problem caused by bandwidth happens everywhere in mechanical, electrical, and optical systems. That has long been obvious for broadcasts such as radio and television over airwaves. The available frequencies are limited, and two stations cannot use the same frequency in the same area at the same time without interference. On the other hand, bandwidth supplied by wires-telephone, cable TV, local area networks, may seem less limited, because wires can always be added or replaced by high-capacity connections. But the high cost of installing new wires or fiber-optic cables and the difficulty of integrating new capacity with old systems place functional limits on the available bandwidth for years and even decades [1].
Chapter 1 Introduction

What is Bandwidth?

The word ‘bandwidth’ has various meanings according to different situations. The IEEE Standard Dictionary of Electrical and Electronics Terms [2] gives for the most relevant cases:

- **Definition 1** Bandwidth of a continuous frequency band: The difference between the limiting frequencies.

- **Definition 2** Bandwidth of a waveform: The least frequency interval outside of which the power spectrum of a time-varying quantity is everywhere less than some specified fraction of its value at a reference frequency.

- **Definition 3** Bandwidth of a signal transmission system: The range of frequencies within which performance, with respect to some characteristics, falls within specific limits (usually 3dB less than the reference or maximum value).

Bandwidth comes in many different forms, and the most obvious way to categorize them is by media:

- Broadcast spectrum (over the air). Bandwidth in the so-called radio spectrum is the most limited and tightly controlled, to prevent interference between two signals. The best-known uses of this spectrum are for one-way broadcast of radio and television either from terrestrial or satellite transmitters.
• Telephone lines. The ubiquitous telephone lines-pairs of copper wire-run from a telephone company switch (formerly called the "central office") to each telephone.

• Cable television. Cable television uses coaxial cable to carry multiple television channels to homes.

• Local area networks. In a local area network (LAN), coaxial, fiber-optic, or twisted-pair telephone cables interconnect an organization’s computers so that information such as electronic mail and computer files can travel between machines.

• Wide area networks. Organizations with more than one location and groups of organizations that need to share information continuously use wide area networks (WANS) to span miles between sites or cross continents.

• Packaged media. A videocassette, a CD-ROM, a floppy disk, and a book all contain information. Although they may seem inert, you can think of them having bandwidth because there is always a rate at which information can be delivered from the package.

What is Bandwidth extension?

Bandwidth extension (BWE) refers to methods that increase the frequency spectrum, or bandwidth, of signals [3]. Such frequency extension is desirable if at some point the frequency content of the signals has been reduced, as can happen, for example,
during recording, transmission (including storage), or reproduction, mostly because of physical constraints. BWE is a field that has attracted increasing attention in recent years. Although some work was done in the early years of the twentieth century, a much more systematic and large-scale approach did not occur until now.

Actually, bandwidth reduction implies a decrease in perceptual quality, and therefore BWE algorithms are employed as tools to enhance the perceived quality of reproduced signals. In most cases, BWE methods are therefore post-processing algorithms, occurring just before reproduction, and this process aims to compensate for the limited bandwidth, no matter low-or high-frequency, which is already known as the prior condition.

1.2 Literature review

The need for the wideband signal transmission arises from the improvement of signal quality in telephone, and other applications such as broadband network, camera imaging. The bandwidth limitation in those cases is resulted primarily from the transducer and transmission channel. One way to extend the bandwidth of output is changing the transmission systems physically by replacing the plain-old-network with new network or enhancing transmission codecs. However, the bandwidth limitation is hard to change in the near future due to both economical and historical reasons.

In mechanical systems, the band-limited response can be inverted using a preceding filter with the inverse of this response, which is not the real inverse of the filtering response actually, but a band-pass filter to enhance the missing part of signals.
In many applications of audio technology, the synthetic frequency components are only obtained from the bandwidth-limited signal, without referring to the transmission system, and in current research two types of methods have been categorized as 'blind' method and statistical model method.

In imaging systems, recovery of input image signals can be solved by camera calibration. There are many different ways for camera calibration, from geometric methods, linear equation techniques, to non-linear optimization.

The above brief literature review is explained in detail from chapter 3 to chapter 5, according to different systems and applications, with particular methods and corresponding references.

1.3 Motivation

Why the Bandwidth extension is necessary and possible nowadays?

As electrical engineering advanced over the years, and especially with the advent of digital technology, signal enhancement became feasible through electronic means; at present, more and more fields in engineering relies heavily on signal processing techniques. Bandwidth extension (BWE) is one of the methods that can be used to improve the quality of transmitted signals, which is especially attractive in the areas of consumer electronics, such as telephones, loudspeakers or digital camera. In this market, signal quality is often less important because of economic constraints on the size and cost of components. Most manufacturers only want to produce as cheaply as possible, yet retain the basic need for acceptable quality. For example, loudspeakers can be built such that they properly reproduce the entire audible frequency spectrum,
down to 20Hz, up to 20 KHz; but such systems would be very expensive and also very bulky. As another example, digital storage and transmission of audio can be done without loss of information, but at a rather high bit rate. To achieve higher storage (coding) efficiency such that more audio can be stored with the same amount of bits, information has to be discarded. Telephony is another example where economic constraints led to the design of transmission system that had the smallest bandwidth that could be used, while ensuring good speech intelligibility at the price of a marked reduced quality.

Nonetheless, quality of signals does suffer from the physical limitation and in many cases a bandwidth reduction results. BWE methods are required because systems they operate with are somehow suboptimal, and usually made so by design. Since electronic means (such as BWE methods) are comparatively cheap and flexible, they play an ever-growing role in maintaining the good transmission of systems.

Why does the algorithm of this thesis exceed other coordinate methods?

The advantages of the BWE algorithm in this thesis are:

- Uniformity: Differing from other specific-methods-to-specific-systems, this BWE algorithm is versatile for multiple systems in, such as mechanical, audio, and optical fields.

- Availability: The implementation of this algorithm relies on the Fourier transform, which is a mature technique, packed in many computing units nowadays. In a word, this method is at hand when you need it and need not to be specially modified because of the general use.
Chapter 1 Introduction

- Familiarity, user-friendly: Not many people have the experience of operating the audio equipments, or metric cameras, while this BWE algorithm is easily interfaced with a computer and simply employed with digital electrical technology.

1.4 Outline of the Thesis

Chapter 2: Preliminaries and principles

In this chapter, mathematics foundation and system theory are presented, on which the thesis is built. Firstly, some basic knowledge on signal and system is introduced and then some useful concepts like transforms, transfer functions are listed respectively. Thirdly, basic principles of the BWE algorithm of this thesis are expressed in detail, as well as its assumptions and applicable conditions.

Chapter 3: BWE algorithm for low-pass filter systems

In this chapter, a new method is proposed to reconstruct the wideband output signal without any extra information of input signal after a low-pass filter. Moreover, a similar method to modify the input before sending out could be applied in the field of signal transmission. Finally, experimental results show that the proposed algorithms outperformed other conventional methods for estimating wideband signal.

Chapter 4: BWE algorithm for microphone systems

In this chapter, the BWE algorithm is used to recover different kinds of wideband audio signals, from tune, noise to speech, after a microphone system. Similarly, this algorithm is not input-dependent, and reconstructs signals without preknowledge of the input. Experimental results are presented both in frequency domain and time.
Chapter 1 Introduction

domain to certify the potency of this algorithm, then standard evaluation terms are also utilized to give persuasive proofs.

Chapter 5: BWE algorithm for imaging systems

In this chapter, the BWE algorithm is used to recover input image signals for imaging systems, which also have bandwidth extension requirements because of physical limitation of lens and/or CCD. It can be seen as the matter of cameral calibration problem. Simulation results are analyzed in both time and frequency domain to testify the feasibility of this algorithm and possible experiments to be done in imaging systems are discussed as well.

Chapter 6: Conclusions

In this chapter, the main contributions of this thesis are summarized and possible future works are proposed.
Chapter 2

Preliminaries and Principles

In this chapter, mathematics foundation and system theory are presented, on which the thesis is built. Firstly, some basic knowledge on signal and system is introduced and then some useful concepts like transforms, transfer functions are listed respectively. Thirdly, basic principles of the BWE algorithm of this thesis are expressed in detail, as well as its extensions and applicable conditions.

This chapter is organized as follows. In section 1, mathematical preliminaries are listed in detail, and BWE principle and its conditions are explained in section 2.

2.1 Mathematical Preliminaries

2.1.1 Signal & System

Signals may stand for a wide variety of physical phenomena. Although signals can be represented in many ways, in all cases the information in a signal is contained in a pattern of variations of some form, and signals are represented mathematically as functions of one or more independent variables [4].

Physical systems in the broadcast sense are an interconnection of components, device or subsystems. Theoretically, a system can be viewed as a process in which input signals are transformed by the system or cause the system to respond in some way,
resulting in other signals as outputs. In order to study a system well, a mathematical model could be constructed.

For this purpose, a system is represented by a black box with a number of accessible terminals as depicted in Figure 2.1. Terminals are divided into two groups: input terminals and output terminals. At input terminals, inputs are applied to the system, while outputs are observed (or measured) at output terminals [5].

![Figure 2.1 Block diagram of system](image)

A system is said to have an input-output description if its output can be expressed completely in terms of the corresponding input function, i.e., for each input there is only one output. Such a description conveys the idea that the output is "caused" by the input: the system transforms each input into a unique corresponding output. Thus for a system with an input-output description, its "action" can be represented by the transformation acting on an input to give a unique output.

For linear time-invariant systems, if \( h_\tau(t) \) denotes the response at time \( t \) to a unit impulse \( \delta(t-\tau) \) located at time \( \tau \), then the response of system in time domain can be expressed in the following equation.

\[
y(t) = \lim_{\Delta \to 0} \sum_{k=-\infty}^{\infty} x(k\Delta)h_{\Delta}(t)\Delta
\]

(2.1)
As $\Delta \to 0$, the summation on the right-hand side becomes an integral and accordingly the summation approaches the area under $x(\tau)h_\tau(t)$ could be viewed as a function of $\tau$. Therefore,

$$y(t) = \int_{-\infty}^{\infty} x(\tau)h_\tau(t) \, d\tau$$

(2.2)

As we know any input $x(t)$ can be represented as

$$x(t) = \int_{-\infty}^{\infty} x(\tau)\delta(t-\tau) \, d\tau$$

(2.3)

That is, $x(t)$ can be intuitively taken as a "sum" of weighted shifted impulses, and the weight on the impulse $\delta(t-\tau)$ is $x(\tau)h_\tau(t)$. With this interpretation, eq. (2.2) represents the superposition of the responses to each of these inputs.

Equation (2.2) represents the general form of the response of a linear system in continuous time. If, in addition to being linear, the system is also time-invariant, then $h_\tau(t) = h_0(t-\tau)$; i.e., the response of an LTI system to the unit impulse $\delta(t-\tau)$, which is shifted by $\tau$ seconds from the origin, is a similarly shifted version of the response to the unit impulse function $\delta(t)$. Again, for notational convenience, the unit impulse response $h(t)$ can be defined as

$$h(t) = h_0(t)$$

(2.4)

i.e., $h(t)$ is the response to $\delta(t)$.

In this case, eq. (2.2) becomes
\[ y(t) = \int_{-\infty}^{\infty} x(\tau) h(t-\tau) d\tau \]  

(2.5)

The convolution integral in eq. (2.5), which is defined as the integral of the product of the two functions after one is reversed and shifted, corresponds to the representation of a continuous-time LTI system in terms of its response to a unit impulse, instead of the summation format. The convolution of \( x(t) \) and \( h(t) \) will be represented symbolically as

\[ y(t) = x(t) * h(t) \]  

(2.6)

Here the symbol * is used to denote both discrete-time and continuous-time convolution.

### 2.1.2 Laplace transform & Fourier transform

The Laplace transform of a general signal \( x(t) \) is defined as

\[ X(s) = L\{x(t)\} = \int_{-\infty}^{\infty} x(t)e^{-st} dt \]  

(2.7)

And it is noted that in particular it is a function of the independent variable \( s \) corresponding to the complex variable in the exponent of \( e^{-st} \). The complex variable \( s \) can be written as \( s = \sigma + j\omega \), with \( \sigma \) and \( \omega \) being the real and imaginary parts, respectively.
Chapter 2 Preliminaries and Principles

According to the convolution theory of system introduced in last section, the response of a linear time-invariant system can be denoted in the way of Laplace transform.

\[ Y(s) = X(s) \cdot H(s) \quad (2.8) \]

where \( H(s) = L\{h(t)\} = \int_{-\infty}^{\infty} h(t)e^{-st} dt \) is the Laplace transform of \( h(t) \), which is the impulse response function of the linear time-invariant system.

Fourier transform is derived from the Fourier series representation of a periodic continuous-time signal \( x(t) \),

\[ x(t) = \sum_{k=-\infty}^{\infty} a_k e^{j\omega_0 t} \quad (2.9) \]

\[ a_k = \frac{1}{T} \int_{-T/2}^{T/2} x(t)e^{-j\omega_0 t} dt \quad (2.10) \]

where \( \omega_0 = 2\pi/T \).

Actually periodic signals rarely exist in practical situations, and a periodic signal can be constructed, for which the aperiodic signal is only one period. And aperiodic signal is identical to periodic signal over a longer interval, and as the period is infinite, aperiodic signal is equal to periodic one. Therefore, the above two equations are still suitable to be applied to aperiodic signals [4].

- 13 -
Defining the envelope $X(j\omega)$ as

$$X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t} dt \quad (2.11)$$

so that, for the coefficients $a_k$,

$$a_k = \frac{1}{T} X(jk\omega_0) \quad (2.12)$$

Combining eqs. (2.12) and (2.9), $x(t)$ is expressed in terms of $X(j\omega)$ as

$$x(t) = \sum_{k=-\infty}^{\infty} \frac{1}{T} X(jk\omega_0)e^{jk\omega t} \quad (2.13)$$

or equivalently, since $\omega_0 = 2\pi/T$,

$$x(t) = \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} X(jk\omega_0)e^{jk\omega t/\omega_0} \quad (2.14)$$

As $T \to \infty$ and then $\omega_0 \to 0$, the right-hand side of eq. (2.14) passes to an integral. Finally, eqs. (2.14) and (2.11) respectively become

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega)e^{j\omega t} d\omega \quad (2.15)$$

and

$$X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t} dt \quad (2.16)$$
Equations (2.15) and (2.16) are referred to as the Fourier transform pair, with the function $X(j\omega)$ referred as the Fourier Transform or Fourier integral of $x(t)$ and (2.15) as the inverse Fourier transform equation. The synthesis equation (2.15) represents that a signal could be reduced to a linear combination of complex exponentials for both periodic and aperiodic signals. For periodic signals, these complex exponentials have amplitudes $\{a_k\}$, and occur at a discrete set of harmonically related frequencies $k\omega_0$, $k = 0, \pm 1, \pm 2, \ldots$. For aperiodic signals, complex exponentials occur at a continuum of frequencies and, according to the synthesis eq. (2.15), have “amplitude” $X(j\omega)(d\omega/2\pi)$. Based on the terminology used for the Fourier series coefficients, the transform $X(j\omega)$ of periodic or aperiodic signal $x(t)$ is commonly referred to as the spectrum of $x(t)$, as it provides us with the information needed for describing $x(t)$ as a linear combination (specifically, an integral) of sinusoidal signals at different frequencies.

The Laplace transform has a close relationship with the Fourier transform, when the complex variable $s$ is purely imaginary.

When $s = j\omega$, eq. (2.7) becomes

$$X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t} dt$$

(2.17)

which corresponds to the Fourier transform of $x(t)$; that is,

$$X(s)|_{s=j\omega} = F\{x(t)\}$$

(2.18)
To illustrate this relationship explicitly, consider $X(s)$ as specified in eq. (2.7) with $s$ expressed as $s = \sigma + j\omega$, so that

$$X(\sigma + j\omega) = \int_{-\infty}^{+\infty} x(t)e^{-(\sigma + j\omega)t} dt \quad (2.19)$$

or

$$X(\sigma + j\omega) = \int_{-\infty}^{+\infty} [x(t)e^{-\sigma t}]e^{-j\omega t} dt \quad (2.20)$$

If the right-hand side of eq. (2.20) is reorganized as the Fourier transform of $x(t)e^{-\sigma t}$; that is, the Laplace transform of $x(t)$ can be interpreted as the Fourier transform of $x(t)$ after multiplication by a real exponential signal. The real exponential $e^{-\sigma t}$ may be decaying or growing in time, depending on whether $\sigma$ is positive or negative.

Therefore, the response of a linear time-invariant system can also be expressed by Fourier transform in the following equation.

$$Y(j\omega) = X(j\omega) \cdot H(j\omega) \quad (2.21)$$

### 2.1.3 Transfer function & Frequency response

The transfer function of a linear system is defined as the ratio of the Laplace transform of the output variable to the Laplace transform of the input variable, with all initial conditions assumed to be zero. A transfer function is a mathematical expression of the relation between the input and output of a linear time-invariant system, and the transfer function of a system (or element) is able to represent the relationship describing the dynamics of the system under specific considerations [6].
It is mainly used in linear, time-invariant system theory, signal processing, communication theory, and control theory.

Let \( x(t) \) be the input to a general linear time-invariant system, and \( y(t) \) be the output, and the Laplace transform of \( x(t) \) and \( y(t) \) be

\[
X(s) = L\{x(t)\} = \int_{-\infty}^{\infty} x(t)e^{-st}dt \tag{2.22}
\]

\[
Y(s) = L\{y(t)\} = \int_{-\infty}^{\infty} y(t)e^{-st}dt \tag{2.23}
\]

Then the output is related to the input by the transfer function \( H(s) \) as

\[
Y(s) = X(s) \cdot H(s) \tag{2.24}
\]

If the system is single-input single-output, the transfer function itself is therefore

\[
H(s) = \frac{Y(s)}{X(s)} \tag{2.25}
\]

Where \( H(s) \) is the symbol for the transfer function, \( Y(s) \) is the output function, and \( X(s) \) is the input function (see the section of Laplace transform).

A unit impulse function is useful for testing the transfer function. The unit impulse is based on a rectangular function

\[
f_\varepsilon = \begin{cases} 
1/\varepsilon & -\frac{\varepsilon}{2} \leq t \leq \frac{\varepsilon}{2} \\
0 & otherwise
\end{cases} \tag{2.26}
\]
Where $\varepsilon > 0$. As $\varepsilon$ approaches zero, the function $f_\varepsilon$ approaches the unit impulse function $\delta(t)$, which has the following properties:

- The Laplace transform of the unit impulse is equal to 1, and therefore the output for an impulse is the transfer function of the closed-loop system.
- The transfer function is simply the Laplace transform of the impulse response.

The transfer function can also be shown using the Fourier transform which is a special case of the bilateral Laplace transform for the case where $s = jw$.

$$H(jw) = \frac{Y(jw)}{X(jw)}$$  \hspace{1cm} (2.27)

The Fourier transform $H(jw)$ of the impulse response function $h(t)$ of a linear time-invariant system is called the Frequency response of the system.

On the other hand, the frequency response of a system is also defined as the steady-state response of the system to a sinusoidal input signal. The sinusoid is a unique input signal, and the resulting output signal for a linear system, as well as signals throughout the system, is sinusoidal in the steady state; it differs from the input waveform only in amplitude and phase angle.

One advantage of the frequency response method is the ready availability of sinusoid test signals for various ranges of frequencies and amplitudes. Thus the experimental determination of the frequency response of a system is easily accomplished and is the most reliable and uncomplicated method for the experimental analysis of a system.
As far as we know, the unknown transfer function of a system can be deduced from the experimentally determined frequency response of a system. Furthermore the design of a system in the frequency domain provides the designer with control of the bandwidth of a system and some measure of the response of the system to undesired noise and disturbances.

In this way, it is possible to get the frequency response of the system using the steady-state response of a system to a sinusoidal input test signal. From the experiment results, it is obvious to see that the response of a linear constant coefficient system to a sinusoidal input signal is an output sinusoidal signal at the same frequency as the input. However, the magnitude and phase of the output signal differ from those of the input sinusoidal signal, and the amount of difference is a function of the input frequency.

2.1.4 The magnitude-phase representation of the Fourier transform

The Fourier transform is in general complex valued and, as discussed before, can be represented in terms of its real and imaginary components or in terms of magnitude and phase. The magnitude-phase representation of Fourier transform $X(j\omega)$ is

$$X(j\omega) = |X(j\omega)| e^{j\angle X(j\omega)}$$  \hspace{1cm} (2.28)

$|X(j\omega)|$ is called magnitude, which provides the information about the relative magnitudes of the complex exponentials that make up $x(t)$, and on the other hand, the $\angle X(j\omega)$ does not effect the amplitudes of the individual frequency components, but instead represents the relative phase of these exponentials. The phase
relationships captured by $\angle X(jw)$ have a significant effect on the nature of the
signal $x(t)$ and thus typically contain a substantial amount of information about the
signal. In particular, depending upon what this phase function is, different-looking
signals might be obtained, even if the magnitude function remains unchanged.

In a similar fashion, this representation could be utilized to analyze transfer function
and frequency response of the system, then the transfer function in the frequency
domain becomes

$$H(jw) = |H(w)| e^{j\theta(w)} \quad (2.29)$$

For example, if there is a complex harmonic signal with a sinusoidal component with
amplitude $|X|$, angular frequency $w$ and phase $\arg(X)$

$$x(t) = |X| e^{j(wt + \arg(X))} = X e^{j\omega t} \quad (2.30)$$

where $X = |X| e^{j\arg(X)}$ is the input to a linear time-invariant system, then the
 entspreching component in the output is:

$$y(t) = |Y| e^{j(wt + \arg(Y))} = Y e^{j\omega t} \quad (2.31)$$

and $Y = |Y| e^{j\arg(Y)}$.

Note that, in a linear time-invariant system, the input frequency $\omega$ has not changed;
only the amplitude and the phase angle of the sinusoid have been changed by the
system. The frequency response $H(j\omega)$ describes this change for every frequency $\omega$
in terms of gain:
Chapter 2 Preliminaries and Principles

\[ |H(w)| = \frac{|Y|}{|X|} \quad (2.32) \]

and phase change:

\[ \phi(w) = \arg(H(w)) = \arg(Y) - \arg(X) \quad (2.33) \]

In graphically displaying continuous-time or discrete-time Fourier transform and system frequency responses in polar form, it is often convenient to use a logarithmic scale for the magnitude of the Fourier transforms [6]. Both the amplitude \( |H(w)| \) and the angle \( \phi(w) \) can be plotted versus the frequency \( w \) by using several different arrangements. The introduction of logarithmic plots, often called Bode plots, simplifies the determination of the graphical description of the frequency response.

As noted before, the phase relationship is additive, while the magnitude relationship involves the product of \( H(jw) \) and \( X(jw) \). Thus if the magnitudes of the Fourier transform are displayed on a logarithmic amplitude scale, an additive relationship can be gained, namely,

\[ \log|Y(jw)| = \log|H(jw)| + \log|X(jw)| \quad (2.34) \]

In addition, plotting the magnitude of the Fourier transform on a logarithmic scale allows detail to be displayed over a wider dynamic range. Typically, the specific logarithmic amplitude scale used is in units of \( 20\log_{10} \), referred to as decibels (abbreviated dB).
2.1.5 Discrete Fourier transform & Fast Fourier transform

In mathematics, the discrete Fourier transform (DFT), is a Fourier transform widely employed in signal processing and related fields to analyze the frequencies contained in a sampled signal, solve partial differential equations, and to perform other operations such as convolutions. The DFT can be computed efficiently in practice using a fast Fourier transform (FFT) algorithm.

The sequence of $N$ complex numbers $x_0, x_1, \ldots, x_{N-1}$ is transformed into the sequence of $N$ complex numbers $X_0, X_1, \ldots, X_{N-1}$ by the DFT according to the formula:

$$X_k = \sum_{n=0}^{N-1} x_n e^{-\frac{2\pi i n k}{N}} \quad k = 0, \ldots, N-1$$

(2.35)

where $e$ is the base of the natural logarithm, $i$ is the imaginary unit.

The inverse discrete Fourier transform (IDFT) is given by

$$x_n = \frac{1}{N} \sum_{k=0}^{N-1} X_k e^{\frac{2\pi i n k}{N}} \quad n = 0, \ldots, N-1$$

(2.36)

Note that the normalization factor multiplying the DFT and IDFT (here 1 and 1/N) and the signs of the exponents are merely conventions, and differ in some treatments.

When the DFT is used for spectral analysis, the $\{x_n\}$ sequence usually represents a finite set of uniformly-spaced time-samples of the signal $x(t)$, where $t$ of course represents time. The conversion from continuous time to samples (discrete-time) changes the underlying Fourier transform of $x(t)$ into a discrete-time Fourier
transform (DTFT), which generally entails a type of distortion called aliasing [7]. Choice of an appropriate sample-rate is the key to minimizing that distortion. Similarly, the conversion from a very long (or infinite) sequence to a manageable size entails a type of distortion called leakage, which is manifested as a loss of detail in the DTFT.

A final source of distortion (or perhaps illusion) is the DFT itself, because it is just a discrete sampling of the DTFT, which is a function of a continuous frequency domain. That can be mitigated by increasing the resolution of the DFT. A general procedure to deduce DFT from continuous Fourier transform is explained as follows.

The continuous Fourier transform is defined as

$$f(v) = \int_{-\infty}^{\infty} f(t)e^{-2\pi i vt} dt$$  \hspace{1cm} (2.37)

Now consider the generalization to the case of a discrete function, $f(t) \rightarrow f(t_k)$ by letting $f_k = f(t_k)$, where $t_k = k\Delta$, with $k = 0, \ldots, N-1$. Writing this out gives the discrete Fourier transform,

$$F_n = \sum_{k=0}^{N-1} f_k e^{-2\pi i nk/N}$$  \hspace{1cm} (2.38)

The inverse transform is then

$$f_k = \frac{1}{N} \sum_{n=0}^{N-1} F_n e^{2\pi ink/N}$$  \hspace{1cm} (2.39)
Chapter 2 Preliminaries and Principles

Discrete Fourier transforms are extremely useful because they reveal periodicities in input data as well as the relative strengths of any periodic components. There are a few subtleties in the interpretation of discrete Fourier transforms according to different references, however.

Fast Fourier transform (FFT) is an efficient algorithm to compute the discrete Fourier transform (DFT) and its inverse, which reduces the number of computations needed for $N$ points from $2N^2$ to $2N \lg N$, where $\lg$ is the base-2 logarithm [8]. The Cooley-Tukey FFT algorithm first rearranges the input elements in bit-reversed order, and then builds the output transform (decimation in time). The basic idea is to break up a transform of length $N$ into two transforms of length $N/2$ using the identity. If the number of points $N$ is not a power of two, a transform can be performed on sets of points corresponding to the prime factors of $N$, which is slightly degraded in speed. An efficient real Fourier transform algorithm is the fast Hartley transform (Bracewell 1999), which gives a further increase in speed by approximately a factor of two. Base-4 and base-8 fast Fourier transforms use optimized code, and can be 20-30% faster than base-2 fast Fourier transforms.

$$\sum_{n=0}^{N-1} a_n e^{-2\pi i n k / N} = \sum_{n=0}^{N/2-1} a_{2n} e^{-2\pi i (2n) k / N} + \sum_{n=0}^{N/2-1} a_{2n+1} e^{-2\pi i (2n+1) k / N} \quad (2.40)$$

$$= \sum_{n=0}^{N/2-1} a_{\text{even}} e^{-2\pi i k (N/2)} + e^{-2\pi i k / N} \sum_{n=0}^{N/2-1} a_{\text{odd}} e^{-2\pi i k (N/2)} \quad (2.41)$$

FFT is of great importance to a wide variety of applications, from digital signal processing to solving partial differential equations to algorithms for quickly multiplying large integers. Since the inverse DFT (IDFT) is of the same principle as
the DFT, but with the opposite sign in the exponential and a $1/N$ factor, any FFT algorithm can be easily adapted for it as well and generate IFFT algorithm.

2.2 Applicable Conditions and BWE Principle

2.2.1 Extension to nonlinear systems

According to the definitions of transfer function and frequency response in above sections, a key assumption has been widely used, i.e., linear time-invariant system. Basically, linear time-invariant systems are nonexistent in practice, so it is essential to extend the algorithm to nonlinear systems. Systems described in this thesis are all nonlinear systems, in a general sense.

The continuous transform is itself actually a generalization of an earlier concept, a Fourier series, which was specific to functions $f(t)$ defined over the finite interval $[0,T]$ and represents these functions as a series of sinusoids [9], [10]

$$f(t) = \sum_{k=-\infty}^{\infty} m_k \cdot e^{i w_k t}$$

(2.42)

where $w_k = 2\pi k / T$, and $m_k$ is a complex amplitude, except that $m_0$ is a real value for the frequency at 0.

$$f(t) = m_0 + [m_1 \ m_2 \ m_3 \ldots][\exp(iw_1 t) \ \exp(iw_2 t) \ \exp(iw_3 t) \ldots]^T$$

$$+ [m'_1 \ m'_2 \ m'_3 \ldots][\exp(-iw_1 t) \ \exp(-iw_2 t) \ \exp(-iw_3 t) \ldots]^T$$

(2.43)

Here, $m'_k$ is the conjugate of $m_k$, that is, if $m_k = a + bi$, then $m'_k = a - bi$. 

- 25 -
And both \([m_1, m_2, m_3, \ldots]\) and \([\exp(iw_1t), \exp(iw_2t), \exp(iw_3t), \ldots]\) are row vectors.

The Fourier coefficients \(m_k\) are given by

\[
m_k = \frac{1}{T} \int_0^T f(t) e^{-jw_k t} \, dt \quad k = 0, 1, 2, 3, \ldots
\] (2.44)

Every system has finite bandwidth, so in other words, it can only respond to the inputs that are in a certain frequency range, and the highest frequency component of its output is within system’s bandwidth [11]. Only the system performance within the bandwidth can be ensured since the signal outside the bandwidth cannot be correctly detected. To obtain an approximate expression for \(f(t)\), the Fourier series can be truncated up to the \((2n+1)\)th terms as

\[
f(t) = \sum_{k=-n}^{n} m_k \cdot e^{jw_k t}, \text{ for } n \text{ sufficiently large}
\] (2.45)

By the vector notation, the function \(f(t)\) is transformed to a vector in Fourier space resulting in

\[
f(t) = \begin{bmatrix} m'_n \cdots m'_2, m'_1, m_0, m_1 \cdots m_n \end{bmatrix}
\]

\[
\begin{bmatrix}
\exp(-iw_n t) \\
\vdots \\
\exp(-iw_2 t) \\
\exp(-iw_1 t) \\
1 \\
\exp(iw_1 t) \\
\vdots \\
\exp(iw_n t)
\end{bmatrix}
\]

\[= F \cdot B(t)
\] (2.46)
with $F = [m'_n \cdots m'_2 m'_1 m_0 m_1 \cdots m_n]$

and $B(t) = \left[ \exp(-iw_nt) \cdots \exp(-iw_2t) \exp(iw_1t) \exp(iw_0t) \cdots \exp(iw_nt) \right]^T$

The elements of $B(t)$ construct an orthonormal family in time interval $[0,T]$. Each element represents one axis in Fourier space. By Fourier transform, the time domain function could be mapped to one point in Fourier space.

For the simplicity, we assume that the system is single-input-single-output (SISO), so both input and output can be expressed by Fourier series with finite terms without loss of generality.

Representing both the input $x(t)$ and output $y(t)$ in the vector form, we have

$$
y(t) = \begin{bmatrix} Y'_n & \cdots & Y'_2 & Y'_1 & Y_0 & \cdots & Y_n \end{bmatrix} B(t) = Y \cdot B(t) \tag{2.47}
$$

where $Y = [Y'_n \cdots Y'_2 Y'_1 Y_0 \cdots Y_n]$

$$
x(t) = \begin{bmatrix} X'_n & \cdots & X'_2 & X'_1 & X_0 & \cdots & X_n \end{bmatrix} B(t) = X \cdot B(t) \tag{2.48}
$$

where $X = [X'_n \cdots X'_2 X'_1 X_0 X_1 \cdots X_n]$

According to the definition of frequency response for linear time-invariant system, there is a certain output signal at the same frequency as the sinusoidal input signal, with different magnitude and phase. For a series of sinusoidal inputs, there is a linear map from input to output.
where \( z_k = |z_k|e^{i\phi_k} \) is a complex number written by magnitude-phase representation of Fourier transform, and also \( z_k' \) is the conjugate of \( z_k \).

On the other hand, \( B(t) \) is an orthonormal basis in the given time duration, and each component of the system input or the output is independent of the other harmonic components.

As a result, for the input \( x(t) \), the output \( y(t) \) could be written as

\[
y(t) = \begin{bmatrix} z'_n \\ \vdots \\ z'_2 \\ z'_1 \\ z_0 \\ z_1 \\ \vdots \\ z_n \end{bmatrix} \begin{bmatrix} X'_n \\ \vdots \\ X'_2 \\ X'_1 \\ X_0 \\ X_1 \\ \vdots \\ X_n \end{bmatrix} B(t)
\]

Consequently, we get the nice relationship between \( X \) and \( Y \) as
where $Z$ is a diagonal matrix that is completely by system itself.

For nonlinear systems, there is not such one-to-one relationship for each frequency component between inputs and outputs. In other words, the input is a unique sinusoidal signal, but the resulting output signal for a nonlinear system, not only has one frequency component the same as input, but also covers various frequencies. The corresponding outputs for a series of single-frequency inputs can be shown in a map as well.

\[
\begin{bmatrix}
\exp(-i\omega_n t) \\
\vdots \\
\exp(-i\omega_1 t) \\
1 \\
\exp(i\omega_1 t) \\
\vdots \\
\exp(i\omega_n t)
\end{bmatrix}
\rightarrow
\begin{bmatrix}
\hat{z}_{m1} & \cdots & \hat{z}_{m1} & \hat{z}_{m0} \\
\vdots & \ddots & \vdots & \vdots \\
\hat{z}_{11} & \cdots & \hat{z}_{11} & \hat{z}_{10} \\
\hat{z}_{01} & \cdots & \hat{z}_{01} & \hat{z}_{00} \\
\vdots & \ddots & \vdots & \vdots \\
\hat{z}_{00} & \cdots & \hat{z}_{00} & \hat{z}_{00} \\
\vdots & \ddots & \vdots & \vdots \\
\hat{z}_{m0} & \cdots & \hat{z}_{m0} & \hat{z}_{m0} \\
\end{bmatrix}
\rightarrow
\begin{bmatrix}
\exp(-i\omega_n t) \\
\vdots \\
\exp(-i\omega_1 t) \\
1 \\
\exp(i\omega_1 t) \\
\vdots \\
\exp(i\omega_n t)
\end{bmatrix}
\]

Similar to the deduction in linear time-invariant system, the following relationship is derived finally.
\[
Y = X \cdot \begin{bmatrix}
\vdots & \vdots & \vdots \\
\vdots & \vdots & \vdots \\
\vdots & \vdots & \vdots \\
\vdots & \vdots & \vdots \\
\vdots & \vdots & \vdots \\
\vdots & \vdots & \vdots \\
\end{bmatrix}
\]

In general, it is possible to get the parameters in \( z \) by using single-frequency test signals in a wide range of frequencies, although there are responses with more than one frequency components for each input. Thus the unknown ‘transfer function’ of a nonlinear system can be deduced from the experimentally determined frequency response of a system. Note the ‘transfer function’ in this thesis is not just limited for LTI systems only, but also a universal denotation for the relationship between inputs and outputs.

### 2.2.2 Basic principles

All practical system can be simplified as input-system-output model. The general principle of the method described in this thesis is to find the optimal representation of system and eliminate its effects on input signals. In time domain, it is impossible to find the suitable representation because a complicated convolution between input and system exists. Instead, the way done in the frequency domain outperforms the way in time domain because of the simple relationship between input and system thanks to the help of Laplace transform and Fourier transform.

If an input signal passes through a system, the transmission process could be represented by the convolution of input and system in time domain.
\[ y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau \]  
(2.52)

According to the convolution theorem and magnitude-phase representation described before, the transform of a convolution is the product of the transforms, which could be simply expressed as following by Fourier transform.

\[ Y(w) = H(w) \cdot X(w) \]  
(2.53)

\[ |Y(w)| = |H(w)||X(w)| \]  
(2.54)

\[ \angle Y(w) = \angle H(w) + \angle X(w) \]  
(2.55)

\[ |X(w)|, \angle X(w) \] are frequency characteristics of \( x(t) \), and \( |Y(w)|, \angle Y(w) \) are frequency characteristics of \( y(t) \) respectively.

\[ H(w) = |H(w)|e^{i\phi(w)} \]  
(2.56)

\( H(w) \) is the transfer function of the system composed of magnitude \( |H(w)| \) and phase \( \phi(w) \).

From the above equations, the relationship of input, transfer function and output could be represented by the product in the frequency domain. Explicitly speaking, the amplitude of the input is multiplied by the factor \( |H(w)| \) and its initial phase adds a phase shift \( \phi(w) \) to generate the output.
In order to avoid bandwidth truncated effects of system and reconstruct the wideband signal from output signal directly, the inverse of transfer function is combined with the Fourier transform of the output signal.

\[ |Y'(w)| = |Y(w)|/|H(w)| \] \hspace{1cm} (2.57)

\[ \angle Y'(w) = \angle Y(w) - \angle H(w) \] \hspace{1cm} (2.58)

These equations indicate that the output could be modified in frequency domain by dividing the magnitude \( |H(w)| \) and subtracting a phase-shift \( \phi(w) \).

Similarly, the input could be rebuilt by eliminating the effects of system on both magnitude and phase in frequency domain, if this BWE algorithm is applied in the field of secure transmission. The new input signal becomes:

\[ |X'(w)| = |X(w)|/|H(w)| \] \hspace{1cm} (2.59)

\[ \angle X'(w) = \angle X(w) - \angle H(w) \] \hspace{1cm} (2.60)
On the theoretical side, the output signal should become the same as the wideband input signal in spectrum due to the cancellation of system response. Two different ways of the implementation of BWE algorithm are depicted in Figure 2.2.

Thus in practical situations output values are given only at discrete points of independent variables, as with physical measurements made at regular time intervals and finite length of input signals. If the sampling starts at time $t(0)$, the sequence of sampling points becomes

$$t(0), t(1), t(2), \ldots, t(N-1)$$

Therefore, it is desirable to use discrete Fourier transform (DFT) to do numerical computing. And given discrete transform, one may recover the time series with the aid of inverse discrete Fourier transform (IDFT), which are expressed in detail in last section.

According to DFT the output signal in frequency domain can be written as:

$$Y(k) = \sum_{n=0}^{N} y(n)e^{-j2\pi(k/N)n}$$  \hspace{1cm} (2.61)

As DFT works at multiples of the fundamental frequency, $w = k/N$ i.e. the transfer function only modifies the magnitude and phase at these sample points, which is called digitizing frequency responses of the system. In order to implement this digitization, only magnitudes and phases of the transfer function at frequencies 0, $1/N$, $2/N$ ... $N-1/N$ are taken, to change corresponding values of the output.

$$H(0), H(1/N), H(2/N) \ldots, H((N-1)/N)$$
Finally, $Y'(w)$ is obtained as the new output in frequency domain. After doing IDFT with the same number of sample points, the modified signal $y'(t)$ is obtained in time domain.

$$y'(n) = \frac{1}{N} \sum_{k=0}^{N} Y'(k) e^{j2\pi(k/N)n} \quad (2.62)$$

In the following chapters regarding practical experiments of BWE algorithm in this thesis, Fast Fourier transform (FFT) is used, because of its high efficiency in computing and easy-access in Matlab and DSP system.

Additionally, the core for applications of this BWE algorithm is that the digital energy could turn to the actual force, which means that physical parameters are able to be changed numerically by programming, supplied by power source accordingly. In order to satisfy this requirement, a computing unit (computer, DSP, FPGA etc) is needed to take the charge of implementing algorithm, and ADC and DAC devices are utilized to digitize inputs and outputs at different sampling rates, which consequently induce relative limitations of this method, a prerequisite computing hardware and the accessible programming for discrete samples.
Chapter 3

BWE Algorithm for Low-Pass Filter System

Recovery of the wideband output signal at the receiving end of transmission system is a fascinating issue, because it could still get the high frequency signal when passing through a low-pass filter (LPF) system. In this chapter, a new method is proposed to reconstruct the wideband output signal without any extra information of input signal after a low-pass filter. Moreover, a similar method to modify the input before sending out could be applied in the field of signal transmission. Finally, experimental results show that the proposed algorithms outperformed other conventional methods for estimating wideband signal.

3.1 Introduction

Traditionally, the band-limited response can theoretically be inverted using a preceding filter with the inverse of this response, which is shown in Figure 3.1. The signal is filtered by $H^{-1}$, the “inverse” of the filtering response, scaled and applied to the output. Usually the $H^{-1}$ does not mean the real inverse of the filtering response, but simply a band-pass filter to boost the truncated-bandwidth part of signals [3]. The subsequent scaling may be manually adjustable or signal dependent. In practice, at high output levels, distortion or even damage and ultimately destruction of the output may occur. Also this solution is quite energy inefficient, due to the intrinsic low efficiency at low frequencies, but the advantages of this approach are its simplicity and linearity.
The method in this chapter to recover the wideband output signal is estimating possible results from the narrowband output of deterministic low-pass filter systems. Since the frequency response of the system could be obtained using tester sine wave signals, it is possible to get the pleasant output of different frequencies by canceling the effects of transmission system, which could be implemented by combining the inverse of the transfer function of the system with the Fourier Transform of the output signal. On the other hand, in the application of signal transmission, the non-distorted transmission is guaranteed by modifying the input at the input terminal.

This chapter is organized as follows. Basic principle of BWE algorithm applied in LPF is explained in section 2, and hardware description is introduced in section 3. Evaluation terms are shown in section 4, and in section 5, experiment results and assessment are presented. At last, we draw our discussion and conclusion.

3.2 Theoretical Part

3.2.1 Algorithm for bandwidth extension

If an input signal consisting of high frequency components passes through the low-pass filter system (LPF), its bandwidth will definitely become narrow. This filtering process could be represented by the convolution in time domain and the product in frequency domain.
\[ y(t) = \int_{-\infty}^{\infty} x(\tau) h(t - \tau) d\tau \] \hspace{2cm} (3.1)

\[ Y(w) = H(w) \cdot X(w) \] \hspace{2cm} (3.2)

As far as we know, the magnitude-phase representation of the transfer function of the system could be written as:

\[ H(w) = |H(w)| e^{i\phi(w)} \] \hspace{2cm} (3.3)

where \( H(w) \) is the transfer function of the LPF composed of magnitude \( |H(w)| \) and phase \( \phi(w) \).

Clearly, the relationship of input, transfer function and output could be represented by the product in the frequency domain. Explicitly speaking, the amplitude of the input is multiplied by the factor \( |H(w)| \) and its initial phase is added by a phase shift \( \phi(w) \) to generate the output [12].

In order to eliminate the effects of LPF and reconstruct the wideband output signal, the inverse of transfer function is combined with the Fourier transform of the original output signal, shown in Figure 3.2.

![Figure 3.2 BWE algorithm applied on output](image-url)
\[ |Y'(w)| = \frac{|Y(w)|}{|H(w)|} \]  

(3.4)

\[ \angle Y'(w) = \angle Y(w) - \angle H(w) \]  

(3.5)

These equations indicate that the output could be modified in frequency domain by dividing the magnitude \(|H(w)|\) and subtracting the phase-shift \(\phi(w)\).

In practical situations output values are given only at discrete points of independent variables, as with physical measurements made at regular time intervals and finite length of input signals. Therefore, it is desirable to use discrete Fourier transform (DFT) to do numerical computing. And given the discrete transform, one may recover the time series with the aid of the inverse discrete Fourier transform (IDFT).

According to DFT the output signal in frequency domain can be written as [13]:

\[ Y(k) = \sum_{n=0}^{N} y(n)e^{-j2\pi(k/N)n} \]  

(3.6)

Finally, \(Y'(w)\) is obtained as the new output in frequency domain. After doing IDFT with the same number of sample points, the modified signal \(y'(t)\) could be obtained in time domain.

\[ y'(n) = \frac{1}{N} \sum_{k=0}^{N} Y'(k)e^{j2\pi(k/N)n} \]  

(3.7)

The key of this method is to modify in frequency but result in time domain. In case where input signal is unknown, it is convenient to do this kind of modification on
output signal if the system is predictable. In Figure 3.3, a diagram of the current system is shown to recover the wideband output signal.

Figure 3.3 Flow chart of implementation 1

### 3.2.2 Algorithm for signal transmission

Similar method can be applied in the area of signal transmission with known input, plotted in Figure 3.4. Actually, what we want is the integral wideband output signal which is affected both by the input and the transmission system. The technique in this section is to modify the input signal using the inverse of transfer function before sending out instead of changing the output signal.
In frequency domain, $X(w)$ can be computed also by DFT.

$$X(k) = \sum_{n=0}^{N} x(n)e^{-j2\pi(k/N)n} \quad (3.8)$$

As mentioned in last chapter, the input in frequency domain could be rebuilt by eliminating the effects of system on both magnitude and phase. The new input signal becomes:

$$|X'(w)| = |X(w)|/|H(w)| \quad (3.9)$$

$$\angle X'(w) = \angle X(w) - \angle H(w) \quad (3.10)$$

Thus, we could create the modified input signal $x'(t)$ via IDFT. Let $x'(t)$ pass through the low-pass filter, the output signal will turn out to be the same as the input signal.

$$x'(n) = \frac{1}{N} \sum_{k=0}^{N} X'(k)e^{j2\pi(k/N)n} \quad (3.11)$$

The modified signal transmission process could be shown in Figure 3.5.
3.3 Hardware Description

The low-pass filter system in this chapter is consisted of a low-pass filter (EF4-03), a DAQ board (AT-MIO-16), and an oscilloscope. The input analog signals were generated by DAQ board and then presented to the low-pass filter on the board, and the modified signals of analog output were also acquired by DAQ board and then sent to the oscilloscope. In addition, for the benefit of good observation and precise analysis of results, it is recommended to use Matlab to plot data. In this experiment, 1) the DAQ board is used for conversion between analog and digital signals, 2) the computer is used to implement the BWE algorithm, and 3) the low-pass filter is used for filtering with a cut-off frequency of 50Hz. Obviously, the input signal has wider bandwidth than the output signal, so the purpose of this experiment is to reproduce the output analog signal with the same bandwidth as the input signal.
3.4 Evaluation Terms

**SNR**

Signal-to-noise ratio is an engineering term for the power ratio of a signal (meaningful information) and the noise.

\[
\text{SNR} = \frac{P_{\text{signal}}}{P_{\text{noise}}} = \left( \frac{A_{\text{signal}}}{A_{\text{noise}}} \right)^2
\]  

(3.12)

Because many signals have a very wide dynamic range, SNRs are usually expressed in terms of the logarithmic decibel scale. In decibels, the SNR is 20 times the base-10 logarithm of the amplitude ratio, or 10 times the logarithm of the power ratio:

\[
\text{SNR}(dB) = 10 \log_{10} \left( \frac{P_{\text{signal}}}{P_{\text{noise}}} \right) = 20 \log_{10} \left( \frac{A_{\text{signal}}}{A_{\text{noise}}} \right)
\]  

(3.13)

where \( P \) is average power and \( A \) is amplitude. Both signal and noise power are measured within the system bandwidth.

Often the signals being compared are electromagnetic in nature, though it is also possible to apply the term to sound stimuli. Due to the definition of decibel, the SNR gives the same result independent of the type of signal which is evaluated (such as power, current, or voltage).

Signal-to-noise ratio is closely related to the concept of dynamic range, which measures the ratio between noise and the greatest non-distorted signal on a channel. Because of this, measuring signal-to-noise ratios requires the selection of a
representative or reference signal. SNR is usually taken to indicate an average signal-to-noise ratio, as it is possible that (near) instantaneous signal-to-noise ratios will be considerably different. The concept can be understood as normalizing the noise level to 1 (0 dB) and measuring how far the signal 'stands out'. In general, higher signal to noise is better, which means the signal is 'cleaner'.

3.5 Experimental Results and Analysis

3.5.1 Frequency response

In experiments, the high frequency tester signal is generated so as to get the frequency response of the low-pass filter.

\[ x = \sin(2\pi t) + \sin(200\pi t) \]  
\[(3.14)\]

Firstly using the tester input signal to transmit through low-pass filter, then the limited-band output is received by DAQ board. After doing DFT on both input and output, the frequency response of the system at 1Hz and 100Hz is obtained based on convolution theorem.

3.5.2 Application I----Recovery of wideband signal

Original signal  
\[ x = 0.4 \cos(2\pi t) + 0.3 \sin(200\pi t) \]  
\[(3.15)\]

In Figure 3.6, only the first 200 points are plotted to make sure that high frequency components could also be seen clearly.
Figure 3.6 Original input signal

Note the output signal, from Figure 3.7.1 to Figure 3.7.3, most of the high frequency parts of the input were filtered through the LPF, although there remained little curves remained due to the environment disturbance or equipment errors. Figure 3.7.1 is plotted by Matlab using the raw data from experiments; Figure 3.7.2 and Figure 3.7.3 are observed directly on the oscilloscope.
Figure 3.7 Output signal without modification

Using the bandwidth extension algorithm to modify the output, the new result of the output signal is generated in Figure 3.8.1 to Figure 3.8.2. It is obvious to see that the output signal is more or less the same as the wideband input signal despite the transient response at the first 30 points. The high frequency of the output signal can be traced up to 100Hz.
Figure 3.8 Modified output by the inverse of transfer function

It is magic to see the high frequency signal even after a low-pass filter, which breakdown all the traditional impressions of LPF. For the purpose of the perfect recovery of input signal, removing the transient response is the next target, which seems common and unavoidable in every filter.

Although, it is impossible to find a universal solution to eliminate any transient response, the problem here could be solved out locally, i.e. one method-to-one case. In these experiments, the frequency of the transient response part is almost the same as the original wideband signal, so it is feasible to modify the magnitudes of these points independently, as shown in Figure 3.9.
Figure 3.9.1

Figure 3.9.2  Figure 3.9.3

Figure 3.9 Modified output without transient response

3.5.3 Application II----Signal transmission

On the other hand, if some modifications are operated on the input using BWE algorithm, satisfactory results could be expected at the end of the output side. It is easy to see in Figure 3.10 that the output signal is generally recovered as original
wideband input signal despite the transient response at first. And Figure 3.11 shows the further modified output with little transient response.

Figure 3.10.1

Figure 3.10.2

Figure 3.10.3

Figure 3.10 Output after LPF by the modification of input
3.6 Discussion & Comments

In this evaluation, the SNR doesn’t necessary represent the common ‘signal’ to ‘noise’ ratio, but the power ratio between different kinds of outputs and original
input signal, which further means that the absolute value becomes small if the output mostly approach the original input. And in application 2, no original output is produced because the input is modified instead before transmitting.

The SNR results of 2 applications are illustrated in the following Table 3.1. As expected, it is observed that the quality of output can be improved both by modifying the input and output. The effects of amending transient response of modified outputs are also conspicuous because the comparison of SNR results indicates a roughly 1dB advantage.

<table>
<thead>
<tr>
<th></th>
<th>( \frac{O_{OPT}}{INP} ) (dB)</th>
<th>( \frac{M_{OPT}}{INP} ) (dB)</th>
<th>( \frac{MTO_{OPT}}{INP} ) (dB)</th>
</tr>
</thead>
<tbody>
<tr>
<td>APP 1</td>
<td>-6.1419</td>
<td>-0.7739</td>
<td>0.2893</td>
</tr>
<tr>
<td>APP 2</td>
<td>N/A</td>
<td>1.1103</td>
<td>0.2187</td>
</tr>
</tbody>
</table>

Table 3.1 SNR results of 2 applications

Note:
INP = Input signal
OOPT = Original output without BWE algorithm
MOPT = Modified output with BWE algorithm
MTOPT = Modified transient response of output signal
APP 1 = Application 1
APP 2 = Application 2

This chapter introduces a new BWE method for two applications of a LPF system: 1) recovery wideband signal from narrowband signal is proposed in this chapter; 2) a modification method for integral signal transmission with given input is also explained. Both of the two methods are based on the deterministic frequency response of the system, which means that the system performance could be known in

- 50 -
advance. It is not appropriate to use these two methods in the wireless and unstable fields since too much computational work has to be finished immediately to catch up with changes. But for a fixed and well established system, computing chips could be incorporated to calculate frequency responses and modify signals on line.
Chapter 4

BWE Algorithm for Microphone System

In this chapter, the BWE algorithm is used to recover different kinds of wideband audio signals, from tune, noise to speech, after a microphone system. Similarly, this algorithm is not input-dependent, and it reconstructs signals without preknowledge of input. Experimental results are presented both in frequency domain and time domain to certify the potency of this algorithm, and then standard evaluation terms are utilized to give persuasive proofs.

4.1 Introduction

In many applications of audio technology, small bandwidth equipments are unavoidable, because of size and/or economic constraints. For example, loudspeakers which are powerful to reproduce the lowest audible frequencies (around 20Hz) at a sufficient level are very expensive and bulky. Telephony is another example, in which very small loudspeakers are employed, leading to the characteristic sound of "telephone-speech".

The bandwidth limitation in those cases is resulted primarily from the transducer and transmission channel. Despite the physical way to compensate bandwidth limitation by improving transmission system, BWE processing methods could be applied to enhance the reproduction of missing frequency components of bandwidth-limited audio signal.
If a given reproduction system can recover wideband signals from narrowband ones, synthetic frequency components are added to the bandwidth-truncated signal, improving the quality thereof. Traditionally, the added synthetic frequency components are only derived from the available bandwidth-limited signal, without referring to the transmission process, and based on this principle two common approaches have been developed:

In the first approach, the BWE algorithm is blind, which means, it has no information regarding the missing frequency part, so only assumptions on the statistics of audio signals can be used to design such systems [14], [15]. The main advantages of this approach are that such a BWE system can be utilized in a wide class of signals, such as music and speech, and that there are no requirements on the signal format because the only necessary information is the actual waveform or spectrum of band-limited signal. The drawback is that the performance of the bandwidth-extended output signal is significantly worse than that of the original full-bandwidth signal, even though it is better than the bandwidth-limited signal without doing BWE.

Figure 4.1 Block diagram of the blind BWE method
Chapter 4 BWE Algorithm for Microphone System

The general example of the first approach presented here is shown in Figure 4.1 [3]. There are two branches, the lower of which simply delays the input signal in order that it is later added exactly in phase with the processed signal from the upper branch. And the upper one consists of two filters and a non-linear device (NLD). The first filter, FIL1, ensures that only frequencies below given frequency of \( x(t) \) enter the NLD. The non-linear processing generates a harmonic signal, which is filtered by FIL2 to obtain a suitable spectral envelope. After scaling of \( g \), the resulting signal is added back to \( x(t) \) to yield the bandwidth-extended output \( y(t) \). The whole implementation is in the time domain, which has the benefit of computational efficiency. Frequency domain algorithms would be possible, but suffer from the drawback that it would be difficult to achieve the desired frequency resolution while at the same time keeping the analysis window sufficiently short to satisfy stationarity of the input signal.

The second method most frequently used in BWE algorithm does require a prior knowledge regarding the missing frequency components. This allows for a much more exact reconstruction of the bandwidth-extended signal, that is, the difference could become transparent or indistinguishable, from the original full-bandwidth signal. The high quality of the output signal is obviously the main advantage of this approach, and the drawback is that more preparations need to be made to provide the BWE algorithm with the required information in advance.

The basic principle of this method is based on the observation that characteristics of wideband signals typically exhibit a close relationship with those of the narrowband signals [16]. Therefore, it is often possible to replace the high-bandwidth with a transposed version of the low-bandwidth, without the need to transmit the wideband
signal at all. According to this theory, if a mathematical model of the source of the signal is available, both the original wideband signal and the band-limited signal are determined by parameters of the same source model. Consequently, exact knowledge of these source parameters would bring up the possibility to reconstruct the complete wideband signal as it was originally created. The parameters of the source, on the other hand, can be estimated from the features of the band-limited signal utilizing statistical models such as Gaussian Mixture Model (GMM) [17] [18], a hidden Markov model (HMM) [19] to predict the lost high-bandwidth spectrum.

Since any mathematical model is only a statistical approximation of the real physical source of a signal, there are several potential drawbacks of such model-based approaches: owing to simplifications introduced by the modeling, there will be estimation errors on both of the parameters of the source model as well as of the reconstructed wideband signal [20]. Another cause of estimation errors follows from the basic properties of the signal source. In general, it is not possible to estimate the parameters of the source with arbitrary accuracy because of the complex attributes of the signal. In addition, if the characteristics of the actual physical source do not match the source model perfectly, that is, if there is any model mismatch, the probability of estimation errors and artifacts in the reproduced signal will increase. Moreover, the utilization of a particular model for the signal source also imposes fundamental limitations on the applications of the bandwidth extension algorithm.

For example, when the algorithm is based on a model of the process of speech production, the algorithm will naturally not have the capability to extend general audio signals, like music, or to reconstruct characteristics of the acoustical environment of the speech signal, such as reverberation or background noise.
As far as we know, the above statistical method is a one-to-many problem because of the little mutual information between low-bandwidth and high-bandwidth spectral envelopes [21], which is shown in Figure 4.2 [22]. Whatever bandwidth extension schemes are used, a common problem is the introduction of artifacts in the extension bands that make the bandwidth extended signal often more annoying than the original narrow-band speech signal [23].

![Diagram](image)

**Figure 4.2 The statistical BWE method**

The method of this chapter to recover the wideband output signal is estimating possible results from the narrowband output of deterministic audio systems. Theoretically, the effects of bandwidth limitation could be inverted using a preceding filter with the inverse of this response. But in time domain, this solution is very energy inefficient and impossible to implement because of the convolution between inputs and audio systems. Since the transfer function of the system could be obtained in frequency domain, it is sufficient to get the pleasant output by canceling the effects of transmission system, which could be carried out by combining the inverse of the transfer function of the system with the Fourier Transform of the output signal. The whole process of implementation is shown in Figure 4.3.
Figure 4.3 Implementation BWE in Microphone system

This chapter is organized as follows. The theory applied in audio system is explained in section 2, and hardware description and test environments are introduced in section 3 and 4 respectively. Evaluation terms are shown in section 5, and in section 6, experiment results and evaluation are presented. In the end, some discussion and conclusion are reached.

4.2 Theoretical Part

A unit impulse function can be used to obtain the transfer function of the system since it is assumed that the transmission system is linear dominant. The unit impulse is based on a rectangular function

$$f_{\varepsilon} = \begin{cases} 1/\varepsilon & -\frac{\varepsilon}{2} \leq t \leq \frac{\varepsilon}{2} \\ 0 & \text{otherwise} \end{cases}$$

(4.1)

Where $\varepsilon > 0$. As $\varepsilon$ approaches zero, the function $f_{\varepsilon}$ approaches the unit impulse function $\delta(t)$, which has the following properties: The Laplace transform of the unit impulse is $R(s) = 1$, and therefore the output for an impulse is the transfer function of the system. In this experiment, the transfer function is simply the Fourier transform of the impulse response.
Chapter 4  BWE Algorithm for Microphone System

The solution to get the precise transfer function could also be applied by modifying some specific frequency responses in the whole spectrum as it is often found that the unknown ‘transfer function’ of a nonlinear system can be deduced from the experimentally determined frequency response for each sinusoidal input.

4.3 Hardware Description

The microphone system used in this thesis consists of speakers, preamplifier circuits, digital signal processing boards installed in a personal computer, and microphones. The input signals from speakers were preamplified and then presented to ADC on the board. The board was programmed to sample the microphone signals at 16 kHz, performing the bandwidth extension algorithms described in the theoretical part. The analog output signals were presented to the subject via DAC and amplifier.

In this experiment, the DSP board is used for implementing algorithm, and the bandwidth of speaker is from 50 Hz to 10 kHz, the bandwidth of microphone is from 200 Hz to 4 KHz. Obviously, the input signal has wider frequency range than the output signal, so the purpose of this experiment is to reproduce the output sound with the same bandwidth as the input sound.

A brief introduction of the hardware system is described as follows:

1. Integrated circuit board:
   I. Digital signal processor (DSP): TMS320VC5402
   II. ADC: PCM1801:
Chapter 4 BWE Algorithm for Microphone System

III. DAC: PCM1770:

IV. External Memory:

1) 256K-W FLASH

2) 64K-W SRAM

V. Amplifier:

TLC2272: 2 Channels

VI. CPLD:

Altera EPM7128S-100

VII. Power Supply:

Linear Regulators: TPS73018

Low-Dropout Voltage Regulators: TPS7333

Triple Processor Supervisors: TPS3307-33

The configuration of integrated circuit board is shown in Figure 4.7, and Figure 4.8 shows the system flow chart.

2. Microphone: Silicon Microphone SPM0103ND3

The basic structure of dynamic microphone is shown in Figure 4.4. Applied acoustic pressure deforms the diaphragm, changing the capacitance of a fixed capacitor. The capacitor inside microphone will produce electric voltage output, and finally the voltage change is preamplified and converted into an audio signal [24].
Figure 4.4 Basic structure of dynamic microphone

The wideband input signal $S_{wb}$ is band-pass filtered and transmitted through speaker to microphone. At the receiving terminal, only the narrowband signal $S_{nb}$ is available. This band-limited signal is analyzed by the bandwidth extension system on DSP board. The missing low and/or high-frequency signal components are estimated and added to the received band-limited components. By this, the algorithm determines an estimate of the wideband signal $S'_{wb}$ that is passed on to the loudspeaker. The whole system configuration from input to output can be expressed in Figure 4.5, and the real experiment setup is shown in Figure 4.6.

Figure 4.5 Block diagram of the proposed algorithm
4.4 Test Environment

The experiments were performed in a normal laboratory with environmental noises from air conditioner or crowd. Test stimuli were delivered from speaker located in the direction of 90 degrees at the distance of 20cm from the microphone. The experimenter, the computer, DSP board, and the output speaker were located beside the microphone to avoid 'whistles' effects.
Figure 4.7 Configuration of Integrated Circuit
Figure 4.8 Flow chart of the system
4.5 Evaluation Terms

4.5.1 Sub-band log spectral distortion measure

For evaluating the performance of a BWE algorithm, the distortion of the spectral envelope of the extended signal shall be measured with respect to the original wideband speech signal. Since the original signal is readily available in the BWE system, ratios of the magnitude and phase envelope in the missing frequency range could be easily derived.

The performance of the estimation of the high-bandwidth spectral envelope shall be defined in terms of the log spectral distortion (LSD) of the missing frequency band. The squared LSD measure is specified in the frequency domain by (e.g. Markel and Gray [25], Gray, Buzo, Gray, and Matsuyama [26])

\[
d^2_{LSD} = \frac{1}{2\pi} \int_{-\pi}^{\pi} (20\log_{10} \left| \frac{\sigma_{rel}}{A_{mb}(e^{j\Omega})} \right| - 20\log_{10} \left| \frac{\overline{\sigma}_{rel}}{\overline{A}_{mb}(e^{j\Omega'})} \right|^2) d\Omega'
\]

Here the quantities \( A_{mb}(e^{j\Omega'}) \) and \( \sigma_{rel} \) refer to the modeled frequency spectrum and relative gain of the missing frequency band of original wideband signal, and \( \overline{A}_{mb}(e^{j\Omega'}) \) and \( \overline{\sigma}_{rel} \) denote the corresponding parameters as decided by a bandwidth extension system. The integration range of \(-\pi\) to \(\pi\) could cover the missing frequency range in the original wideband speech signal since the LSD measure is evaluated for the critically sub-band signal containing only the missing frequency band. And the unit of \( d_{LSD} \) is dB.
Unfortunately, the evaluation of the LSD measure in the frequency domain is quite complicated in general. Therefore, an alternative representation by a mean-square error criterion in the cepstral domain, following the definition from eq. (4.2) will be used in the following equation:

$$\ln \frac{\sigma^2_{rel}}{|A_{mb}(e^{j\omega r})|^2} = \sum_{j=-\infty}^{\infty} c_j e^{-j\Omega r}$$  \hspace{1cm} (4.3)

With this definition, for a sequence of signal, the root mean square (RMS) average of the LSD is given by

$$\bar{d}_{LSD} = \frac{\sqrt{210}}{\ln 10} \sqrt{E\left\{ \frac{1}{2}(c_0 - \bar{c}_0)^2 + \sum_{i=1}^{d} (c_i - \bar{c}_i)^2 \right\}}$$  \hspace{1cm} (4.4)

Here, the function $E\{\}$ denotes the expectation operation.

Now, the output representation $\bar{y}$ of the estimation shall be defined in such a manner that the estimation performance can be determined by a mean-square error criterion. For this, the quantities $y$ are defined as weighted cepstral coefficients $c_0, c_1, \ldots$ to represent the spectral envelope of the missing frequency band.

$$y_i = \begin{cases} 
\frac{1}{\sqrt{2}} c_i, & \text{if } i = 0 \\
\frac{1}{\sqrt{2}} c_i, & \text{if } 1 \leq i \leq d.
\end{cases}$$  \hspace{1cm} (4.5)

The scalar values constitute a $d$-dimensional vector. The dimension should be at least such that all non-redundant cepstral coefficients are considered. Inserting the definition of RMS sub-band log spectral distortion measure yields the relationship
\[ d_{LSD}^2 = \left( \frac{\sqrt{210}}{\ln 10} \right)^2 \sum_{i=0}^{d-1} (y_i - \bar{y}_i)^2 \]  \quad (4.6)

Note that the term on the right-hand side of eq. (4.6) is only an approximation of the log spectral distortion because only the first \( d \) summands of the sum are considered.

### 4.5.2 Ease of listening

Relative ease of listening was assessed using a comparison procedure [27]. Each trial compared three signals, randomly assigned to be “A”, “B” and “C”, which are the original input signal, the output without BWE algorithm and the output after BWE algorithm respectively. The subject listened through “A”, “B” and “C”, switching among them at will. The subject was requested to decide the ease of listening and to select a rating corresponding to the following grading scale.

- 3 – Best result with clear sound and little noise
- 2 – Satisfying result with identifiable sound and acceptable noise
- 1 – Not good result with unnoticeable sound and overwhelming noise.

### 4.6 Experimental Results and Analysis

#### 4.6.1 Transfer function

The BWE algorithm is operated on real sound data which were recorded with a given microphone. The first step in the bandwidth extension algorithm of microphone system is the estimation of the transfer function using impulse signal. In this experiment, a series of impulse signals is used to avoid arbitrary spectrum truncation.
of impulse response, and thus single meaningful range of transfer function can be subtracted from the repetitive impulse response of the system.

The following four pictures identify the frequency response vividly by using sine waves from different frequencies, like 5k, 6k, 7k and 7.5k, as inputs, and observe the change of outputs.

5k

6k

7k

7.5k

Figure 4.9 shows the frequency characteristic of impulse signal, and in Figure 4.10, a signal consisting of repetitive impulse signals is depicted as the testing input of the experiments.
Figure 4.9.1 Magnitude characteristic of impulse signal

Figure 4.9.2 Phase characteristic of impulse signal
Figure 4.10 Repetitive impulse signals in experiments

As far as we know, theoretically there is no upper and lower band of frequency for impulse response, but it is necessary to get helpful frequency range according to different practical systems. In this experiment, the properties (Magnitude and Phase) of impulse response, or the transfer function of system is shown in Figure 4.11.1 and Figure 4.11.2 with the range to 0~8 KHz, which is determined by the Nyquist–Shannon sampling theorem as the sampling rate of hardware is 16 KHz.
Figure 4.11.1 Magnitude characteristic of transfer function

Figure 4.11.2 Phase characteristic of transfer function
4.6.2 Experiment results-1

Next, the behavior of BWE method is studied by a complex input sound (UFO whistling) as shown in Figure 4.12, which has the specific properties in both magnitudes and phases, and corresponding results without doing BWE algorithm are shown in Figure 4.13.

![Figure 4.12.1 Magnitude properties of complex input sound](image)

Figure 4.12.1 Magnitude properties of complex input sound

![Figure 4.12.2 Phase properties of complex input sound](image)

Figure 4.12.2 Phase properties of complex input sound
Figure 4.13.1 Magnitude properties of output sound

Figure 4.13.2 Phase properties of output sound

The original output clearly shows detrimental frequency performance through the microphone when compared to the complex input sound. In contrast, this BWE method provides relatively consistent benefits, despite some fluctuations in small
area of the low frequency and high frequency part. Moreover, after modification on the original output, it is possible to get ease-of-listening and intelligible results, so the modified output could have wider bandwidth and without maximum value occurring at the resonant frequency.

Figure 4.14.1 Magnitude properties of modified output

Figure 4.14.2 Phase properties of modified output
The results of modified output signal, shown in Figure 4.14, follow the general trends predicted by the analysis of algorithm. However, there are some substantial discrepancies between theoretical results and practical results, which may be attributed to a combination of matters. In these experiments, the input signals are not ideally broadcasted by speaker, the environment noise interferes the input; and the results are not recorded precisely. All of these factors severely affect the performance of modified output generated from implementing BWE algorithm.

4.6.3 Experiment results-2

Another complex input sound (random noise) has the following properties in both magnitude and phases, seen in Figure 4.15.

![Magnitude properties of complex input sound](image)
Figure 4.15.2 Phase properties of complex input sound

In Figure 4.16, the original output signal without modification can be analyzed in aspects of both magnitude and phase.

Figure 4.16.1 Magnitude properties of output sound
The modified output signal after doing the bandwidth extension algorithm is shown in Figure 4.17, and exhibits two distinctive characteristics. First, the general spectral shape is more or less the same as the original input, which certifies the success of this BWE method. A second feature is that small distortions of magnitude and phase are found with respect to different frequencies. Needless to say, minor distortion is caused by various factors and could be ignored for an effective restoration of input sound.
Figure 4.17.1 Magnitude properties of modified output

Figure 4.17.2 Phase properties of modified output
4.6.4 Experiment results-3

In order to certify the all-round performance of this BWE algorithm, an English speech signal said by a woman is analyzed to support this method, see Figure 4.18.

Figure 4.18.1 Magnitude properties of English speech by female

Figure 4.18.2 Phase properties of English speech by female
In Figure 4.19, the original output signal without modification can be analyzed in aspects of both magnitude and phase.

![Magnitude properties of output speech](image1)

Figure 4.19.1 Magnitude properties of output speech

![Phase properties of output speech](image2)

Figure 4.19.2 Phase properties of output speech
The modified speech signal after doing the bandwidth extension algorithm is illustrated in Figure 4.20. Observe that the BWE algorithm has increased the amplitudes in the high frequency part, resulting in satisfactory output because the transfer function has saved the missing information of the frequency response of high frequency part. For speech signal, this BWE algorithm yields a consistent better output speech in comparison to the original output in the frequency domain and also could be confirmed by informal listening test afterwards.

---

**Figure 4.20.1** Magnitude properties of modified speech output

**Figure 4.20.2** Phase properties of modified speech output
4.7 Discussion & Comments

To define the frequency distortion measure, first the wideband speech signal is split into two sub-band signals, which contain only the base band components and only the extended frequency components of the wideband speech respectively. For this purpose, the two sub-band signals are determined via low-pass filtering and high-pass filtering respectively. In the next step, the estimation of LSD in the missing frequency band is done with two individual signals.

<table>
<thead>
<tr>
<th>Experiments</th>
<th>RMS LSD (dB)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Before</td>
</tr>
<tr>
<td>EXP 1</td>
<td>15.0</td>
</tr>
<tr>
<td>EXP 2</td>
<td>14.8</td>
</tr>
<tr>
<td>EXP 3</td>
<td>14.3</td>
</tr>
</tbody>
</table>

Table 4.1 RMS LSD results of 3 experiments

Table 4.1 further shows that the accuracy of the recovered signal is improved considerably by this BWE algorithm.

Ease of listening results are provided by 10 people as subjects, see Table 4.2. The ease-of-listening data indicate that all subjects generally preferred modified output results over original output results for all different input sounds with the same noise condition. An interesting discovery is that subjects all voted for better restoration if speech signals are used as inputs in the experiment.
<table>
<thead>
<tr>
<th></th>
<th>INP</th>
<th>OOPT</th>
<th>MOPT</th>
</tr>
</thead>
<tbody>
<tr>
<td>EXP 1</td>
<td>3</td>
<td>1.5</td>
<td>1.8</td>
</tr>
<tr>
<td>EXP 2</td>
<td>3</td>
<td>1.7</td>
<td>2.2</td>
</tr>
<tr>
<td>EXP 3</td>
<td>3</td>
<td>1.8</td>
<td>2.6</td>
</tr>
</tbody>
</table>

Table 4.2 Ease of listening results of 3 experiments

Note:
INP = Input signal
OOPT = Original output without BWE algorithm
MOPT = Modified output with BWE algorithm
EXP 1 = Experiment results 1
EXP 2 = Experiment results 2
EXP 3 = Experiment results 3

The results of experiments and evaluations in this chapter clearly show that BWE algorithms can provide substantial benefits for recovering sound signals in background noise, compared with original output without any modification. And the overall high-bandwidth enhancement scheme produced a significant quality improvement in different kinds of narrowband signals, and demonstrates potential as a post-processor to any band limited sound source.
Chapter 5

BWE Algorithm for Imaging System

In this chapter, the BWE algorithm is used to recover input image signals for imaging systems, which also have bandwidth extension requirements because of physical limitation of lens and/or CCD. It can be seen as the matter of cameral calibration problem. Simulation results are analyzed in both time and frequency domain to testify the feasibility of this algorithm and possible experiments to be done in imaging systems are discussed as well.

5.1 Introduction

Camera calibration in the context of three-dimensional (3D) machine vision is the process of determining the internal camera geometric and optical characteristics (intrinsic parameters) and/or the 3D position and orientation of the camera frame relative to a certain world coordinate system (extrinsic parameters) [28]. There are varied categories of methods for camera calibration. Some techniques involve full-scale nonlinear optimization [29], [30], [31], and some techniques use linear equation for perspective transformation matrix [32], [33], [34], where two planes [35] and geometric methods [36] are employed as well.

An imaging system is also a transmission system, which has bandwidth limitation as well as audio systems and filters. Moreover, in Fourier optics, aperture and/or lens have a specific transfer function, and have exhibited analogous results like Fourier
Transform in the imaging plane. The method introduced in this chapter concentrates on the relation between standard imaging system and practical systems. Comparing the 2D outputs of standard system and practical systems, the difference of optical transfer function could be obtained, which is essential for the camera calibration in real imaging systems. Consequently, given suboptimal results from practical systems, the recovered pretty output image is generated with the help of compensation for optical transfer function. In a word, it is worthwhile to do simulations to recover image signals using this BWE algorithm, although it is not easy to implement the experiments in practice, which is shown in the Figure 5.1.

![Diagram](image)

Figure 5.1 The process of compensation for object imaging system

### 5.2 Multidimensional Fourier Transform

The ordinary DFT computes the transform of a "one-dimensional" dataset: a sequence (or array) $x_n$ that is a function of one discrete variable $n$. More generally,
one can define the multidimensional DFT of a multidimensional array $x_{n_1,n_2,\cdots,n_d}$ that is a function of $d$ discrete variables $n_l = 0,1,\cdots,N_l-1$ for $l$ in $1,2,\cdots,d$:

$$X_{k_1,k_2,\cdots,k_d} = \sum_{n_1=0}^{N_1-1} \omega_{N_1}^{k_1 n_1} \cdots \sum_{n_d=0}^{N_d-1} \omega_{N_d}^{k_d n_d} x_{n_1,n_2,\cdots,n_d}$$  \hspace{1cm} (5.1)$$

where $\omega_{N_l} = \exp(-2\pi i / N_l)$ as above and the $d$ output indices run from $k_l = 0,1,\cdots,N_l-1$. The above equations could be more compactly expressed in vector notation, where $n \equiv (n_1,n_2,\cdots,n_d)$ and $k \equiv (k_1,k_2,\cdots,k_d)$ are d-dimensional vectors of indices from 0 to $N - 1 \equiv (N_1 - 1,N_2 - 1,\cdots,N_d - 1)$:

$$X_k = \sum_{n=0}^{N-1} e^{-2\pi i k (n / N)} x_n$$  \hspace{1cm} (5.2)$$

where the division $n / N \equiv (n_1 / N_1,n_2 / N_2,\cdots,n_d / N_d)$ is performed element-wise, and the sum denotes the set of nested summations above.

Analogous to the one-dimensional case, the inverse of the multi-dimensional DFT is, given by:

$$x_n = \frac{1}{\prod_{l=1}^{d} N_l} \sum_{k=0}^{N-1} e^{2\pi i n k / N} X_k$$  \hspace{1cm} (5.3)$$

It is obvious to see that the multidimensional DFT has a simple interpretation [37]. Just as the one-dimensional DFT expresses the input $x_n$ as a superposition of sinusoids, the multidimensional DFT expresses the input as a superposition of plane
waves, or sinusoids oscillating along the direction $k / N$ in space and having amplitude $X_k$. Such decomposition is of great importance for everything from digital image processing ($d = 2$) to solving partial differential equations in three dimensions ($d = 3$) by breaking the solution up into plane waves.

From a computational point of view, the multidimensional DFT is simply the composition of a sequence of one-dimensional DFTs along each dimension. For example, in the two-dimensional case $x_{n1,n2}$, one can first compute the $N1$ independent DFTs of the rows (i.e., along $n2$) to form a new array $y_{n1,n2}$, and then compute the $N2$ independent DFTs of $y$ along the columns (along $n1$) to form the final result $X_{k1,k2}$. Or, one can transform the columns and then the rows—the order is immaterial because the nested summations above commute.

Because of this, given a way to compute a one-dimensional DFT (e.g. an ordinary one-dimensional FFT algorithm), one immediately has a way to efficiently compute the multidimensional DFT. This is known as a row-column algorithm, although there are also intrinsically multidimensional FFT algorithms.

### 5.3 Simulation Results

The facular of a black background image could be considered as the input test signal, which has similar functions as the impulse signal in audio systems, shown in Figure 5.2, and the frequency characteristics of facular are shown in Figure 5.3.
Figure 5.2 The facular of a black background image

Figure 5.3.1 Magnitude properties of facular
Figure 5.3.2 Phase properties of facular

The frequency can be represented by different types of grids. For example, in Figure 5.4, the following grid is created by sampling by 2, and the according features of frequency are shown in Figure 5.5.

Figure 5.4 Grid created by sampling by 2
Figure 5.5.1 Magnitude properties of grid

Figure 5.5.2 Phase properties of grid
Chapter 5  BWE Algorithm for Imaging System

Next, the image of tiger is used as the 2D image of standard imaging system to do the simulation, shown in Figure 5.6, and the output 2D image of practical imaging system is displayed as Figure 5.7. After doing the BWE algorithm, the simulation result of the modified output image is shown in Figure 5.8.

Figure 5.6 2D image from standard system

Figure 5.7 2D image from practical system
Figure 5.8 Modified output image after simulation
Chapter 6

Conclusions

This chapter is a summary of the major work of this thesis. Also, some possible directions are given for future research.

6.1 Future Work

- Experiments of imaging system in practical situation are valuable to justify the BWE method and simulation results. The whole process of camera calibration is firstly getting relationship of transfer function from the ideal undistorted image coordinate, distorted image coordinate and 3D coordinate system, and then compensating the distortion with obtained transfer function.

- More comparison between this BWE algorithm and other related methods is recommended to be done for further discussion and application. Since this BWE algorithm is generally applied for multiple systems and effects vary accordingly, it might not outperform other methods in specific areas.

- Note that the algorithm in this thesis can only be modified on line for a fixed and well established system, but not appropriate for the use in wireless and unstable fields. Possible applications are: Fixed-line telephone, Network calls, Encryption & Decryption, Cameras.
6.2 Contributions

Main contents of this thesis are summarized and an overview of the main contributions is given.

- The problem of bandwidth limitation in multiple systems is formulated and solved in frequency domain. A novel bandwidth extension algorithm with the help of transfer function and frequency response of deterministic systems is proposed.

- This BWE algorithm is universal for multiple systems, from mechanical systems, audio systems, to imaging systems.

- The original definition of transfer function is expanded from linear systems into nonlinear systems. In addition, necessary conditions are described to satisfy the BWE algorithm.

- Detailed theoretical description, mathematical derivation and physical understandings have been presented to support the principle and applications of the BWE algorithm in this thesis. And different evaluation approaches are forwarded to testify the algorithm, which ensures that it is a comprehensive solution to diverse categories of signals with knowable characteristics of systems.

- Based on the theory of this algorithm, experiments and simulations are operated on various systems, and thus comparisons between modified results and original results are saved and analyzed.
Bibliography


