Wulff Steiner Polynomial for Ovaloid

by

FAN, Wai Tong

A Thesis Submitted to
The Hong Kong University of Science and Technology
in Partial Fulfillment of the Requirements for
the Degree of Master of Philosophy
in Department of Mathematics

June 2005, Hong Kong

Copyright © by FAN, Wai Tong 2005
Authorization

I hereby declare that I am the sole author of the thesis.

I authorize the Hong Kong University of Science and Technology to lend this thesis to other institutions or individuals for the purpose of scholarly research.

I further authorize the Hong Kong University of Science and Technology to reproduce the thesis by photocopying or by other means, in total or in part, at the request of other institutions or individuals for the purpose of scholarly research.

[Signature]

FAN, Wai Tong
Wulff Steiner Polynomial for Ovaloid

by

FAN, Wai Tong

This is to certify that I have examined the above MPhil thesis and have found that it is complete and satisfactory in all respects, and that any and all revisions required by the thesis examination committee have been made.

Professor CHENG, Shiu Yuen
Supervisor

Professor LI, Jian Shu
Head of Department

Department of Mathematics
June, 2005
Acknowledgements

I would like to express my heartfelt gratitude to Prof. Shiu Yuen Cheng, who generously shared his time to teach me. It has been a privilege to study and work under his guidance. He is very nice and patient and his ideas are inspiring. I learned much from him, both in Mathematics and in the way of being a good teacher. "Mondays with Cheng" will be a most treasurable memory in my life.

I also wish to thank my lifelong teacher, Prof. Kin Yin Li, for his sincere concern and guidance. His encouragement and passion was a great source of power to me.

Many thanks go to Prof. Guo Wu Meng, Prof. Wei Ping Li and Prof. Yang Xiang for their kind support. I would also acknowledge my friends Kelvin Yu and Lo Wing Cheong for many helpful and enjoyable discussion. They are very nice and their ideas are inspiring.

Last but not least, I express my sincere gratitude to my family for their unconditional love.
Contents

Authorization Page ii
Signature Page iii
Acknowledgements iv
Table of Contents v
Abstract vii

1 Convex bodies and support functions 1
   1.1 Support function ................................................. 1
   1.2 Orthonormal Frame .............................................. 2
   1.3 The 3rd fundamental form of $\partial K$ ...................... 4
   1.4 Volume in terms of the support function .................. 5
   1.5 Other properties ................................................. 6

2 A motivation for Wulff-Steiner Polynomial 7
   2.1 Wulff Shape .................................................. 7
   2.2 Wulff flow and crystal growth ................................. 10
3 Mixed Volume

3.1 Definitions and basic properties .................................. 13
3.2 The Brunn-Minkowski Theorem ..................................... 14

4 Wulff-Steiner Polynomial of degree 2 ................................. 16

4.1 Invariant of the Wulff flow ......................................... 16
4.2 Properties of the roots ............................................... 17
4.3 Relative positions .................................................... 18
4.4 Characterization of degree 2 Wulff-Steiner Polynomials .......... 24

5 Wulff-Steiner Polynomial of degree 3 ................................ 27

5.1 Invariants .......................................................... 28
5.2 Properties of the roots ............................................... 29
5.3 Relative positions among the roots, r and R ......................... 31

6 Appendix ...................................................................... 33

6.1 Useful inequalities ...................................................... 33
6.2 Matlab results for ellipsiod ........................................... 34

Bibliography .................................................................... 37
Wulff Steiner Polynomial for Ovaloid

by FAN, Wai Tong

Department of Mathematics
The Hong Kong University of Science and Technology

Abstract

The equilibrium shape of a crystal is the celebrated Wulff shape $W$, which can be obtained via the Wulff flow. When the initial crystal $K$ is convex, the image under the Wulff flow is $K + tW$ after time $t$. The $n + 1$-dimensional volume $V_{n+1}(K + tB)$ is the well known Steiner polynomial, where $B$ is the closed unit ball.

Wulff flow and Wulff shape has been highly studied in crystal growth. Geometric information encoded in the the associated Wulff-Steiner polynomial was far from being thoroughly understood.

In this thesis, Wulff-Steiner polynomial of degree 2 was surveyed. We obtained a characterization of them. The relative positions among the roots, the $W$-inradius, $W$-outradius and other geometric quantities are studied. Wulff-Steiner polynomial have no positive real root by its definition. For degree 2, the fact that it must have negative real roots is equivalent to the classical isoperimetric inequality. For degree 3, we showed that complex roots can occur and that the real parts of any complex root must be negative. As an application of the Wulff-Steiner polynomial, we found an invariant under the Wulff flow in any dimension $n \geq 2$. 

vii
Chapter 1

Convex bodies and support functions

The aim of this chapter is to introduce some basic concepts and notations about convex bodies.

Let $\Phi^{n+1} = \{K \subset \mathbb{R}^{n+1} : K$ is a compact convex set with nonempty interior$\}$ and $\Psi^{n+1} = \{K \in \Phi^{n+1} :$ the boundary $\partial K$ of $K$ is a smooth, strictly convex hypersurface$\}$. 

By a convex body in $\mathbb{R}^{n+1}$, we mean an element in $\Phi^{n+1}$. An element in $\Psi^{n+1}$ is said to be an ovaloid. Clearly, $\Phi^{n+1} \supset \Psi^{n+1}$. We will restrict our attention to $\Psi^{n+1}$ most of the time, where we can apply knowledge of differential geometry to smooth hypersurfaces.

1.1 Support function

Definition Consider $K \in \Phi^{n+1}$, the support function of $K$ is defined to be a function $p : S^n \rightarrow \mathbb{R}$ such that $p(u) = \sup\{\langle x, u \rangle : x \in \partial K\}$ for any $u \in S^n$, where $\langle \cdot, \cdot \rangle$ is the usually inner product of $\mathbb{R}^{n+1}$.

- Note that $p$ depends on the origin. If $K$ contains the origin in its interior, then $p$ has positive value everywhere on $S^n$.

- The intuitive meaning of the support function $p$ is as follows: For a unit vector $u \in S^n$, $p(u)$ is the signed distance from the origin
to the supporting hyperplane $\{x \in \mathbb{R}^{n+1} : <x, u> = p(u)\}$ of $K$ with outward unit normal vector $u$. (Figure 1.1)

![Figure 1.1: Support function](image)

- Any convex body can conveniently be described by specifying the position of its supporting planes. This is done if we know its support function. Thus all information about $K$ can be represented by $p$ theoretically.

- Define the generalized Gauss map $G : \partial K \rightarrow S^n$ to be the set-valued map such that $G(x)$ is the set of all outward normals of the supporting planes of $\partial K$ passing through $x$, $x \in \partial K$. Clearly, $p(u) = <z, u>$ for each $u \in G(z)$. When $K \in \Psi^{n+1}$, $G$ is the Gauss map $g$, which maps $x \in \partial K$ to the (uniquely defined) outward unit normal vector. Then $p(u) = <g^{-1}(u), u>$.

In the remaining parts of Chapter 1, unless otherwise stated, $K$ denotes a convex body in $\Psi^{n+1}$, $p$ is the support function of $K$, and $g$ is the Gauss map of $\partial K$.

### 1.2 Orthonormal Frame

When $K \in \Psi^{n+1}$, the Gauss map of $\partial K$, $g : \partial K \rightarrow S^n$, defines a smooth homeomorphism between $\partial K$ and $S^n$ with functional determinant everywhere nonzero. $g$ is therefore a diffeomorphism. We can therefore identify $\partial K$ with $S^n$ via $g$.

Consider the diffeomorphism $x = g^{-1} : S^n \rightarrow \partial K$, for each point $z \in S^n$, let $e_{n+1}(z)$ be the unit outward normal vector of $\partial K$ at $x(z)$. (Figure 1.2)
Let \( \{e_1, e_2, \ldots, e_n\} \) be a local orthonormal frame of \( \partial K \) such that the determinant \( (e_1, e_2, \ldots, e_n, e_{n+1}) = 1 \). Such a frame can be defined smoothly almost everywhere on \( \partial K \).

Throughout this section, small Latin indices run from 1 to \( n \) and small Greek indices from 1 to \( n + 1 \). We also use the Einstein summation convention. Let \( \{\omega^\alpha\} \) be the local coframe associated to \( \{e_\alpha\} \), \( \{\omega^\beta_\alpha\} \) be the connection forms associated to \( \{e_\alpha\} \).

Then \( dx = \omega^i e_i \) and \( de_\alpha = \omega^\beta_\alpha e_\beta \).

- As \( \langle e_\alpha, e_\beta \rangle = \delta^\alpha_\beta \), exterior differentiating both sides gives us \( \omega^\beta_\alpha = -\omega^\alpha_\beta \). In particular, \( \omega^\alpha_\alpha = 0 \).
- Exterior differentiating the vector valued 1-form \( dx \), we have \( 0 = ddx \). This gives us \( d\omega^i = \omega^j \wedge \omega^i_j \).
- On \( \partial K \), \( \omega^{n+1} = 0 \). Thus \( 0 = d\omega^{n+1} = \omega^i \wedge \omega^n_i \). Since \( \{\omega^i\} \) are linearly independent 1-forms, by Cartan's lemma, we have \( \omega^n_{i+1} = h_{ij} \omega^i \) for some \( \{h_{ij}\} \) with \( h_{ij} = h_{ji} \).

The three quadratic fundamental forms of \( \partial K \) are

\[
I = dx \cdot dx = \sum_i (\omega^i)^2
\]
\[
II = de_{n+1} \cdot dx = \sum_i \omega^i_{n+1} \omega^i = \sum_{i,k} h_{ik} \omega^i \omega^k
\]
\[
III = de_{n+1} \cdot de_{n+1} = \sum_i (\omega^i_{n+1})^2
\]
Their matrices with respect to \( \{ e_i \} \) are \( (\delta_{ij}), (h_{ij}), (\Sigma_k h_{ik} h_{jk}) \) respectively.

The eigenvalues \( k_1, k_2, \ldots, k_n \) of II relative to I (i.e. the eigenvalues of the matrix \( (h_{ij}) \)) are called the **principal curvatures**. Their product \( k = k_1 k_2 \cdots k_n \) is called the **Gauss-Kronecker curvature**. Their arithmetic mean \( h = \frac{k_1 + k_2 + \cdots + k_n}{n} \) is called the **mean curvature**.

Since \( \partial K \) is strictly convex, we have \( k_i > 0 \) for \( i = 1, 2, \ldots, n \). \( \frac{1}{k_1}, \frac{1}{k_2}, \ldots, \frac{1}{k_n} \) are called the **radii of principal curvature**, they are the eigenvalues of II relative to III. (Note that \( (h_{ij})(\Sigma_k h_{ik} h_{jk})^{-1} = (h_{ij})^{-1} \))

### 1.3 The 3rd fundamental form of \( \partial K \)

The support function of \( \partial K \) is given by \( p(z) = \langle x(z), e_{n+1}(z) \rangle \) for all \( z \in S^n \). For simplicity, we use the "\( . \)" to represent the inner product when no ambiguity is caused.

The smooth section

\[
dp = d(x \cdot e_{n+1}) = x \cdot de_{n+1} + dx \cdot e_{n+1} = x \cdot (\omega_{n+1}^i e_i) \quad \text{as} \quad dx \cdot e_{n+1} = 0 = (x \cdot e_i)\omega_{n+1}^i
\]

Let \( p_i = x \cdot e_i \), i.e. \( dp = p_i \omega_{n+1}^i \), then by the definition of covariant derivative \( \nabla \) on \( S^n \), \( \nabla_{e_i} dp = dp_i - \omega_i^j p_j = p_{ij} \omega_{n+1}^j \)

Since

\[
\begin{align*}
dp_i - \omega_i^j p_j &= dx \cdot e_i + x \cdot de_i - p_j \omega_i^j \\
&= (\omega^k e_k) \cdot e_i + x \cdot (\omega_i^n) - p_j \omega_i^j \\
&= \omega^i + (x \cdot e_j)\omega_i^j + (x \cdot e_{n+1})\omega_{n+1}^i - p_j \omega_i^j \\
&= \omega^i + p\omega_{n+1}^i,
\end{align*}
\]

we have

\[
\begin{align*}
p_{ij} \omega_{n+1}^j &= \omega^i + p\omega_{n+1}^i \\
p_{ij} \omega_{n+1}^j + p\omega_{n+1}^i &= \omega^i \\
(p_{ij} + p\delta_{ij})\omega_{n+1}^j &= \omega^i.
\end{align*}
\]
Thus, we obtain the following important matrix relation:

$$\left( p_{ij} + p\delta_{ij} \right) = \left( h_{ij} \right)^{-1} \tag{1.1}$$

### 1.4 Volume in terms of the support function

Now we are going to express some common geometric quantities (e.g. Gauss-Kronecker curvature, Mean curvature, Volume, Surface area) of a convex body $K \in \Psi^{n+1}$ in terms of its support function $p$.

From equation (1.1), we immediately have

- Gauss-Kronecker curvature $k = \frac{1}{\det(p_{ij} + p\delta_{ij})}$.

- Considering the trace of the matrices, we have $\Delta p + np = \frac{1}{k_1} + \frac{1}{k_2} + \cdots + \frac{1}{k_n} = \frac{nh}{k}$, where $\Delta p$ is the Laplace operator of $p$ on $S^n$. Hence the mean curvature $h = \frac{k(\Delta p + np)}{n} = \frac{\Delta p + np}{n \det(p_{ij} + p\delta_{ij})}$.

Denote $V_{n+1}(K)$ and $S_{n+1}(K)$ to be the $(n+1)$-dimensional volume and the $(n+1)$-dimensional surface area of $K$ respectively. The volume element $dA$ of $\partial K$ pull back to $S^n$ is $\frac{1}{k} d\sigma$, where $d\sigma$ is the volume element of $S^n$. Thus,

- $S_{n+1}(K) = \int_{\partial K} dA = \int_{S^n} \frac{1}{k} d\sigma = \int_{S^n} \det(p_{ij} + p\delta_{ij}) d\sigma$.

Consider the function $x \mapsto \sum_{i=1}^{n+1} x_i^2$, $x = (x_1, x_2, \cdots, x_{n+1}) \in \mathbb{R}^{n+1}$. Applying the Laplace operator $\Delta$ on $\mathbb{R}^{n+1}$ to it, we easily get

$$2(n + 1) = \Delta \left( \sum_{i=1}^{n+1} x_i^2 \right)$$

Integrate both sides over $K$ and then apply Stoke’s Theorem,

$$2(n + 1) V_{n+1}(K) = \int_K \Delta \left( \sum_{i=1}^{n+1} x_i^2 \right) dx_1 \cdots dx_{n+1}$$

$$= \int_K \nabla \cdot \left( 2x_1, 2x_2, \cdots, 2x_{n+1} \right) dx_1 \cdots dx_{n+1}$$

$$= \oint_{\partial K} \left( 2x_1, 2x_2, \cdots, 2x_{n+1} \right) \cdot \vec{n} dA$$
where $n$ is the outward unit normal vector of $\partial K$ at $(x_1, x_2, \cdots, x_{n+1}) \in \partial K$.

Therefore,

\[
V_{n+1}(K) = \frac{1}{n+1} \int_{\partial K} x \cdot g(x) \, dA = \frac{1}{n+1} \int_{\partial K} p \, dA = \frac{1}{n+1} \int_{S^n} p \det(p_{ij} + p \delta_{ij}) \, d\sigma
\]

1.5 Other properties

In this section, we collect some definitions and properties that we need later.

**Lemma 1.5.1.** The support function of the translate $K + \eta$ is given by $u \mapsto p(u) + \eta \cdot u$ for all $u \in S^n$, where $p$ is the support function of $K \in \Phi^{n+1}$.

**Definition** Let $K_1, K_2 \in \Phi^{n+1}$ and $t_1, t_2 \geq 0$, the Minkowski sum $t_1 K_1 + t_2 K_2$ is defined to be the set $\{t_1 x_1 + t_2 x_2 : x_1 \in K_1, x_2 \in K_2\}$.

**Lemma 1.5.2.** $t_1 K_1 + t_2 K_2 \in \Phi^{n+1}$ and its support function is $t_1 p_1 + t_2 p_2$, where $p_1, p_2$ are support functions of $K_1, K_2$ respectively.

**Definition** $K_1 \in \Phi^{n+1}$ is said to be **homothetic** to $K_2 \in \Phi^{n+1}$ if $K_1 = \lambda K_2 + \eta$ for some $\lambda > 0$ and $\eta \in \mathbb{R}^{n+1}$.

If $K_1 = \lambda K_2 + \eta$, then $K_2 = \frac{1}{\lambda} (\cdot - \frac{\eta}{\lambda})$. Hence $K_2$ is also homothetic to $K_1$. It is easy to check that homothety is an equivalent relation.

**Definition** $K \in \Phi^{n+1}$ is said to be **symmetric** if there is a point $v \in \mathbb{R}^{n+1}$ such that the support function $p$ of $K$ satisfies $p(u) = p(-u) + 2v \cdot u \forall u \in S^n$. Clearly, $v$ must be unique if it exists. We call it the **point of symmetry** of $K$.

**Lemma 1.5.3.** Suppose $K_i \in \Phi^{n+1}$ has support function $p_i$, $i=1,2$. Then

(i) $K_2 \subseteq K_1$ if and only if $p_2 \leq p_1$ on $S^n$,

(ii) $K_2 \subset K_1 \setminus \partial K_1$ if and only if $p_2 < p_1$ on $S^n$.

**Remark** The proofs of Lemma 1.5.1, Lemma 1.5.2 and Lemma 1.5.3 easily follow from the definition of the support function.
Chapter 2

A motivation for Wulff-Steiner Polynomial

In 1901 ([14]), Wulff first posed and solved the following problem in the field of material science:

*What is the equilibrium shape of a perfect crystal of one material in contact with a single surrounding medium?*

It is physically given by a shape which minimizes the total energy. Wulff gave an ingenious geometric construction of the solution, which is the celebrated Wulff shape. In 1974 [15], L.E. Taylor gave a Mathematical proof that $W$ does yield a minimizer of the energy, and this shape is unique up to translation.

2.1 Wulff Shape

**Definition** The first Legendre transform of $\gamma : S^n \rightarrow (0, \infty)$ is the function $\gamma_* : S^n \rightarrow (0, \infty)$ given by $\gamma_*(u) = \inf \left\{ \frac{\gamma(\theta)}{\theta \cdot u} : \theta \in S^n, \theta \cdot u > 0 \right\}$ $\forall u \in S^n$.

The second Legendre transform of $\gamma : S^n \rightarrow (0, \infty)$ is the function $\gamma^* : S^n \rightarrow (0, \infty)$ given by $\gamma^*(u) = \sup \left\{ \gamma(\theta)(\theta \cdot u) : \theta \in S^n, \theta \cdot u > 0 \right\}$ $\forall u \in S^n$.

**Definition** The Wulff shape associated to $\gamma : S^n \rightarrow (0, \infty)$ is defined by $W = \{ r\theta \in \mathbb{R}^{n+1} : r \leq \gamma_*(\theta), \theta \in S^n \}$.

- When $\gamma \equiv 1$, $W$ is the closed unit ball $B = \{ x \in \mathbb{R}^{n+1} : \| x \| \leq 1 \}$. 
• When $n = 1$ and $\gamma : S^1 \rightarrow (0, \infty)$ is given by $\gamma(\theta) = \sin(\theta) + | \cos(\theta) |$ (Figure 2.1), $W$ is a square of side length 2 units and centered at the origin. (Figure 2.2)

![Polar plot of $\gamma$](image)

**Figure 2.1: Polar plot of $\gamma$**

**Definition** A continuous function $\gamma : S^n \rightarrow (0, \infty)$ is said to be **convex** if the polar plot of $\frac{1}{\pi}$, which is $\{ r\theta : r \leq \frac{1}{\pi}(\theta), \theta \in S^n \}$, is a convex subset of $\mathbb{R}^{n+1}$. A continuous function $\gamma : S^n \rightarrow (0, \infty)$ is said to be **polar convex** if the polar plot of $\gamma$ is a convex subset of $\mathbb{R}^{n+1}$.

**Lemma 2.1.1.** $\gamma$ is convex if and only if its homogeneous extension of degree one is a convex function in $\mathbb{R}^{n+1}$.

**Lemma 2.1.2.** $\gamma$ is convex if and only if $(\gamma_*)^* = \gamma$. $\gamma$ is polar convex if and only if $(\gamma^*)_* = \gamma$.

For the proofs of the above lemmata, please refer to [4].

**Claim:** $\Phi^{n+1}$ is the set of all Wulff shapes, up to translation.

**Proof** Note that $W$ can be written as $\{ x \in \mathbb{R}^{n+1} : x \cdot \theta \leq \gamma(\theta) \forall \theta \in S^n \}$. For $x_1$ and $x_2 \in \mathbb{R}^{n+1}$, and $\alpha \in (0, 1)$, $(\alpha x_1 + (1 - \alpha)x_2) \cdot \theta = \alpha(x_1 \cdot \theta) + (1 - \alpha)x_2 \cdot \theta \leq \gamma$. 

8
Hence, $\alpha x_1 + (1 - \alpha)x_2 \in W$. Thus $W$ is a convex body in $\mathbb{R}^{n+1}$ containing the origin.

Conversely, given $K \in \Phi^{n+1}$ which contains the origin, we can suppose $K = \{r\theta : r \leq \zeta(\theta), \theta \in S^n\}$, where $\zeta$ is polar convex. Then $K$ is the Wulff shape associated to $\zeta^*$ by Lemma (2.1.2).

With the notion of convexity, we obtain a correspondence between Wulff shapes and convex functions.

**Proposition 2.1.3.** For a continuous function $\gamma : S^n \rightarrow (0, \infty)$, $\gamma$ is the support function of a convex body in $\Phi^{n+1}$ if and only if $\gamma$ is convex.

**Proof** If $\gamma$ is the support function of $\{r\theta : r \leq \zeta(\theta), \theta \in S^n\}$, then $\gamma = \zeta^*$. Hence, $\frac{1}{\gamma} = \frac{1}{\zeta} = (\frac{1}{\zeta})^*$. Note that the polar plot of the first Legendre transform $\zeta^*$ is convex for any continuous $\zeta$, thus $\gamma$ is convex.

Conversely, if $\gamma$ is convex, then by Lemma (2.1.2), $\gamma = (\gamma^*)^*$. Note that the Second Legendre transform $\zeta^*$ is the support function of the region $\{r\theta : r \leq \zeta(\theta), \theta \in S^n\}$ for any continuous $\zeta$. Thus $\gamma$ is the support function of the Wulff shape associated to itself. \qed
2.2 Wulff flow and crystal growth

Consider the following set:

\[ \mathcal{Y} = \{ \Omega \subseteq \mathbb{R}^{n+1} : \Omega = \bigcup \Omega_i, \text{ where each } \Omega_i \text{ is the closure of a nonempty bounded open set, } \Omega_i \cap \Omega_j = \emptyset \text{ for } i \neq j, \text{ each } \partial \Omega_i \text{ is piecewise smooth} \} \]

**Definition** Given an initial region \( \Omega \in \mathcal{Y} \) and a uniformly continuous function \( \gamma : S^n \to (0, \infty) \). At each point where \( \partial \Omega \) is smooth, we move the hypersurface \( \partial K \) outwards with normal velocity \( \gamma(n) \), where \( n \) is the outward unit normal. We call this the Wulff flow associated to \( \gamma \), with initial region \( \Omega \). In particular, when \( \gamma \equiv 1 \) on \( S^n \), we call this the **unit-speed outward normal flow**.

Wulff flow is of considerable interest in the study of crystal growth. In many solid materials, the surface tension function \( \gamma \) of a crystalline structure depends only on how the surface is directed relative to the crystalline lattice structure (i.e. \( \gamma \) is a function on \( S^n \)). The total surface energy of a crystal \( \Omega \) is given by \( \int_{\partial \Omega} \gamma \).

It was shown in [2] that

1. the asymptotic growth shape under the Wulff flow is precisely the Wulff shape \( W \), appropriately scaled in time, and

2. \( W \) minimizes the surface energy among all shapes with a given volume.

In other words, \( W = \{ r \theta \in \mathbb{R}^{n+1} : r \leq \gamma_*(\theta), \theta \in S^n \} \) is the *equilibrium shape of a growing crystalline structure* with surface tension \( \gamma \) and any initial region that lies in \( \mathcal{Y} \). 

10
The following Theorem serves as a motivation for the Wulff-Steiner Polynomial.

**Theorem 2.2.1.** Consider the Wulff flow associated to a convex uniformly continuous \( \gamma : S^n \rightarrow (0, \infty) \), with initial region \( K \in \Phi^{n+1} \). Let \( \Omega(t) \) be the image of \( K \) at time \( t \geq 0 \), \( \Gamma(t) = \partial \Omega(t) \). Let \( A(t) = \int_{\Gamma(t)} \gamma(n) \, dS \) be the surface integral of \( \gamma(n) \), where \( n \) is the outward normal. \( V(t) = V_{n+1}(\Omega(t)) \), \( E(t) = \frac{A(t)}{V(t)^{\frac{n}{n+1}}} \). Then

(i) \( \Omega(t) = K + tW \)

(ii) \( E(t) \) strictly decreases under the Wulff flow if \( K \) is not homothetic to \( W \).

If \( K \) and \( W \) are homothetic, then \( E(t) \) is constant in time.

For detail of the proof, please refer to [2]. We outline the idea of the proof here for motivation.

*Proof Outline*

(1) We first need to formulate the Wulff flow mathematically. This can be accomplished by using the level set method and the viscosity solution to the Hamilton-Jacobi equation.

Consider a Lipschitz continuous function \( \phi_0 : \mathbb{R}^{n+1} \rightarrow \mathbb{R} \) such that

\[
\phi_0(x) = \begin{cases} 
< 0, & x \in K \setminus \partial K \\
0, & x \in \partial K \\
> 0, & \text{otherwise}
\end{cases}
\]  

(2.1)

Such function exists (e.g. it can be the signed distance function to \( \partial K \)).

The evolution of \( \partial K \) can be formulated as the Hamilton-Jacobi equation

\[
\begin{cases} 
\phi_t = -\gamma(\frac{\nabla \phi}{|\nabla \phi|}) \cdot \nabla \phi, & x \in \mathbb{R}^{n+1}, t > 0 \\
\phi(x, 0) = \phi_0(x), & x \in \mathbb{R}^{n+1}
\end{cases}
\]

(2.2)

The image of \( \partial K \) under the Wulff flow at time \( t \) can be identified as the zero level set \( \{x \in \mathbb{R}^{n+1} : \phi(x, t) = 0\} \).

(2) Since \( \gamma \) is convex, the Hamiltonian \( H(\nabla \phi) = \gamma(\frac{\nabla \phi}{|\nabla \phi|}) \cdot \nabla \phi \) is a convex function on \( \mathbb{R}^{n+1} \) by Lemma(2.1.1). Then we can apply the first Hopf formula [7] to equation (2.2), giving us the unique uniformly continuous viscosity solution to (2.2). (i) can then be easily obtained.
(3) For the proof of (ii), first check that $E = \frac{\partial V}{\partial t} V^{\frac{1}{n+1}} = (n+1) \frac{\partial (V^\frac{1}{n+1})}{\partial t}$, so
$\frac{\partial E}{\partial t} = (n+1) \frac{\partial^2 V}{\partial t^2} (t)^{\frac{1}{n+1}}$. We want to show that $\frac{\partial E}{\partial t} \leq 0$, i.e. $V(t)^{\frac{1}{n+1}}$ is concave in $t$. The key tool is the Brunn-Minkowski Theorem.

Remark By Lemma (1.5.2), we know that the support function of the convex body under the Wulff flow has a very clean formula, namely $p + t \gamma$. From (i), the region $\frac{1}{t} \Omega(t)$ converges to $W$ in the strong sense that its support function converges to that of $W$. From (ii), the surface energy to volume ratio decreases under the Wulff flow.

$V_{n+1}(\Omega(t))$ turns out to be a polynomial by Theorem (3.1.1). Though the Wulff flow is defined for $t \geq 0$, the polynomial $V_{n+1}(K + tW)$ can be formally defined for any complex number $t$. The Wulff-Steiner Polynomial is defined by

$$V_{n+1}(K + tW) \overset{\text{def}}{=} \sum_{j=0}^{n+1} C^j_{n+1} V_K (K, \ldots, K, W, \ldots, W) t^j$$

for $t \in \mathbb{C}$.

$V_{n+1}(K + tB)$ is known simply as the Steiner Polynomial. Note that $V_{n+1}(K + tW)$ remains unchanged if we translate $K$ or $W$. We will consider $V_{n+1}(K + tW)$ for $K, W \in \Phi^{n+1}$. 

12
Chapter 3

Mixed Volume

Studying the behaviour of volume under (Minkowski) addition of convex bod-
ies, we are led to the notion of mixed volume and to the Brunn-Minkowski
Theorem. The mixed volume is a useful tool, and the Brunn-Minkowski The-
orem is a fundamental property.

3.1 Definitions and basic properties

Theorem 3.1.1. Suppose $K_1, K_2 \in \Phi^{n+1}$ and $t_1, t_2 \geq 0$. Then there is a
symmetric function $V : (\Phi^{n+1})^{n+1} \to (0, \infty)$ such that

$$V_{n+1}(t_1 K_1 + t_2 K_2) = \sum_{j=0}^{n+1} C_{j+1}^{n+1} t_1^{n+1-j} t_2^j V(K_1, \ldots, \underbrace{K_1, K_2, \ldots, K_2}_{n+1-j})$$  \hspace{1cm} (3.1)

where $V_{n+1}(\cdot)$ is the $(n+1)$-dimensional volume.

Definition The terms $V(K_1, \ldots, \underbrace{K_1, K_2, \ldots, K_2}_{n+1-j}, \underbrace{K_2, \ldots, K_2}_j)$, $0 \leq j \leq n+1$, are called
mixed volumes.

For $K \in \Phi^{n+1}$, let $G$ be the generalized Gauss map and define a Borel
measure $\mu_K$ on $S^n$ as follows:

$$\mu_K(E) = V_n(\{x \in \partial K : G(x) \cap E \neq \emptyset\}) \forall \text{ Borel subset } E \text{ of } S^n.$$  \hspace{1cm} (3.2)

Then we have an important integral formula for mixed volume:
Theorem 3.1.2.

\[ V(K_1, \ldots, K_1, K_2) = \frac{1}{n+1} \int_{S^n} p_2 \, d\mu_{K_1} \]  

where \( p_2 \) is the support function of \( K_2 \).


We now present several interesting properties of mixed volumes.

1. \( V \) is linear in each of its arguments. That is,
\[ V(\lambda K + \lambda' K', W, \ldots, W) = \lambda V(K, W, \ldots, W) + \lambda' V(K', W, \ldots, W) \]
2. \( V(K, \ldots, K) = V_{n+1}(K) \)
3. \( V(T(K_1), \ldots, T(K_1), T(K_2), \ldots, T(K_2)) = V(K_1, \ldots, K_1, K_2, \ldots, K_2) \) for any volume-preserving linear map \( T : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1} \) (e.g. translation, rotation, reflection)
4. \( V \) is monotone, in the sense that if \( W_1 \subset W_2 \), then
\[ V(W_1, K, \ldots, K) \leq V(W_2, K, \ldots, K) \]

They can be easily checked: (1) follows if we expand \( V(t_1(\lambda K + \lambda' K') + t_2 W) \) in 2 different ways using Theorem(3.1.1). (2) can be obtained directly by putting \( t_1 = 1, t_2 = 0 \) into Theorem(3.1.1). (3) follows if we note that
\[ V_{n+1}(t_1 T(K_1) + t_2 T(K_2)) = V_{n+1}(T(t_1 K_1 + t_2 K_2)) = V_{n+1}(t_1 K_1 + t_2 K_2) \] and expand \( V_{n+1}(t_1 T(K_1) + t_2 T(K_2)) \) and \( V_{n+1}(t_1 K_1 + t_2 K_2) \) respectively using Theorem(3.1.1). (4) follows from the integral formula in Theorem(3.1.2).

3.2 The Brunn-Minkowski Theorem

The Brunn-Minkowski Theorem, which we now state, is the starting point of a rich theory of geometric inequalities. In particular, it can be used to prove some classical isoperimetric inequalities. For a nice motivation, see the chapter "Convex bodies" in [9]. Readers may also consult [1] or [5] for the proof.

Theorem 3.2.1. For \( K_1, K_2 \in \Phi^n \) and for \( \lambda \in [0,1] \), let \( K_\lambda = (1-\lambda)K_1 + \lambda K_2 \), then the function \( f(\lambda) = (V_n(K_\lambda))^{\frac{1}{n}} \) is concave on \( [0,1] \). If \( K_1 \) and \( K_2 \) do not lie on parallel hyperplanes, then \( f \) is linear if and only if \( K_1 \) and \( K_2 \) are homothetic.
An equivalent statement of the Brunn-Minkowski Theorem is as follows:

\textbf{Theorem 3.2.2.} For $K_1, K_2 \in \Phi^n$ and for $\lambda \in [0,1]$, we have $(V_n(K_\lambda))^{\frac{1}{n}} \geq (1-\lambda)(V_n(K_1))^{\frac{1}{n}} + \lambda(V_n(K_2))^{\frac{1}{n}}$ If $K_1$ and $K_2$ do not lie on parallel hyperplanes, then equality holds if and only if $K_1$ and $K_2$ are homothetic.

\textbf{Corollary 3.2.3. (Minkowski's inequalities)} For $K_1, K_2 \in \Phi^n$, (i) and (ii) below both hold:

(i) $V(K_1, \ldots, K_1, K_2)^n \geq V_n(K_1)^{n-1}V_n(K_2)$

Equality holds if and only if $K_1$ and $K_2$ are homothetic.

(ii) $V(K_1, \ldots, K_1, K_2)^2 \geq V_n(K_1)V(K_1, \ldots, K_1, K_2, K_2)$

\textbf{Remark} Inequality (ii) is a special case of the Aleksandrov-Fenchel inequality [1] p.327-332:

$$V(K_1, K_2, K_3, \ldots, K_n)^2 \geq V(K_1, K_1, K_3, \ldots, K_n)V(K_2, K_2, K_3, \ldots, K_n)$$

The investigation of the case of equality in general is still incomplete. Here is a partial result [1] p.359: If $K_i$ is smooth for $i = 3, 4, \ldots, n$, then equality holds in the Aleksandrov-Fenchel inequality if and only if $K_1$ and $K_2$ are homothetic.
Chapter 4

Wulff-Steiner Polynomial of degree 2

We first concentrate on plane regions in $\Psi^2$, in which case the Wulff-Steiner polynomial is of degree 2, which is easier to analyse.

For $K$ and $W \in \Psi^2$ having support functions $p$ and $\gamma$ respectively, we have $p + p'' > 0$ and $\gamma + \gamma'' > 0$ on $S^1$. By Lemma(1.5.2), the support function of $K + tW$ is $p + t\gamma$. Clearly, $(p + t\gamma) + (p + t\gamma)' > 0$ on $S^1$, hence $K + tW \in \Psi^2$ and

$$V_2(K + tW) = \frac{1}{2} \int_{S^1} (p + t\gamma)(p'' + t\gamma'' + p + t\gamma) \, d\sigma. \tag{4.1}$$

On the other hand, using the notion of mixed volume, we have

$$V_2(K + tW) = V_2(K) + 2V(K, W)t + V_2(W)t^2 \tag{4.2}$$

4.1 Invariant of the Wulff flow

Definition A quantity $q(K, W)$ is said to be invariant under the Wulff flow if $q(K, W) = q(K + sW, W) \forall s \in \mathbb{R}$.

By considering the Wulff-Steiner polynomial, we obtain an invariant easily:

Theorem 4.1.1. $V(K, W)^2 - V_2(K)V_2(W)$ is invariant under the Wulff flow.
Proof If we shift the graph of $f(t) = V_2(K + tW)$ horizontally in the direction of the negative $t$-axis by $s$ units, we get the graph of $f(t + s) = V_2(K + sW + tW)$. Clearly, $\min\{f(t+s): t \in \mathbb{R}\} = \min\{f(t): t \in \mathbb{R}\}$. This gives $V(K_s, W)^2 - V_2(K_s)V_2(W) = V(K, W)^2 - V_2(K)V_2(W)$ for all $s \in \mathbb{R}$, where $K_s = K + sW$. \qed

4.2 Properties of the roots

**Proposition 4.2.1.** The polynomial $V_2(K + tW)$ must have negative real roots. The 2 roots are equal if and only if $K$ and $W$ are homothetic to each other.

Proof It can be proved in 2 ways.

*(Method 1)*

Applying Corollary (3.2.3), we have

$$\min\{V_2(K + tW) : t \in \mathbb{R}\} = \frac{V_2(K)V_2(W) - V(K, W)^2}{V_2(W)} \leq 0$$

with equality holds if and only if $K$ and $W$ are homothetic.

*(method 2)*

Let $t_0 \in \mathbb{R}$ be such that $\int_{S^1} p + t_0 \gamma = 0$ (explicitly, $t_0 = -\frac{\int_{S^1} p}{t_\gamma}$).

By Wirtinger’s inequality (See Appendix), $\int_{S^1} (p + t_0 \gamma)^2 \leq \int_{S^1} (p' + t_0 \gamma')^2$, with equality holds if and only if $p(u) + t_0 \gamma(u) = d \cdot u \quad \forall u \in S^1$, for a fixed $d \in \mathbb{R}^2$.

Rewrite $V_2(K + tW)$ as $\frac{1}{2} \int_{S^1} (p + t \gamma)^2 - (p' + t \gamma')^2$ by using integration by parts on the R.H.S. of (4.1). We have

$$\min\{V_2(K + tW) : t \in \mathbb{R}\} \leq V_2(K + t_0 W)$$

$$= \frac{1}{2} \int_{S^1} (p + t_0 \gamma)^2 - (p' + t_0 \gamma')^2$$

$$\leq 0.$$ 

If equality holds, then $K$ and $W$ are homothetic.

Conversely, if $K$ and $W$ are homothetic, (i.e. $K = aW + \eta$ for $a > 0$ and $\eta \in \mathbb{R}^{a+1}$), then $V_2(K + tW) = (a + t)^2V_2(W)$ which has equal negative real roots.
4.3 Relative positions

**Definition** \( r(K, W) = \max \{ t : \text{some translate of } tW \text{ is contained in } K \} \) is called the \textbf{W-inradius}, \( R(K, W) = \min \{ t : \text{some translate of } tW \text{ contains } K \} \) is called the \textbf{W-outradius}, \( \rho_W = \frac{p + p''}{\gamma + \gamma''} \) is called the \textbf{Wulff radius of curvature}.

Let \( \sup \left \{ \frac{1}{V_2(W)} \int_I \rho_W \gamma(\gamma + \gamma'') : I \subset S^1, \int_I \gamma(\gamma + \gamma'')d\theta = V_2(W) \right \} \) be obtained on \( I_1(K, W), I_2(K, W) = S^1 \setminus I_1 \).
Let also \( \rho_i(K, W) = \frac{1}{V_2(W)} \int_{I_i} \rho_W \gamma(\gamma + \gamma'') \), \( i = 1, 2 \).

Then there exists a positive real number \( a \) such that \( I_1(K, W) \subset \{ \theta \in S^1 : \rho_W(\theta) \geq a \} \) and \( I_2(K, W) \subset \{ \theta \in S^1 : \rho_W(\theta) \leq a \} \).

For simplicity, we write \( r, R, \rho_1, \rho_2, I_1 \) and \( I_2 \) for \( r(K, W), R(K, W), \rho_1(K, W), \rho_2(K, W), I_1(K, W) \) and \( I_2(K, W) \) respectively.

Note that \( \rho_1 + \rho_2 = \frac{2V(K, W)}{V_2(W)} \) and \( \rho_1 \geq \rho_2 \). So we can write
\[
\rho_1 = \frac{V(K, W)}{V_2(W)} + b, \quad \rho_2 = \frac{V(K, W)}{V_2(W)} - b \text{ for some } b \geq 0.
\]

Let \( -t_1 \) and \( -t_2 \) \( (t_2 \geq t_1 > 0) \) be the roots of the equation \( V_2(K + tW) = 0 \).
Then
\[
-t_1 = -\frac{V(K, W)}{V_2(W)} + c, \quad -t_2 = -\frac{V(K, W)}{V_2(W)} - c,
\]
where \( c = \frac{\sqrt{V(K, W)^2 - V_2(K)V_2(W)}}{V_2(W)} \).

We also let \( \rho_{\min} = \inf \{ \rho_W(\theta) : \theta \in S^1 \} \), \( \rho_{\max} = \sup \{ \rho_W(\theta) : \theta \in S^1 \} \).

Before comparing the positions among \( t_1, t_2, r, R, \rho_1, \rho_2, \rho_{\min}, \rho_{\max} \) and \( \frac{V(K, W)}{V_2(W)} \),
We generalize a lemma given by Mark Green and Stanley Osher in [3].

**Proposition 4.3.1.** Suppose \( K_2 \subseteq K_1 \) where \( K_i \in \Phi^{n+1} \) has support function \( p_i, i=1,2 \). If \( p_2 < p_1 \) on the hemisphere \( N_w = \{ u \in S^n : u \cdot w \geq 0 \} \), where \( w \in S^n \), then
(i) \( \exists \epsilon > 0 \) such that \( K_2 + \epsilon w \subseteq K_1 \setminus \partial K_1 \), and

(ii) for this \( \epsilon \), \( \exists \lambda_1 \in (0, 1) \) and \( \lambda_2 > 0 \) such that \( K_2 + \epsilon w \subseteq (1 - \lambda_1)K_1 \) and \( (1 + \lambda_2)K_2 + \epsilon w \subseteq K_1 \).

Proof Since \( p_1 - p_2 \) is continuous and positive on the compact set \( N_w \), \( \exists \epsilon > 0 \) such that \( p_1 > p_2 + \epsilon \) on \( N_w \).

Note that \( u \cdot w \in [0, 1] \forall u \in N_w \), hence \( p_2(u) + \epsilon u \cdot w \leq p_2(u) + \epsilon < p_1(u) \forall u \in N_w \). On \( S^n \setminus N_w \), \( u \cdot w < 0 \), so \( p_2(u) + \epsilon u \cdot w < p_2(u) \leq p_1(u) \).

Therefore, \( p_2(u) + \epsilon u \cdot w < p_1(u) \forall u \in S^n \). By Lemma(1.5.3), we have \( K_2 + \epsilon w \subseteq K_1 \setminus \partial K_1 \).

Since \( p_1 - (p_2 + \epsilon u \cdot w) \) is continuous and positive on the compact set \( S^n \), \( \exists \epsilon_1 > 0 \) such that \( p_1 - (p_2 + \epsilon u \cdot w) \geq \epsilon_1 \) on \( S^n \).

Let \( M_i = \sup \{ p_i(u) : u \in S^n \} \) which is positive and finite by the compactness of \( S^n \), \( i=1,2 \). Choose \( \lambda_i = \frac{\epsilon_1}{M_i} > 0, i=1,2 \). Then \( p_1 - (p_2 + \epsilon u \cdot w) \geq M_i \lambda_i \geq p_i(u) \lambda_i(i) \), \( i=1,2 \), which implies

\[
\begin{align*}
(1 - \lambda_1)p_1(u) & \geq p_2 + \epsilon u \cdot w \\
p_1(u) & \geq (1 + \lambda_2)p_2(u) + \epsilon u \cdot w
\end{align*}
\]

The assertion follows by Lemma(1.5.3).

\( \square \)

Remark From the above proposition and the definition of \( W \)-inradius and \( W \)-outradius, we know that: On each hemisphere, there is a point \( v \) such that \( p(v) - r\gamma(v) = 0 \) and a point \( w \) such that \( R\gamma(w) - p(w) = 0 \) (i.e. \( \partial(rW) \) is tangent to \( \partial K \) at a point with normal vector \( v \) and \( \partial(RW) \) is tangent to \( \partial K \) at a point with normal vector \( w \)).

Theorem 4.3.2. For \( K,W \in \Psi^2 \), \( K \) and \( W \) are not homothetic to each other if and only if

\[
-t_2 < -R < -\frac{V(K,W)}{V_2(W)} < -r < -t_1 < 0
\]

Proof By Proposition(4.2.1), we have \( t_2 > \frac{V(K,W)}{V_2(W)} > t_1 > 0 \) if \( K \) and \( W \) are not homothetic.

Since \( RW \supset K \supset rW \), we have \( R\gamma \geq p \geq r\gamma \) on \( S^1 \). Also, \( \gamma + \gamma'' > 0 \) on \( S^1 \) by hypothesis. Hence \( \frac{1}{2} \int_{S^1} R\gamma(\gamma + \gamma'') > \frac{1}{2} \int_{S^1} p(\gamma + \gamma'') > \frac{1}{2} \int_{S^1} r\gamma(\gamma + \gamma'') \).

i.e. \( RV_2(W) > V(K,W) > rV_2(W) \). Thus \( R > \frac{V(K,W)}{V_2(W)} > r \).
It remains to show that $-R$ and $-r$ both lie in the open interval $(-t_2, -t_1)$. i.e. $V_2(K - RW)$ and $V_2(K - rW)$ are both negative.

From the remark following Proposition 4.3.1, we know that:

If $\partial(rW)$ (or $\partial(RW)$) is tangent to $\partial K$ at points where the unit outward normals are $(\cos \theta_j, \sin \theta_j) \in S^1$, where $0 \leq \theta_1 < \theta_2 < \ldots < \theta_\ell \leq 2\pi$. Then $\theta_{j+1} - \theta_j \leq \pi$ for $j = 1, 2, \ldots, \ell - 1$.

Now

$$V_2(K - rW) = \frac{1}{2} \int_{S^1} (p - r\gamma)^2 - (p - r\gamma)^2 d\theta$$

$$= \frac{1}{2} \sum_{j=1}^{\ell-1} \int_{\theta_j}^{\theta_{j+1}} (p - r\gamma)^2 - (p - r\gamma)^2 d\theta$$

$$\leq 0 \quad \text{by Poincare's inequality}$$

By continuity of $p - r\gamma$

Equality holds $\iff \exists d \in \mathbb{R}^2$ such that $p(u) - r\gamma(u) = d \cdot u \forall u \in S^1$

$\iff K = rW + d$ for some $d \in \mathbb{R}^2$.

Since $K$ and $W$ are not homothetic by assumption, equality does not hold and thus $V_2(K - rW) < 0$.

Applying the same argument for $R$, we have $V_2(K - RW) < 0$.

The converse is trivial.

\[ \square \]

**Corollary 4.3.3.** $V_2(K) + 2V(K, W)t + V_2(W)t^2 \leq 0$ for $t \in [-R, -r]$

**Corollary 4.3.4.** $(R - r)^2 \leq 4\frac{V(K, W)^2 - V_2(K)V_2(W)}{V_2(W)^2}$. This is known as Bonnesen's Inequality

*Proof* By Theorem 4.3.2, $t_2 \geq R \geq r \geq t_1$. Hence $(R - r)^2 \leq (t_2 - t_1)^2 = 4c^2 = 4\frac{V(K, W)^2 - V_2(K)V_2(W)}{V_2(W)^2}$.

*Remark* When $W = B$, Corollary 4.3.4 tells us that

$$L^2 - 4\pi A \geq \pi^2 (R - r)^2$$

for a convex body with area $A$ and perimeter $L$. This was obtained by T. Bonnesen during the 1920's.
Theorem 4.3.5. If $K, W \in \Psi^{n+1}$ with $W$ symmetric about the origin, then $\rho_1 \geq t_2$ (i.e. $b \geq c$).

Proof The proof consists of 2 cases:

(Case 1) $K$ is symmetric.

Note that $\rho_1$ and $t_2$ are unchanged under any translation of $K$. Hence we may assume the origin is also the point of symmetry of $K$. Then $RW \supset K \supset rW$ and they are all symmetric about the origin. Hence,

\[
\begin{align*}
\{ & \quad p(\theta) = p(-\theta) \\
& \quad \gamma(\theta) = \gamma(-\theta) \\
& \quad R\gamma(\theta) \geq p(\theta) \geq r\gamma(\theta) \\
& \quad \forall \theta \in S^1
\end{align*}
\]

By Theorem (4.3.2), we have $t_2 \geq R \geq \frac{p(\theta)}{\gamma(\theta)} \geq r \geq t_1 \quad \forall \theta \in S^1$. That is

\[c\gamma(\theta) \geq p(\theta) - \frac{V(K,W)}{V_2(W)}\gamma(\theta) \geq -c\gamma(\theta) \quad \forall \theta \in S^1 \tag{4.4}\]

On $I_1$, $\rho_W - a \geq 0$. Using the right inequality in (4.4), we have

\[(\rho_W - a)c\gamma \geq - (\rho_W - a)(p - \frac{V(K,W)}{V_2(W)}\gamma) \quad \text{on } I_1.
\]

Multiply both sides by $\gamma + \gamma''$ and then integrate over $I_1$,

\[- \int_{I_1} (\rho_W - a)(p - \frac{V(K,W)}{V_2(W)}\gamma)(\gamma + \gamma'') d\theta \\
\leq c \int_{I_1} (\rho_W - a)\gamma(\gamma + \gamma'') d\theta \\
= c(\int_{I_1} \rho_W \gamma(\gamma + \gamma'') d\theta - a \int_{I_1} \gamma(\gamma + \gamma'') d\theta) \\
= cV_2(W)(\rho_1 - a) \tag{4.5}
\]

On $I_2$, $\rho_W - a \leq 0$. Using the left inequality in (4.4), we have

\[-(\rho_W - a)(p - \frac{V(K,W)}{V_2(W)}\gamma) \leq (\rho_W - a)(-c\gamma) \quad \text{on } I_2.
\]

21
Multiply both sides by $\gamma + \gamma''$ and then integrate over $I_2$,
\[
- \int_{I_2} (\rho_W - a)(p - \frac{V(K, W)}{V_2(W)}\gamma)(\gamma + \gamma'') \, d\theta \\
\leq -c \int_{I_2} (\rho_W - a)\gamma(\gamma + \gamma'') \, d\theta \\
= -cV_2(W)(\rho_2 - a) 
\]
(4.6)

Adding (4.5) and (4.6), we have
\[
- \int_{S^1} (\rho_W - a)(p - \frac{V(K, W)}{V_2(W)}\gamma)(\gamma + \gamma'') \, d\theta \leq cA_2(W)(\rho_1 - \rho_2) 
\]
(4.7)

\[
\text{R.H.S.} = 2cbV_2(W) \\
\text{L.H.S.} = \frac{-1}{V(W)} \left\{ \int_{S^1} (p + p'')(V_2(W)p - V(K, W)\gamma) \, d\theta \\
- a \left( V_2(W) \int_{S^1} p(\gamma + \gamma'') \, d\theta - V(K, W) \int_{S^1} \gamma(\gamma + \gamma'') \, d\theta \right) \right\} \\
= \frac{-1}{V_2(W)} \{2V_2(W)V_2(K) - V(K, W)^2\} \\
= -2c^2V_2(W) 
\]

Putting into (4.7), we have $2c^2V_2(W) \leq 2cbV_2(W)$. Hence, $c \leq b$.

(Case2) $K$ is not symmetric.

Then we apply a symmetrization argument and then apply (case1).
\forall \theta \in S^1, by joining the 2 points $M(\theta) = g^{-1}(\theta)$ and $N(\theta) = g^{-1}(-\theta)$ where $g$ is the Gauss map, we can divide $K$ into 2 convex parts, $K_1^\theta$ and $K_2^\theta$, lying on the left and the right of the oriented line segment $MN$ respectively.

The function $D : S^1 \to \mathbb{R}$ defined by
\[
D(\theta) = [2V(K_1^\theta, W)^2 - 2V_2(K_1^\theta)V_2(W)] - [2V(K_2^\theta, W)^2 - 2V_2(K_2^\theta)V_2(W)] 
\]
is continuous and satisfies $D(0) = -D(\pi)$. By Intermediate Value Theorem, $\exists \mu \in S^1$ such that
\[
(2V(K_1^\mu, W))^2 - 2V_2(K_1^\mu)V_2(W) = (2V(K_2^\mu, W))^2 - 2V_2(K_2^\mu)V_2(W) 
\]
(4.8)
For simplicity, we write \( A_i \) and \( B_i \) for \( V_2(K_i^n) \) and \( V(K_i^n, W) \) respectively. Then
\[
(2B_1)^2 - 2A_1V_2(W) = (2B_2)^2 - 2A_2V_2(W) \quad (= \lambda) \quad (4.9)
\]
Clearly, \( V(K, W) = B_1 + B_2 \) and \( V_2(K) = A_1 + A_2 \).

Let \( L_i \) be the symmetric convex body obtained by joining \( K_i^n \) to a copy of \( K_i^n \) rotated by 180 degrees. Then \( V(L_i, W) = 2B_i \) and \( V_2(L_i) = 2A_i \).

Applying the result of (Case1) to \( L_1 \) and \( L_2 \) respectively, we have
\[
\rho_1(L_i, W) \geq \frac{2B_i}{V_2(W)} + \frac{\sqrt{(2B_i)^2 - 2A_iV_2(W)}}{V_2(W)}, \quad i = 1, 2 \quad (4.10)
\]

Since \( \int_{S^1} \rho_1 \gamma(\gamma + \gamma^n) \, d\theta \) is maximized by \( I_1(K, W) \) among all subset \( I \subset S^1 \) with \( \int_{S^1} \gamma(\gamma + \gamma^n) \, d\theta = V_2(W) \), we have \( \rho_1(K, W) \geq \rho_1(L_i, W) \) for \( i = 1, 2 \). Thus
\[
\rho_1(K, W) \geq \frac{1}{2} (\rho_1(L_1, W) + \rho_1(L_2, W)) \\
\geq \frac{1}{2} \left( \frac{2(B_1 + B_2)}{V_2(W)} + \frac{2\sqrt{\lambda}}{V_2(W)} \right) \\
= \frac{V(K, W)}{V_2(W)} + \frac{\sqrt{\lambda}}{V_2(W)}
\]

Therefore, the inequality that we want to show, which is
\[
\rho_1(K, W) \geq \frac{V(K, W)}{V_2(W)} + \frac{\sqrt{V(K, W)^2 - V_2(K)V_2(W)}}{V_2(W)},
\]
will follow if we can show that \( \lambda \geq V(K, W)^2 - V_2(K)V_2(W) \).

Since \( (B_1 + B_2)^2 \leq 2(B_1^2 + B_2^2) \),
\[
V(K, W)^2 - V_2(K)V_2(W) = (B_1 + B_2)^2 - (A_1 + A_2)V_2(W) \\
\leq 2(B_1^2 + B_2^2) - A_1V_2(W) - A_2V_2(W) \\
= \frac{(2B_1)^2 - 2A_1V_2(W)}{2} + \frac{(2B_2)^2 - 2A_2V_2(W)}{2} \\
= \lambda.
\]

We are done.

\( \square \)
From Theorem (4.3.2) and Theorem (4.3.5), we obtain a more detailed ordering:

\[ \rho_{\text{max}} \geq \rho_1 \geq t_2 \geq R \geq \frac{V(K, W)}{V_2(W)} \geq r \geq t_1 \geq \rho_2 \geq \rho_{\text{min}} > 0 \]  

(4.11)

In [3], Mark Green and Stanley Osher obtained a collection of isoperimetric inequalities for convex plane curves using Theorem (4.3.5). The novel feature is that they considered the roots of the Wulff-Steiner Polynomial.

4.4 Characterization of degree 2 Wulff-Steiner Polynomials

Lemma 4.4.1. Given \( A, L > 0 \) with \( L^2 \geq 4\pi A \), we can find \( K \in \Phi^2 \) such that \( V_2(K) = A \) and \( 2V(K, B) = L \).

Proof By hypothesis, \( r = \frac{L - \sqrt{L^2 - 4\pi A}}{2\pi} > 0 \) and \( s = \frac{\sqrt{L^2 - 4\pi A}}{2} \geq 0 \). We consider the Minkowski sum \( K = K_1 + K_2 \), where \( K_1 \) is a straight line segment of length \( s \), \( K_2 \) is a circular disc of radius \( r \). \( K \) is a "sausage-like" body: A rectangle of length \( s \) and width \( 2r \) with a semi-disc of radius \( r \) attached to each of its widths. Clearly, \( K \) lies in \( \Phi^2 \) and has area \( A \) and perimeter \( L \). On other hand, area of \( K = V_2(K) \) by definition, and perimeter of \( \partial K = \int_{\partial K} ds = 2V(K, B) \). We are done. \( \square \)

Remark Note that \( -r \) is a root of \( V_2(K + tB) \) in the above case.

Proposition 4.4.2. Given \( a, b, c > 0 \) with \( b^2 \geq 4ac \), we can find \( K, W \in \Phi^2 \) such that \( at^2 + bt + c = V_2(K + tW) \).

Proof First, we choose \( W \) to be \( \lambda B \) where \( \lambda = \sqrt{\frac{a}{c}} \). Then \( \lambda^2 \pi = a \).
Since \( \left( \frac{b}{\lambda} \right)^2 - 4\pi c = \pi \left( \frac{b^2 - 4ac}{a} \right) \geq 0 \), by Lemma 4.4.1, we can find \( K \in \Phi^2 \) such that \( V_2(K) = c \) and \( 2V(K, B) = \frac{b}{\lambda} \). Then

\[
V(K + tW) = (\lambda^2 \pi) t^2 + 2V(K, \lambda B)t + V_2(K) \\
= at^2 + bt + c
\]

The last equality holds since \( 2V(K, \lambda B) = \int_{\partial K} \lambda ds = \lambda \int_{\partial K} ds = \lambda 2V(K, B) \). \( \square \)
In view of Proposition (4.2.1) and Proposition (4.4.2),
\[ \{at^2 + bt + c : a, b, c > 0 \text{ and } b^2 \geq 4ac\} = \{V_2(K + tW) : K, W \in \Phi^2\} \quad (4.12) \]


It is natural to ask the following

**Question:** Given \(a, b, c > 0\) with \(b^2 \geq 4ac\), is the choice of the pair \((K, W)\) in Proposition (4.4.2) unique up to translation?

**Answer:** Negative!

Let \(T : \mathbb{R}^{n+1} \rightarrow \mathbb{R}^{n+1}\) be a volume-preserving linear map which is not the identity or a translation, we can choose \(K \in \Phi^{n+1}\) such that \(K \neq T(K)\) up to translation. Then by property (3) of mixed volume, \(V_{n+1}(K + tW) = V_{n+1}(T(K) + tT(W)) \forall t, \) but \(K \neq T(K)\) up to translation.

**Remark** If we choose \(W \in \Phi^{n+1}\) such that \(W = T(W)\), we know that

- \(V_{n+1}(K_1 + tW) = V_{n+1}(K_2 + tW)\) for some \(W \in \Phi^{n+1}\) does not imply \(K_1 = K_2\), although \(K_1 + tW = K_2 + tW\) for some \(W \in \Phi^{n+1}\) implies that \(K_1 = K_2\) (because \(p_1 + t\gamma = p_2 + t\gamma\) implies \(p_1 = p_2\)).

**Proposition 4.4.3.** For \(n \geq 1\), if \(V(K_1, \underbrace{W, \ldots, W}_n) = V(K_2, \underbrace{W, \ldots, W}_n) \forall W \in \Phi^{n+1}\), then \(K_1 = K_2 + \eta\) for some \(\eta \in \mathbb{R}^{n+1}\).

**Proof** By the integral formula in Theorem (3.1.2), we have
\[ \int_{S^n} p \, d\mu_W = 0 \quad \forall W \in \Phi^{n+1} \quad (4.13) \]
where \(p = p_1 - p_2\).

Let \(\{u_i\}_{i=1}^\ell\) be unit vectors linearly spanning \(\mathbb{R}^{n+1}\), \(\{c_i\}_{i=1}^\ell\) be positive real numbers such that \(\sum_{i=1}^\ell c_i u_i = 0\). By Minkowski’s existence theorem (see [1], p.390-392), there exists a polytope \(W \in \Phi^{n+1}\) having facets \(\{F_i\}_{i=1}^\ell\) such that \(c_i = V_n(F_i)\) and \(u_i\) is precisely the outward normal of \(F_i, i = 1, 2, \ldots, \ell\). Then \(\mu_W = \sum c_i \delta_{u_i}\) where \(\delta_{u_i}\) is the unit point mass at \(u_i\). From (4.13), we have
\[ \sum c_i p(u_i) = 0, \text{ whenever } \{u_i\} \text{ spans } \mathbb{R}^{n+1} \text{ and } \{c_i\} \text{ are positive and satisfies } \sum c_i u_i = 0. \]
Claim: \( p(w) + p(-w) = 0 \) \( \forall w \in S^n \)

**Proof of the claim:** Let \( \{w, -w, u_3, u_4, \ldots, u_\ell\} \) be unit vectors spanning \( \mathbb{R}^{n+1} \) and \( \{c_i\}_{i=3}^\ell \) be positive real numbers such that \( \sum_{i=3}^\ell c_i u_i = 0 \). Note that 
\[
 w + (-w) + \sum_{i=3}^\ell \frac{c_i}{k} u_i = 0 \quad \text{for} \quad k \in (0, \infty).
\]

Thus
\[
p(w) + p(-w) + \sum_{i=3}^\ell \frac{c_i}{k} p(u_i) = 0 \quad \text{for} \quad k \in (0, \infty).
\]

Let \( k \to \infty \), we have \( p(w) + p(-w) = 0 \). The proof of the claim is complete.

By the claim, we can drop the condition that \( c_i \) be positive. \( \forall u \in S^n \),
\[
u = \sum_{i=1}^{n+1} (u \cdot e_i) e_i,
\]
where \( \{e_i\}_{i=1}^{n+1} \) is an orthonormal basis of \( \mathbb{R}^{n+1} \). Then
\[
p(u) - \sum_{i=1}^{n+1} (u \cdot e_i) p(e_i) = 0
\]
from the statement before the claim. Hence
\[
p(u) = u \cdot d,
\]
where \( d = \sum_{i=1}^{n+1} p(e_i) e_i \in \mathbb{R}^{n+1} \). As a result, \( K_1 = K_2 + d \).

\( \square \)

**Corollary 4.4.4.** For \( n \geq 1 \), if \( V_{n+1}(K_1 + tW) = V_{n+1}(K_2 + tW) \) for all \( W \in \Phi^{n+1} \), then \( K_1 = K_2 \).
Chapter 5

Wulff-Steiner Polynomial of degree 3

For $K, W \in \Psi^3$, we have

$$V_3(K + tW) = V_3(K) + 3V(K, K, W)t + 3V(K, W, W)t^2 + V_3(W)t^3 \quad (5.1)$$

For simplicity, let

$$V_1 = V_3(K) = \frac{1}{3} \int_{\partial K} p$$

$$V_{112} = V(K, K, W) = \frac{1}{3} \int_{\partial K} \gamma$$

$$V_{122} = V(K, W, W) = \frac{1}{3} \int_{\partial W} p$$

$$V_2 = V_3(W) = \frac{1}{3} \int_{\partial W} \gamma$$

Then

$$V_3(K + tW) = V_1 + 3V_{112}t + 3V_{122}t^2 + V_2t^3 \quad (5.2)$$

By Corollary(3.2.3) and the remark that follows, we have

$$V_{112}^3 \geq V_1^2V_2$$ and $$V_{112}^2 \geq V_1V_{122} \quad (5.3)$$

Equality holds in either inequality if and only if $K$ and $W$ are homothetic.

By symmetry, we have

$$V_{221}^3 \geq V_2^2V_1$$ and $$V_{221}^2 \geq V_2V_{211} \quad (5.4)$$

Equality holds in either inequality if and only if $K$ is homothetic to $W$. 

27
5.1 Invariants

Proposition 5.1.1. \( V_3(K + tW) \) have 2 critical points if and only if \( K \) and \( W \) are not homothetic.

Proof Consider the derivative \( V_3'(K + tW) = \frac{d}{dt} V_3(k + tW) = 3(V_{112} + 2V_{122}t + V_{2}t^3) \) which is a quadratic polynomial. By (5.4), its roots \( -c_1 \) and \( -c_2 \) are real and negative, where
\[
-c_1 = \frac{-V_{122} + \sqrt{V_{122}^2 - V_{112}V_2}}{V_2} \quad \text{and} \quad -c_2 = \frac{-V_{122} - \sqrt{V_{122}^2 - V_{112}V_2}}{V_2}.
\]
The 2 roots coincide if and only if \( K \) and \( W \) are homothetic.

Remark If \( K \) is homothetic to \( W \), say \( K = \lambda W + \eta \) with \( \lambda > 0 \), then \( V_3(K + tW) = V_3(W)(\lambda + t)^3 \) has exactly one critical point and exactly one root (both are equal to \( -\lambda \)).

Applying the same argument as in the proof of Theorem (4.1.1), we see that the critical values, \( V_3(K - c_1W) \) and \( V_3(K - c_2W) \), are invariants under the Wulff flow. After simplification, we have the following 2 invariants:

\[
\begin{align*}
\delta_1 & = V_{122}^2 - V_{112}V_2 \\
\delta_2 & = 3V_2V_{112}V_{122} - V_1V_2^2 - 2V_{122}^3
\end{align*}
\]

(5.5)

(5.6)

Note that \( \delta_1 \geq 0 \) and that \( \delta_1 = 0 \) if and only if \( K \) and \( W \) are homothetic by Corollary (3.2.3).

In fact, the invariant \( \delta_1 \) can be obtained more easily from the critical value of the quadratic polynomial \( \frac{d}{dt} V_3(k + tW) \). This idea can be applied to higher dimensions.

Proposition 5.1.2. For \( n \geq 2 \),

\[
V(K, \underbrace{W, \ldots, W}_{n-1}) - V_n(W) V(K, \underbrace{W, \ldots, W}_{n-2})
\]

(5.7)

is a non-negative invariant under the Wulff flow associated to \( W \in \Psi^n \) with initial region \( K \in \Psi^n \).

Proof By Theorem (3.1.1), \( V_n(K + tW) = \sum_{r=0}^{n} C^n_r V_{1 \ldots 1 \underbrace{2 \ldots 2}_{r}} t^r \), where

\[
V_{1 \ldots 1 \underbrace{2 \ldots 2}_{r}} = V(K, \underbrace{\ldots, K, W, \ldots, W}_{n-r})
\]

\[ r \]

28
It is easy to check that \( \frac{d^{(n-2)}}{dt^{n-2}} V_n(K + tW) = \frac{n!}{2^t} [V_2 \cdots 2 t^2 + 2V_{12} \cdots 2 t + V_{112} \cdots 2]. \)

If we replace \( K \) by \( K + sW \) for \( s \in \mathbb{R} \), the quadratic polynomial \( \frac{d^{(n-2)}}{dt^{n-2}} V_n(K + tW) \) still has the same critical value, which gives us the invariant.

The fact that the invariant is non-negative follows directly from Corollary (3.2.3)(ii). □

5.2 Properties of the roots

Recall that all roots of every Wulff-Steiner polynomial of degree 2 must be real and negative. However, complex roots can occur for degree 3. For example, if \( K \) is the one-cap body (i.e. the convex hull of the unit ball and a point outside it) and \( W \) is the closed unit ball, then the W-S polynomial has exactly one real root and two complex roots in conjugate pair. (Refer to Appendix for more examples such as ellipsoids)

By the Cardano-Tartaglia formula, the roots of the cubic equation

\[
V_3(K + tW) = V_2 \left( t^3 + \frac{3V_{122}}{V_2} t^2 + \frac{3V_{112}}{V_2} t + \frac{V_1}{V_2} \right) = 0
\]

are

\[
\begin{align*}
z_1 &= \frac{-V_{122}}{V_2} + S + T \\
z_2 &= \frac{-V_{122}}{V_2} - \frac{1}{2} (S + T) + \frac{\sqrt{3}}{2} (S - T)i \\
z_3 &= \frac{-V_{122}}{V_2} - \frac{1}{2} (S + T) - \frac{\sqrt{3}}{2} (S - T)i,
\end{align*}
\]

where

\[
\begin{align*}
Q &= \frac{S_1}{V_2^2}, \quad R = \frac{S_2}{2V_3^2} \\
D &= Q^3 + R^2 \\
S &= \sqrt[3]{R + \sqrt{D}}, \quad T = \sqrt[3]{R - \sqrt{D}} \text{ with } ST = -Q
\end{align*}
\]
Besides,

\[ D > 0 \iff z_1 \text{ is real and } z_2 = \bar{z}_3 \text{ are complex numbers} \]
\[ D = 0 \iff z_1, z_2 = z_3 \text{ are real numbers} \]
\[ D < 0 \iff z_1, z_2, z_3 \text{ are distinct real numbers} \]

Since all the coefficients are positive, any real root must be negative. For complex roots, we claim that their real parts must all be negative:

**Proposition 5.2.1.** The real part of any root of the cubic equation \( V_3(K + tW) = 0 \) is negative.

**Proof** Since all the coefficients are positive, any real root must be negative.

Assume that complex roots occur, i.e. \( D > 0 \). Since \( Q \leq 0 \), we have \( R^2 = D - Q^3 \geq D \). We have two possible cases:

Case 1) If \( R \geq 0 \), then
\[ R \geq \sqrt{D} > 0 \]. Hence \( S > T \geq 0 \). The real part of the complex root
\[ -\frac{V_{122}}{V_2} - \frac{1}{2}(S + T) \] is then negative.

Case 2) If \( R < 0 \), then

\[
\text{the real part is negative} \iff -\frac{V_{122}}{V_2} - \frac{1}{2}(S + T) < 0
\]
\[
\iff \frac{V_{122}}{V_2} > -\frac{1}{2}(S + T)
\]
\[
\iff \frac{V_{122}}{V_2} > \frac{-R}{(S - T)^2 - Q}
\]

The last inequality is true if \( \frac{V_{122}}{V_2} > \frac{-R}{Q} \). It is easy to check that
\[
\frac{V_{122}}{V_2} > \frac{R}{Q} \iff V_{112}V_{122} - V_1V_2 > 0.
\]

From (5.3) and (5.4), we have \( V_{112}^3 \geq V_1^2V_2 \) and \( V_{122}^3 \geq V_2^2V_1 \).
\( V_{112}V_{122} \geq V_1V_2 \) easily follows.

\[ \square \]

**Remark** Instead of checking directly as above, we can also apply the Routh-Hurwitz criterion.
We then obtain a condition for the position of the negative real part relative to critical points \(-c_1\) and \(-c_2\). For simplicity, let \(z_2 = -a + bi, z_1 = -d\), then \(a, d > 0\) by Proposition (5.2.1).

**Proposition 5.2.2.**

\[
\frac{d}{dt} V_3(K + tW) \bigg|_{t=-a} > 0
\]

Hence \(a \in (0, c_1) \cup (c_2, \infty)\) where \[-c_1 = \frac{-V_{122} + \sqrt{V_{122}^2 - V_{112}V_2}}{V_2}\]

and \[-c_2 = \frac{-V_{122} - \sqrt{V_{122}^2 - V_{112}V_2}}{V_2} .\]

**Proof** By direct expansion,

\[
0 = V(K + z_2W) = [V_3(K - aW) - 3b^2(V_{122} - aV_2)] + ib[V_3'(K - aW) - b^2V_2] ,
\]

we get

\[
\begin{align*}
V_3(K - aW) &= 3b^2(V_{122} - aV_2) \\
V_3'(K - aW) &= b^2V_2 > 0
\end{align*}
\]

(5.8)

The assertion follows from the second inequality.

\[\square\]

### 5.3 Relative positions among the roots, \(r\) and \(R\)

For the positions of \(R\) and \(r\), since \(rW \subset K \subset RW\), by the linearity and monotonicity of mixed volume,

\[
\begin{align*}
r^3V_2 &\leq V_1 \leq R^3V_2 \quad (5.9) \\
r^2V_2 &\leq V_{112} \leq R^2V_2 \quad (5.10) \\
rV_2 &\leq V_{122} \leq RV_2 \quad (5.11)
\end{align*}
\]

Let \(\tilde{R} = R(W, K)\), \(\tilde{r} = r(W, K)\). Then similarly we have

\[
\begin{align*}
\tilde{r}^3V_1 &\leq V_2 \leq \tilde{R}^3V_1 \quad (5.12) \\
\tilde{r}^2V_1 &\leq V_{122} \leq \tilde{R}^2V_1 \quad (5.13) \\
\tilde{r}V_1 &\leq V_{112} \leq \tilde{R}V_1 \quad (5.14)
\end{align*}
\]

31
Note that \( \tilde{R} = \frac{1}{r} \) and \( \tilde{r} = \frac{1}{\tilde{R}} \).

Combining (5.3), (5.4) and (5.10)-(5.15), we have

\[
R \geq \frac{V_{122}}{V_2} \geq \frac{V_1}{V_{112}} \geq r
\]

(5.15)

**Remark** Hence \(-R\) and \(-r\) cannot lie on the same side of the point of inflection \(-\frac{V_{122}}{V_2}\). Moreover, \(R - r\) is bounded below by the non-negative number \(\frac{V_{122}}{V_2} - \frac{V_1}{V_{112}}\).

J.R. Sangwine-Yager obtained a Bonnesen style inradius inequality in \(\mathbb{R}^3\) in [11]. For more about inequalities involving \(R\) and \(r\), please refer to [12].

We end this section with an open conjecture due to B. Teissier [8].

**Conjecture:** For \(K \in \Phi^n\), if \(-a_1 \geq -a_2 \geq \cdots \geq -a_n\) are the real parts of the roots of \(V_n(K + tB)\), then \(0 > -a_1 \geq -r(K, B) \geq -R(K, B) \geq -a_n\).

**Remark** Theorem (4.3.2) shows that the conjecture must be true for \(n=2\). For \(n=3\), we verify the conjecture for several ellipsoids in Appendix 6.2. J.M. Wills also considered the roots of \(V_{n+1}(K + tB)\) in [13].

In view of our experimental results using MatLab, we make a further conjecture for \(n=3\):

**Conjecture:** For \(K \in \Phi^3\), if \(-a_3 < -a_2 < -a_1\) are 3 distinct real roots of \(V_3(K + tB)\), then \(-a_3 \leq -R(K, B) \leq -a_2 \leq -r(K, B) \leq -a_1 < 0\).
Chapter 6

Appendix

6.1 Useful inequalities

**Theorem 6.1.1. (Poincare's inequality)** Let \( f : [0, a] \rightarrow \mathbb{R} \) be a differentiable function with \( f'(x) \in L^2([0, a]) \). If \( f(0) = f(a) = 0 \), then

\[
\int_{[0,a]} f'(x)^2 \, dx \geq \left( \frac{\pi}{a} \right)^2 \int_{[0,a]} f(x)^2 \, dx
\]

, with equality holds if and only if \( f(x) = c_1 \cos \left( \frac{\pi}{a} x \right) + c_2 \sin \left( \frac{\pi}{a} x \right) \) for some constants \( c_1 \) and \( c_2 \).

One of the proofs of Poincare's inequality applies the Fourier expansion on \( f \) and also the Parseval's identity. We will leave it to the reader.

**Theorem 6.1.2. (Wirtinger's inequality on \( S^n \))** If \( f \) is a twice continuously differentiable function on \( S^n \) and \( \int_{S^n} f = 0 \), then

\[
\int_{S^n} | \nabla f |^2 \geq n \int_{S^n} f^2
\]

, with equality holds if and only if \( f(u) = u \cdot d \forall u \in S^n \), for some \( d \in \mathbb{R}^{n+1} \).

6.2 Matlab results for ellipsoid

For the case \( W=\)the closed unit ball \( B, K=\) the ellipsoid \( \{(x,y,z) \in \mathbb{R}^3 : \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} - 1 \leq 0\} \), The Steiner polynomial is \( V_3(K+tB) = V + At + Mt^2 + \frac{4\pi t^3}{3} \), where \( V \) is the volume of \( K \), \( A \) is the surface area of \( K \), \( M = V(K,B,B) = \int_{S^2} p \, d\sigma \), and \( p : S^2 \rightarrow \mathbb{R} \) is the support function of \( K \).

We parametrize \( \partial K \) by \( (\theta, \psi) \mapsto (a \sin \psi \cos \theta, b \sin \psi \sin \theta, c \cos \psi), \theta \in (0,2\pi), \psi \in (0,\pi) \). Then by direct calculation, we get

- \( V = \frac{4\pi}{3} abc \)
- \( A = \int_0^\pi \int_0^{2\pi} \sin \psi \sqrt{T} \, d\theta d\psi \)
- \( M = \int_{S^2} p \, d\sigma = (abc)^3 \int_0^\pi \int_0^{2\pi} \frac{\sin \psi}{T^2} \, d\theta d\psi \)

where \( T = (bc \sin \psi \cos \theta)^2 + (ac \sin \psi \sin \theta)^2 + (ab \cos \psi)^2 \).

We will plot the graphs of the Steiner polynomials corresponding to the cases \((a,b,c) = (4,4,8), (4,6,8) \) and \((4,8,8)\) respectively. This illustrates that the following cases can occur.

Case(1) Both critical values are positive
(Complex roots occur) See figure 6.1.

Case(2) One critical value is positive and the other is negative
(3 distinct real roots) See figure 6.2.

Case(3) Both critical values are negative
(Complex roots occur) See figure 6.3.

Moreover, we verify B.Teissier’s conjecture for \( n=3 \) in each case. i.e.
\[ -a_3 \leq -R \leq -r \leq -a_1 < 0 \]

- The upward-pointing triangle ”\( \Delta \)” marks the position of \(-a_1\),
- The downward-pointing triangle ”\( \nabla \)” marks the position of \(-a_3\),
- The plus sign ”+” marks the position of \(-R\),
- The cross ”\( \times \)” marks the position of \(-r\),
- The dot ”.” marks the position of \( \frac{M}{4\pi} \) which is the point of inflection.
Figure 6.1: \((a,b,c)=(4,4,8)\)

\[(-a_3, -R, \frac{-M}{4\pi}, -r, -a_1) = (-9.077, -8, -5.520, -4, -3.742)\]
Figure 6.2: $(a, b, c) = (4, 6, 8)$

$$(-a_3, -R, \frac{-M}{4\pi}, -r, -a_1) = (-8.673, -8, -6.134, -4, -3.629)$$
Figure 6.3: \((a, b, c) = (4, 8, 8)\)

\((-a_3, -R, \frac{-M}{4\pi}, -r, -a_1) = (-8.492, -8, -6.836, -4, -3.525)\)
Bibliography


