Marcinkiewicz's Theorem and its Generalizations

by

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Marcinkiewicz's Theorem and its Generalizations

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Abstract

In this thesis, we obtained some generalizations of a theorem of Marcinkiewicz. The main tools of the proofs in this thesis are Weierstrass' approximation theorem, Lusin’s theorem and Egoroff’s theorem. In the one variable generalization, we will see that the statement holds even when we replace the closed and bounded interval \( I \) by \( \mathbb{R} \), the real line. We will also prove the statement with the “first difference form” replaced by the “second difference form”. In the two variables generalization, we will first construct a “Cantor function” on \([0,1] \times [0,1]\). With this function in hand, we can prove on \( \mathbb{R}^2 \) a similar statement with some variations. Finally we will see some linkages between the one variable generalization and the two variable generalization.
Chapter 1

Preliminaries

1.1 Stone-Weierstrass Theorem

Let $X$ be a compact space and let $C^r(X)$ denote the real-valued continuous functions on $X$. Let $A$ be a subalgebra of $C^r(X)$ that separates the points of $X$. Then the uniform closure $\overline{A}$ of $A$ satisfies either

$$\overline{A} = C^r(X)$$

or there is some $p \in X$ such that

$$\overline{A} = \{ f \in C^r(X) : f(p) = 0 \}.$$

This implies the Weierstrass Approximation Theorem as follow.

Let $D$ be compact in $\mathbb{R}^n$, $f : D \to \mathbb{R}$ be continuous, and let $\epsilon > 0$ be given. Then there exists a polynomial $P$ of $n$ variables such that

$$|P(x) - f(x)| < \epsilon \text{ for } x \in D.$$

1.2 The Cantor Function

When we construct the Cantor set $K$, in the $n$-th process we delete the middle third open interval from each of the remaining intervals with length $1/3^n$. Let $I_{n,k}$ be the $k$-th deleted interval in the $n$-th process.

Now we define

$$h(x) = \frac{2k - 1}{2^n} \text{ on } I_{n,k}.$$ 

For $x \in K$, we define

$$h(x) = \sup\{h(t) : t \in [0, 1]\setminus P, t < x\}.$$ 

This $h$ is called the Cantor function, which is continuous on $[0, 1]$, and $h' = 0$ almost everywhere on $[0, 1]$. For details, please refer to [2,p129,3.92].

1.3 Lusin’s Theorem

Let $f$ be measurable on $\mathbb{R}$. Then for each $\delta > 0$ there exists a continuous complex-valued function $g$ on $\mathbb{R}$ such that

$$\lambda(\{x \in \mathbb{R} : f(x) \neq g(x)\}) < \delta.$$ 

Moreover, if $|f(x)| \leq \beta$ almost everywhere, we can choose $g$ such that $|g(x)| \leq \beta$ for all $x$.

For details, please refer to [2,p303,6.76].

1.4 Egoroff’s Theorem

Let $E$ be a measurable set with $\lambda(E) < \infty$. Suppose that $\{f_n\}_{n=1}^\infty$ be measurable functions and that $f_n \to f$ almost everywhere on $E$, where $f$ is a complex-valued measurable function defined almost everywhere on $E$. Then for each $\delta > 0$ there exists a compact set $C$ such that $C \subseteq E$, $\lambda(E\setminus C) < \delta$, and $f_n \to f$ uniformly on $C$.

For details please refer to [2,p302,6.74].

In the entire paper, $\lambda$ and $\mu$ denote Lebesgue measure on $\mathbb{R}$ and $\mathbb{R}^2$ respectively.
1.5 Baire Category Theorem

Let $X$ be a complete metric space and $\{U_n\}_{n=1}^{\infty}$ be open dense sets in $X$. Then $\bigcap_{n=1}^{\infty} U_n$ is dense in $X$.

For details please refer to [2, p302, 6.74].

In the entire paper, residual denotes a set whose complement is of first category.

1.6 Tietze Extension Theorem

Let $A$ be a closed subset of a metric space $X$. Every bounded continuous function on $A$ can be extended continuously to $X$ with the same sup-norm.
Chapter 2

Marcinkiewicz Function on $\mathbb{R}^1$

2.1 Original Problem

The following result was first proven by J. Marcinkiewicz in [1].

**Theorem.** Let $I \subset \mathbb{R}$ be a closed interval and let $\{h_n\}_{n=1}^{\infty}$ be any fixed sequence of nonzero real numbers having limit 0. Then there exists a continuous function $F : I \to \mathbb{R}$ having the following property. If $\phi : I \to \mathbb{R}$ is any measurable function, then there is a subsequence $\{h_{n_j}\}_{j=1}^{\infty}$ of $\{h_n\}_{n=1}^{\infty}$ such that

$$\lim_{j \to \infty} \frac{F(x + h_{n_j}) - F(x)}{h_{n_j}} = \phi(x)$$

for almost every $x \in I$. One and the same $F$ works for all $\phi$. Of course the subsequence depends on $\phi$. Here in the entire paper, we call the functions satisfying the above conditions Marcinkiewicz functions.

We will first give a proof of the original problem. We will show later that $I$ in the statement can be replaced by $\mathbb{R}$ without changing the conclusion. The proof will be given in the section Corollaries and Consequences.

**Proof of theorem.**

(a) Given $G \in C(I)$ and $\epsilon > 0$. Let $h$ be the Cantor function on $[0, 1]$. As $G$ is uniformly continuous, we can divide $I$ into $k$ intervals $I_1, I_2, \ldots, I_k$ such that $|G(s) - G(t)| < \epsilon$ for $s, t \in I_i$. Let $[a, b] = I_i, c = G(b) - G(a)$, and

$$H(a + (b - a)x) = G(a) + ch(x), \text{ where } 0 \leq x \leq 1.$$
Then $H(a) = G(a), H(b) = G(b)$, and the monotonicity of $H$ shows
$|H - G| < \epsilon$ on $I_t$. Define in the same way for $H$ on each $I_i$ then we get
$H$ satisfied $H' = 0$ almost everywhere on $I$.

(b) Given $Q, G_1 \in C(I), Q$ is differentiable almost everywhere on $I$ and
$\epsilon > 0$. By (a), there exists $H$ such that $|H - (G_1 - Q)| < \epsilon$ on $I$ and
$H' = 0$ almost everywhere. Define $G_2 = H + Q$. Then $G_2' = Q'$ almost
everywhere and $|G_2 - G_1| < \epsilon$ on $I$.

(c) There is an enumeration $\{P_k\}_{k=1}^\infty$ of the set of all polynomials having
rational coefficients for which $P_1 = 0$. (Hereafter, polynomial will mean
polynomial with rational coefficients.)

(d) Set $F_1 = 0, E_1 = \phi, n_1 = 1$ and $t_1 = h_1$. By (b) and let $\epsilon = \frac{|t_{k-1}|}{k-1}$, for
each $F_{k-1}$, we get $F_k$ such that $F_k = P_k$ almost everywhere and
$|F_k - F_{k-1}| < \epsilon$ on $I$.

(e) Define $f_n(x) = \frac{F_k(x + h_n) - F_k(x)}{h_n}$ on the interior of $I$. Then $f_n$ converges to
$P_k$ almost everywhere on $I$. By Egoroff’s Theorem, $f_n$ converges
uniformly to $P_k$ on $I\setminus E_k$ for some $E_k$ with measure less than $2^{-k}$. Hence
we can choose $t_k = h_n$ for $n$ large enough so that $|t_{k-1}| > 2|t_k|$ and
$$|F_k(x + t_k) - F_k(x) - P_k(x)| < \frac{1}{k}$$
for $x \in I \setminus E_k$.

(f) This defines $F_k, E_k, n_k$ and $t_k$ for all $k \in \mathbb{N}$.

(g) For every $x \in I$, we have
$$|F_{k+n} - F_k| \leq |F_{k+n} - F_{k+n-1}| + |F_{k+n-1} - F_{k+n-2}| + \cdots + |F_{k+1} - F_k|$$
$$< \sum_{i=k}^{k+n-1} |t_i|$$
$$< \frac{|t_k|}{k} \left(1 + \frac{1}{2} + \cdots + \frac{1}{2^{n-1}}\right)$$
$$< \frac{2|t_k|}{k}$$

(h) $\{F_k\}_{k=1}^\infty$ is now a Cauchy sequence hence has a limit $F$ defined on $I$.
Then $|F - F_k| = \lim_{n \to \infty} |F_{k+n} - F_k| \leq \frac{2|t_k|}{k}$ on $I$. 

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(i) By (h), for every $x \in I \setminus E_k$, we get

$$\frac{|F(x + t_k) - F(x)|}{t_k} - P_k(x) \leq \frac{F(x + t_k) - F(x) - (F_k(x + t_k) - F_k(x))}{t_k} + \frac{F_k(x + t_k) - F_k(x)}{t_k} - P_k(x) \leq \frac{4t_k}{kt_k} + \frac{1}{k} = \frac{5}{k}$$

Let $\phi : I \to \mathbb{R}$ be a measurable function.

(j) By Lusin’s theorem, there exists a continuous $g$ on $I$ so that $g = \phi$ on $I \setminus B_j$ for some set $B_j$ with measure less than $2^{-j}$. Then by Weierstrass approximation theorem, we have $|P_{k_j} - g| < 1/j$ on $I$. Thus we have $|P_{k_j} - \phi| < 1/j$ on $I \setminus B_j$.

(k) By (i) and (j), write $s_j = t_{k_j}$, we have

$$\left| \frac{F(x + s_j) - F(x)}{s_j} - \phi(x) \right| < \frac{5}{k_j} + \frac{1}{j} < \frac{6}{j}$$

for all $j \in \mathbb{N}$ and $x \in I \setminus (E_{k_j} \cup B_j)$.

(l) Set $A = \bigcap_{r=1}^{\infty} \bigcup_{j=r}^{\infty} (E_{k_j} \cup B_j)$. As

$$\lambda\left(\bigcup_{j=r}^{\infty} (E_{k_j} \cup B_j)\right) \leq \sum_{j=r}^{\infty} \lambda(E_{k_j} \cup B_j) < \sum_{j=r}^{\infty} 2^{-(j-1)} = 2^{-(r-2)}$$

we see that $\lambda(A) = 0$.

(m) By (k) and (l), for $x \in I \setminus A$,

$$\lim_{j \to \infty} \frac{F(x + s_j) - F(x)}{s_j} = \phi(x)$$

and the proof is completed.
2.2 Marcinkiewicz Functions are Residual

Furthermore, we can prove that the set of functions satisfying the same properties is residual in $C^r(I)$. The main tool of the proof is the Baire Category theorem.

First define

$$A_k = \left\{ F \in C(I) : \left| \frac{F(x+t)-F(x)}{t} - P_k(x) \right| < \frac{1}{k}, t = h_n \text{ for some } n > k, x \in I \setminus E_k \text{ where } \lambda(E_k) < 2^{-k} \right\}$$

$$C_k = \{ F : F' = P_k \text{ almost everywhere} \}$$

By (b) and (c), $C_k$ is dense in $C(I)$ and since $C_k \subseteq A_k$, $A_k$ is dense.

Furthermore, for $F \in A_k$ and $\epsilon > 0$, set $r = \frac{1}{k} - \sup_{x \in I} \left| \frac{F(x+t)-F(x)}{t} - P_k(x) \right|$ and $\delta < \frac{\epsilon}{2}$. Then for any $G$ satisfying $|F - G| < \delta$ on $I$,

$$\left| \frac{G(x+t) - G(x) - P_k(x)}{t} \right| \leq \left| \frac{G(x+t) - G(x) - (F(x+t) - F(x))}{t} \right| + \left| \frac{F(x+t) - F(x)}{t} - P_k(x) \right|$$

$$< \frac{r}{2} \times 2 + \left| \frac{F(x+t) - F(x)}{t} - P_k(x) \right|$$

$$= \frac{1}{k}$$

This shows $A_k$ is open. Hence by the Baire Category theorem, $A = \bigcap_{k=1}^{\infty} A_k$ is dense. Now pick $F \in A$ and $\phi : I \to \mathbb{R}$ measurable.

As in (j), apply Weierstrass and Lusin’s theorem to choose $k_j$ so that $|\phi(x) - P_{k_j}(x)| < j^{-1}$ on $I \setminus B_j$ with $\lambda(B_j) < 2^{-j}$. When $j = 1$, as $F \in A_{k_1}$, we have $n_1 > k_1$ such that if $t_1 = h_{n_1}$, $\left| \frac{F(x+t_1)-F(x)}{t_1} - P_{k_1}(x) \right| < \frac{1}{k_1}$ on $I \setminus (E_{k_1} \cup B_1)$. Inductively, after we get $t_{j-1}$, as $F \in A_{k_j}$ and without loss of generality we assume $k_j \geq n_{j-1}$, we have $n_j > n_{j-1}$ such that if $t_j = h_{n_j}$, $\left| \frac{F(x+t_j)-F(x)}{t_j} - P_{k_j}(x) \right| < \frac{1}{k_j}$. Hence $\left| \frac{F(x+t_j)-F(x)}{t_j} - P_{k_j}(x) \right| < \frac{\epsilon}{2}$ on $I \setminus (E_{k_j} \cup B_j)$.

Finally, we can proceed as in step (j) and (m) to show that $F$ has the required properties.
2.3 Corollaries and Consequences

(1) One may wonder in the problem if there exists a continuous $F$ on $[a, b]$ such that for any real-valued measurable $\phi$,

$$\lim_{j \to \infty} \frac{F(x + h_n) - F(x)}{h_n} = \phi(x)$$

holds for not only almost everywhere on $[a, b]$, but for all $x \in [a, b]$. The answer is negative. This can be seen by considering the cardinality of the set $\mathbb{M}_{[a,b]}$ of measurable functions on $[a, b]$ and the set $S$ of subsequences of $\{h_n\}_{n=1}^\infty$.

Clearly $\text{card}(S) = \text{card}([\mathbb{R}])$. Now we want to show $\text{card}([\mathbb{M}_{[a,b]}]) > \text{card}([\mathbb{R}])$. Without loss of generality, we let $[a, b] = [0, 1]$, $K$ be the Cantor set, $S(K)$ be the set of all subsets of $K$ and $\chi$ be the characteristic function. Then

$$B = \{\chi_A : A \subseteq K\}$$

has the same cardinality as $S(K)$, and

$$\text{card}([\mathbb{M}_{[a,b]}]) \geq \text{card}(B) = \text{card}(S(K)) > \text{card}([\mathbb{R}]).$$

This shows there is no function which can map $S$ onto $\mathbb{M}_{[a,b]}$, and this answers the question.

Actually if we set $\tilde{\mathbb{M}}_{[a,b]}$ to be the collection of the all equivalent classes $[f]$, where

$$[f] = \{g \in \mathbb{M}_{[a,b]} : g = f \text{ almost everywhere on } [a, b]\},$$

we can see that $\text{card}(\tilde{\mathbb{M}}_{[a,b]}) \leq \text{card}([\mathbb{R}])$, as we have proved that there is $F$ which maps $S$ onto $\tilde{\mathbb{M}}_{[a,b]}$. The reverse inequality is obvious. Hence the cardinalities of $\tilde{\mathbb{M}}_{[a,b]}$ and $\mathbb{R}$ are the same. In other words, if we treat two measurable functions which agrees almost everywhere to be the same function, the amount of measurable functions is as much as the real number.

(2) For positive integer $k$, let $\{s_{k,j}\}$ be a subsequence of $\{h_n\}_{n=1}^\infty$ such that

$$\lim_{j \to \infty} \frac{F(x + s_{k,j}) - F(x)}{s_{k,j}} = k$$

for $x \notin I \setminus E_k$ with $\lambda(E_k) = 0$. Define $E = \bigcup_{k=1}^\infty (E_k \cup E_{-k})$. Then $\lambda(E) = 0$. Now for $x \in I \setminus E$, we have

$$\limsup_{h \to 0} \frac{F(x + h) - F(x)}{h} = +\infty \text{ and } \liminf_{h \to 0} \frac{F(x + h) - F(x)}{h} = -\infty.$$
Furthermore, there exists \( n_k \) such that
\[
\frac{F(x + s_{k,n_k}) - F(x)}{s_{k,n_k}} > k - 1 \text{ for all } x \notin I \setminus E.
\]

Then
\[
\lim_{k \to \infty} \frac{F(x + s_{k,n_k}) - F(x)}{s_{k,n_k}} = +\infty \text{ for all } x \notin I \setminus E.
\]

Similarly we can carry out the same process to have
\[
\lim_{k \to \infty} \frac{F(x + s_{k,n_k}) - F(x)}{s_{k,n_k}} = -\infty \text{ for all } x \notin I \setminus E.
\]

(3) Choose \( h_n = 1/n \) and any closed and bounded interval \( I \) and get the required \( F \) and the set \( E \) as in (2). Then \( F \) does not attain a local maximum or local minimum for every \( x \in I \setminus E \), hence almost \( x \in I \). If \( x \) is a local minimum, \( \frac{F(x + 1/n) - F(x)}{1/n} > 0 \) for large \( n \). This contradicts (2) as
\[
\lim_{j \to \infty} \frac{F(x + 1/n_j) - F(x)}{1/n_j} \geq 0.
\]

The case of local maximum is similar. From the same argument we can show that \( F \) attain its maximum and minimum at some point in \( I \setminus E \).

(4) Suppose \( I = [a, b] \) and \( \{h_n\}_{n=1}^\infty \) are given as in the problem and we get the required \( F \). Let \( G(x) = F(x) - L(x) \), where \( L \) is linear and \( L(x) = F(a) + \frac{x-a}{b-a} (F(b) - F(a)) \). Then \( G(a) = G(b) = 0 \). Extend \( G \) to a periodic function on the whole real line, with cycle on \( I \). Note that \( G \) is a Marcinkiewicz function on \( I \), as we pick \( \phi : I \to \mathbb{R} \) and let \( K = L'(x) \), we get \( \{h_n\}_{n=1}^\infty \) such that
\[
\lim_{j \to \infty} \frac{F(x + h_{n_j}) - F(x)}{h_{n_j}} = \phi(x) + K \text{ almost everywhere on } I.
\]

Hence
\[
\lim_{j \to \infty} \frac{G(x + h_{n_j}) - G(x)}{h_{n_j}} = \phi(x) \text{ almost everywhere on } I.
\]

From this we conclude that if \( \phi(x) \) is a periodic measurable function with cycle on \([a, b]\), then there exists \( \{h_{n_j}\}_{j=1}^\infty \) such that
\[
\lim_{j \to \infty} \frac{G(x + h_{n_j}) - G(x)}{h_{n_j}} = \phi(x) \text{ almost everywhere on } \mathbb{R}.
\]

Moreover, any measurable function with period \( b - a \) can be written as
\[
\lim_{j \to \infty} \frac{H(x + h_{n_j}) - H(x)}{h_{n_j}} \text{ where } H(x) = G(x - c) \text{ and } c \text{ is the phase difference.}
\]
(5) $F$ can be chosen to be nowhere differentiable on $I = [a,b]$. This is because the set of continuous nowhere differentiable functions on $[a,b]$ is also residual in $C^r(I)$ and the intersection of two residual sets in $C^r(I)$ is nonempty.

(6) We do not need $I$ to be a closed and bounded interval in the problem. Once $I$ is bounded measurable, we choose $[a,b]$ which contains $I$, and apply the result to $[a,b]$ to obtain the required $F$. Then for any measurable $\phi : I \to \mathbb{R}$, extend $I$ to $[a,b]$ by $\phi(x) = 0$ for $x \notin I$, we get
\[
\lim_{j \to \infty} \frac{F(x + h_{n_j}) - F(x)}{h_{n_j}} = \phi(x)
\]
almost everywhere on $[a,b]$ hence on $I$.

However it is not so obvious that we can have the same conclusion with the closed and bounded $I$ replaced by $\mathbb{R}$.

Define
\[
A_k = \left\{ F \in C(\mathbb{R}) : \left| \frac{F(x+t) - F(x)}{t} - P_k(x) \right| < \frac{1}{k}, t = h_n \text{ for some } n > k, x \in [-k,k] \setminus E_k, \text{ where } \lambda(E_k) < 2^{-k} \right\}
\]

For any continuous $G$ on $\mathbb{R}$, differentiable $Q$ on $[-k,k]$ and $\epsilon > 0$, we can get a continuous $F$ on $[-k,k]$ such that $F' = Q'$ almost everywhere on $[-k,k]$ and $|F - G| < \epsilon$ on $[-k,k]$.

Extend $F(x) = G(x) + F(k) - G(k)$ for $x \geq k$ and $F(x) = G(x) + F(-k) - G(-k)$ for $x \leq -k$. Hence $|F - G| < \epsilon$ on $\mathbb{R}$.

By the same argument as before, we can show $A_k$ is open and dense.

Pick $F \in \bigcap_{k=1}^{\infty} A_k$, and $\phi : \mathbb{R} \to \mathbb{R}$, a measurable function.

For each $j \in \mathbb{N}$, apply Lusin's theorem to get a continuous $g$ such that $g = \phi$ on $\mathbb{R} \setminus B_j$ with $\lambda(B_j) < 2^{-j}$. There exists $|P_{k_j}(x) - \phi(x)| / j$ on $[-j,j]$, where $k_j > k_{j-1}$, by Weierstrass theorem. Now as $F \in A_{k_j}$, we can get $n_j > k_j$ such that setting $t_j = h_{n_j}$ we have
\[
\left| \frac{F(x + t_j) - F(x)}{t_j} - \phi(x) \right| < \frac{2}{j}
\]
on $[-j,j] \setminus (E_{k_j} \cup B_j)$. Without loss of generality, we assume $k_j > n_{j-1}$ so that $n_j > n_{j-1}$.

Let $A = \bigcap_{r=1}^{\infty} \bigcup_{j=r}^{\infty} (E_{k_j} \cup B_j)$. Then $\lambda(A) = 0$.

Now pick $x_0 \in \mathbb{R} \setminus A$. Then $x_0 \in [-r, r]$ and $x_0 \notin \bigcup_{j=r}^{\infty} (E_{k_j} \cup B_j)$ for some $r \in \mathbb{N}$. As
\[
\left| \frac{F(x_0 + t_j) - F(x_0)}{t_j} - \phi(x_0) \right| < \frac{2}{j}
\]
holds when \( j \geq r \), we get

\[
\lim_{j \to \infty} \frac{F(x_0 + s_j) - F(x_0)}{s_j} = \phi(x_0).
\]

Thus replacing the closed and bounded interval \( I \) by any measurable set in \( \mathbb{R} \) has the same conclusion.

### 2.4 Second Difference Form of Marcinkiewicz’s Result

We can generalize the result to a second difference form: With the same setup, we have exactly the same conclusion with \( \lim_{j \to \infty} \frac{F(x + h_{n_j}) - F(x)}{h_{n_j}} \) replaced by \( \lim_{j \to \infty} \frac{F(x + h_{n_j}) - 2F(x) + F(x - h_{n_j})}{h_{n_j}^2} \).

To prove this, we need to modify the arguments used before. The details are as follows.

(a) Given \( G \in C(I) \) and \( \epsilon > 0 \). Let \( h \) be the Cantor function on \([0, 1]\). As \( G \) is uniformly continuous, we can divide \( I \) into \( k \) intervals \( I_1, I_2, \ldots, I_k \) such that \( |G(s) - G(t)| < \epsilon \) for \( s, t \in I_i \).

Let \([a, b] = I_i, c = G(b) - G(a), d = \int_0^1 h(t) dt, \) and

\[
H(a + (b - a)x) = \frac{c}{d} \int_0^x h(t) dt + G(a), \text{ where } 0 \leq x \leq 1.
\]

Then \( H(a) = G(a), H(b) = G(b), \) and the monotonicity of \( H \) shows \( |H - G| < \epsilon \) on \( I_i \). Define in the same way for \( H \) on each \( I_i \) then we get \( H \) satisfied \( H'' = 0 \) almost everywhere on \( I \).

(b) Given \( Q, G_1 \in C(I), Q \) is twice differentiable almost everywhere on \( I \) and \( \epsilon > 0 \). By (a), there exists \( H \) such that \( |H - (G_1 - Q)| < \epsilon \) on \( I \) and \( H'' = 0 \) almost everywhere. Define \( G_2 = H + Q \). Then \( G_2'' = Q'' \) almost everywhere and \( |G_2 - G_1| < \epsilon \) on \( I \).

(c) There is an enumeration \( \{P_k\}_{k=1}^\infty \) of the set of all polynomials having rational coefficients for which \( P_1 = 0 \).

(d) Set \( F_1 = 0, E_1 = \phi, n_1 = 1 \) and \( t_1 = h_1 \). By (b) and let \( \epsilon = \frac{|t_{k-1}|^2}{k-1} \), for each \( F_{k-1} \), we get \( F_k \) such that \( F_k' = P_k \) almost everywhere and \( |F_k - F_{k-1}| < \epsilon \) on \( I \).
(e) Define \( f_n(x) = \frac{F_k(x + h_n) - 2F_k(x) + F_k(x - h_n)}{h_n^2} \) on the interior of \( I \). Then \( f_n \) converges to \( P_k \) almost everywhere on \( I \). By Egoroff’s Theorem, \( f_n \) converges uniformly to \( P_k \) on \( I \setminus E_k \) for some \( E_k \) with measure less than \( 2^{-k} \). Hence we can choose \( t_k = h_n \) for \( n \) large enough so that \(|t_{k-1}|^2 > 2|t_k|^2\) and

\[
\frac{F_k(x + t_k) - 2F_k(x) + F_k(x - t_k)}{t_k^2} - P_k(x) < \frac{1}{k}
\]

for \( x \in I \setminus E_k \).

(f) This defines \( F_k, E_k, n_k \) and \( t_k \) for all \( k \in \mathbb{N} \).

(g) For every \( x \in I \), we have

\[
|F_{k+n} - F_k| \leq \sum_{i=k}^{k+n-1} |t_i|^2 \leq \frac{|t_k|^2}{k} \left( 1 + \frac{1}{2} + \cdots + \frac{1}{2^{n-1}} \right) < \frac{2|t_k|^2}{k}
\]

(h) \( \{F_k\}_{k=1}^{\infty} \) is now a Cauchy sequence hence has a limit \( F \) defined on \( I \).

Then \( |F - F_k| = \lim_{n \to \infty} |F_{k+n} - F_k| \leq \frac{2|t_k|^2}{k} \) on \( I \).

(i) By (h), for every \( x \in I \setminus B_k \), we get

\[
\left| \frac{F(x + t_k) - 2F(x) + F(x - t_k)}{t_k^2} - P_k(x) \right| \leq \frac{F(x + t_k) - 2F(x) + F(x - t_k) - (F_k(x + t_k) - 2F_k(x) + F_k(x - t_k))}{t_k^2} + \left| \frac{F_k(x + t_k) - 2F_k(x) + F_k(x - t_k)}{t_k^2} - P_k(x) \right|
\]

\[
\leq \frac{8t_k^2}{kt_k^2} + \frac{1}{k} = \frac{9}{k}
\]

Let \( \phi : I \to \mathbb{R} \) be a measurable function.

(j) By Lusin’s theorem, there exists a continuous \( g \) on \( I \) so that \( g = \phi \) on \( I \setminus B_j \) for some set \( B_j \) with measure less than \( 2^{-j} \). Then by Weierstrass approximation theorem, we have \( |P_{k_j} - g| < 1/j \) on \( I \). Thus we have \( |P_{k_j} - \phi| < 1/j \) on \( I \setminus B_j \).
(k) By (i) and (j), we have
\[
\left| \frac{F(x + s_j) - 2F(x) + F(x - s_j)}{s_j^2} - \phi(x) \right| < \frac{9}{k_j} + \frac{1}{j} < \frac{10}{j}
\]
for all \(j \in \mathbb{N}\) and \(x \in I \setminus (E_{k_j} \cup B_j)\).

(k) Set \(A = \bigcap_{r=1}^{\infty} \bigcup_{j=r}^{\infty} (E_{k_j} \cup B_j)\). As
\[
\lambda\left(\bigcup_{j=r}^{\infty} (E_{k_j} \cup B_j)\right) \leq \sum_{j=r}^{\infty} \lambda(E_{k_j} \cup B_j) < \sum_{j=r}^{\infty} 2^{-(j-1)} = 2^{-(r-2)},
\]
we see that \(\lambda(A) = 0\).

(m) By (k) and (l), for \(x \in I \setminus A\),
\[
\lim_{j \to \infty} \frac{F(x + s_j) - 2F(x) + F(x - s_j)}{s_j^2} = \phi(x)
\]
and the proof is completed.

Similarly, we can prove that the set of functions satisfying the properties above is residual in \(C(I)\) as before.

First define
\[
S_k = \left\{ F \in C(I) : \begin{array}{l}
\left| \frac{F(x+t) - 2F(x) + F(x-t)}{t^2} - P_k(x) \right| < \frac{1}{k}, t = h_n \text{ for some} \\
\ n > k, x \in I \setminus E_k \text{ where } \lambda(E_k) < 2^{-k}
\end{array} \right\}
\]

\(T_k = \{ F : F'' = P_k \text{ almost everywhere on } I \}\)

By (b) and (e), \(T_k\) is dense in \(C(I)\) and since \(T_k \subset S_k, S_k\) is dense.

Furthermore, for \(F \in S_k\) and \(\epsilon > 0\), set
\[
r = \frac{1}{k} - \sup_{x \in I} \left| \frac{F(x+t) - 2F(x) + F(x-t)}{t^2} - P_k(x) \right| < \frac{\pi^2}{4}. \quad \text{Then for any } G \text{ satisfying } |F - G| < \delta \text{ on } I,
\]
\[
\begin{align*}
&\left| \frac{G(x+t) - 2G(x) + G(x-t)}{t^2} - P_k(x) \right| \\
\leq &\left| \frac{G(x+t) - 2G(x) + G(x-t)}{t^2} \right| \\
&+ \left| \frac{F(x+t) - 2F(x) + F(x-t)}{t^2} - P_k(x) \right| \\
< &\frac{\pi^2}{4} \times 4 + \left| \frac{F(x+t) - 2F(x) + F(x-t)}{t^2} - P_k(x) \right|
\end{align*}
\]
\[
= \frac{1}{k}
\]

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This shows $S_k$ is open. Hence by the Baire Category theorem, $A = \bigcap_{k=1}^{\infty} S_k$ is dense. Now pick $F \in A$ and $\phi : I \to \mathbb{R}$ measurable.

As in (j), apply Weierstrass and Lusin’s theorem to choose $k_j$ so that $|\phi(x) - P_{k_j}(x)| < j^{-1}$ on $I \setminus B_j$ with $\lambda(B_j) < 2^{-j}$. When $j = 1$, as $F \in A_{k_1}$, we have $n_1 > k_1$ such that if $t_1 = h_{n_1}$, $\left| \frac{F(x+t_1) - 2F(x) + F(x-t_1)}{t_1^2} - P_{k_1}(x) \right| < \frac{1}{k_1}$ on $I \setminus (E_{k_1} \cup B_1)$. Inductively, after we get $t_{j-1}$, as $F \in A_{k_j}$, we have $n_j > k_j$ such that if $t_j = h_{n_j}$, $\left| \frac{F(x+t_j) - 2F(x) + F(x-t_j)}{t_j^2} - P_{k_j}(x) \right| < \frac{1}{k_j}$. Here without loss of generality, we can assume $k_j \geq n_{j-1}$ so that $n_j > n_{j-1}$. Hence

$$\left| \frac{F(x+t_j) - 2F(x) + F(x-t_j)}{t_j^2} - \phi(x) \right| < \frac{2}{j}$$

on $I \setminus (E_{k_j} \cup B_j)$.

Finally, we can proceed as in step (j) and (m) to show that $F$ has the required properties.

Similarly, as in (6) in the section Corollaries and Consequences, modifying the above argument we can prove the statement with $I$ replaced by $\mathbb{R}$.
Chapter 3

Marcinkiewicz’s Results on $\mathbb{R}^2$

3.1 Cantor Function on $[0,1] \times [0,1]$

To prove similar results in $\mathbb{R}^2$, we need a real-valued function $G$ on $D = [0,1] \times [0,1]$ similar to the Cantor function on $[0,1]$ which is continuous but its “derivative” is zero almost everywhere on $D$. Here we define “derivative” by

$$G'(x,y) = \lim_{(r,s)\to(0,0)} \frac{G(x+r,y+s) - G(x,y)}{\sqrt{r^2 + s^2}}.$$

Indeed, we have the following lemma.

Lemma Given $a, b, c, d \in \mathbb{R}$. There exists $G \in C(D)$ such that $G(0,0) = a, G(1,0) = b, G(0,1) = c, G(1,1) = d$ and $G' = 0$ almost everywhere on $D$.

Proof. Let $h$ be the Cantor function on $[0,1]$.

Define

\begin{align*}
G(0,y) &= a + (c - a)h(y) \\
G(1,y) &= b + (d - b)h(y) \\
G(x,0) &= a + (b - a)h(x) \\
G(x,1) &= c + (d - c)h(x) \\
\text{and } G(x,y) &= G(0,y) + [G(1,y) - G(0,y)]h(x) \quad \text{for } (x,y) \in D.
\end{align*}
Note that

$$G(x + r, y + s) - G(x, y)$$

$$= G(x + r, y + s) - G(x, r + s) + G(x, y + s) - G(x, y)$$

$$= \left\{ G(0, y + s) + [G(1, y + s) - G(0, y + s)]h(x + r)
- [G(0, y + s) + [G(1, y + s) - G(0, y + s)]h(x)] \right\}$$

$$+ \left\{ G(0, y + s) + [G(1, y + s) - G(0, y + s)]h(x)
- [G(0, y) + [G(1, y) - G(0, y)]h(x)] \right\}$$

As $h(x + r)$ converges to $h(x)$, $G(0, y + s)$ converges to $G(0, y)$ and $G(1, y + s)$ converges to $G(1, y)$ when $(r, s) \to (0, 0)$, we see that the two braces converge to 0, hence their sum. This shows the continuity of $G$.

Let $A = (K \cup [0, 1]) \times ([0, 1] \cup K)$, where $K$ is the Cantor set. Then $A$ is closed and $\mu(A) = \int_0^1 \int_0^1 \chi_A dxdy = 0$, where $\mu$ is the Lebesgue measure on $\mathbb{R}^2$ and $\chi$ is the characteristic function. Now it is ready to talk about the “derivative” of $G$.

For $(x, y) \in D \setminus A$,

$$\frac{G(x + r, y + s) - G(x, y)}{\sqrt{r^2 + s^2}}$$

$$= \frac{G(x + r, y + s) - G(x, y + s) + G(x, y + s) - G(x, y)}{\sqrt{r^2 + s^2}}$$

$$= \frac{[G(1, y + s) - G(0, y + s)]h(x + r) - h(x)}{r} \cdot \frac{r}{\sqrt{r^2 + s^2}}$$

$$+ \frac{G(0, y + s) - G(0, y)}{s} \cdot \frac{s}{\sqrt{r^2 + s^2}}$$

$$+ \frac{G(1, y + s) - G(1, y) - [G(0, y + s) - G(0, y)]}{s} \cdot \frac{sh(x)}{\sqrt{r^2 + s^2}}$$

As $h'(x) = 0$, $G_y(0, y) = 0$, $G_y(1, y) = 0$ outside $K$ and $\frac{s}{\sqrt{r^2 + s^2}} \leq 1$, the three parts converge to 0 when $(r, s) \to (0, 0)$. This shows $G_x' = 0$ on $D \setminus A$, hence almost everywhere on $D$. This completes the proof.

The function $G$ constructed satisfies the inequality

$$\min(a, b, c, d) \leq G(x, y) \leq \max(a, b, c, d).$$

The right side of the inequality can be seen by noting that $G(x, y) \leq \max(G(0, y), G(1, y))$, $G(0, y) \leq \max(a, c)$ and

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\( G(1, y) \leq \max(b, d) \). The left side of the inequality follows similarly. This inequality will be used in the very beginning of the proof of the results in \( \mathbb{R}^2 \). Furthermore, it is obvious to see that the partial derivatives of \( G, G_x \) and \( G_y \) is zero almost everywhere on \( D \).

**Remark.** By the above lemma we can construct a similar function on \([u, v] \times [s, t]\), by setting \( H(u + (v - u)x, s + (t - s)y) = G(x, y) \) where \( x \) and \( y \) vary from 0 to 1. Then \( H \) agrees with \( G \) at the vertices, and

\[
\frac{H(u + (v - u)x + r, s + (t - s)y + w) - H(u + (v - u)x, s + (t - s)y)}{\sqrt{r^2 + w^2}}
= \frac{G\left(x + \frac{r}{v-u}, y + \frac{w}{t-s}\right) - G\left(x, y + \frac{w}{t-s}\right)}{\sqrt{r^2 + w^2}} \cdot \frac{r}{v-u}
+ \frac{G\left(x, y + \frac{w}{t-s}\right) - G(x, y)}{\sqrt{r^2 + w^2}} \cdot \frac{w}{t-s}
\]

which converges to 0 as \((r, w)\) converges to \((0, 0)\) on \([u, v] \times [s, t]\) \( \setminus \{(u, s) + [(v - u)K \times [s, t]] \cup [u, v] \times [(t - s)K]\}\). This means \( H' = 0 \) almost everywhere on \([u, v] \times [s, t]\).

### 3.2 Various Forms of Marcinkiewicz’s Results

We can generalize the result to \( \mathbb{R}^2 \), with the help of the lemma in section 3.1, as follow.

**Theorem.** Given \( D = [a, b] \times [a, b] \subseteq \mathbb{R}^2 \), \( \{s_n\}_{n=1}^\infty \) and \( \{t_n\}_{n=1}^\infty \) which are nonzero and converging to 0. There exists a continuous \( F : D \to \mathbb{R} \) such that given \( \phi_1, \phi_2 : D \to \mathbb{R} \) measurable, there are subsequences \( \{s_{n_j}\}_{j=1}^\infty \) and \( \{t_{m_j}\}_{j=1}^\infty \) such that

\[
\lim_{j \to \infty} \frac{F(x + s_{n_j}, y) - F(x, y)}{s_{n_j}} = \phi_1(x, y)
\]

and

\[
\lim_{j \to \infty} \frac{F(x, y + t_{m_j}) - F(x, y)}{t_{m_j}} = \phi_2(x, y)
\]

almost everywhere on \( D \).
To prove this result, we first need the following.

**Claim.** Given \( D = [a, b] \times [a, b] \subseteq \mathbb{R}^2 \) and \( \{h_n\}_{n=1}^{\infty} \) which is nonzero and converging to 0. There exists a continuous \( F : D \to \mathbb{R} \) such that given \( \phi : D \to \mathbb{R} \) measurable, there are subsequences \( \{h_{n_j}\}_{j=1}^{\infty} \) such that

\[
\lim_{j \to \infty} \frac{F(x + h_{n_j}, y) - F(x, y)}{h_{n_j}} = \phi(x, y)
\]

almost everywhere on \( D \).

The argument is similar to previous proofs.

**Proof.**

(a) Given \( G \in C(D) \) and \( \epsilon > 0 \). Divide \( D \) into squares \( S_k \) with length small enough so that for \( (x_1, y_1), (x_2, y_2) \in S_k \), \( |G(x_1, y_1) - G(x_2, y_2)| < \epsilon/2 \).

By the lemma in section 3.1, we can define \( H \) on each square, where \( H \) is continuous on \( D \). Furthermore, for \( (x, y) \in D \), \( (x, y) \in S_k \) for some \( k \), hence

\[
|H(x, y) - G(x, y)| < |H(x, y) - H(a, a)| + |G(a, a) - G(x, y)| < \epsilon,
\]

by the choice of squares and the comment under the lemma in section 3.1. Note that \( H' = 0 \) almost everywhere on each square, thus almost everywhere on \( D \).

(b) Given \( Q, G_1 \in C(D), Q_x \) exists and \( \epsilon > 0 \). Let \( G = G_1 - Q \) in (a) to get \( H \), and \( G_2 = H + Q \). Then \( |G_2 - G_1| = |H - G| < \epsilon \) and \( (G_2)_x = Q_x \) almost everywhere on \( D \).

(c) Let \( \{P_k\}_{k=1}^{\infty} \) be the set of all polynomials of \((x, y)\) with \( P_1 = 0, F_1 = 0, E_1 = \phi \) and \( t_1 = h_1 \).

(d) By (b), there exists \( F_k \in C(D) \) such that \( (F_k)_x = P_k \) almost everywhere and \( |F_k - F_{k-1}| < \frac{t_{k-1}}{k-1} \).

(e) Apply Egoroff's theorem to

\[
f_n(x, y) = h_n^{-1}[F_k(x + h_n, y) - F_k(x, y)]
\]

to get \( E_k \subset D \) with measure less than \( 2^{-k} \) and \( t_k = h_{nk} \), where \( |t_k| < |t_{k-1}|/2 \), such that

\[
\left| \frac{F_k(x + t_k, y) - F_k(x, y)}{t_k} - P_k(x, y) \right| < \frac{1}{k}
\]

on \( D \setminus E_k \).
(f) This defines $F_k, E_k, n_k$ and $t_k$ for all $k \in \mathbb{N}$.

(g) For every $x \in D$, we have
\[
|F_{k+n} - F_k| \leq \sum_{i=k}^{k+n-1} |t_i| |F_{k+i-n} - F_{k+i-n-1}| + \cdots + |F_{k+n} - F_{k+n-1}| + |F_{k+n-1} - F_{k+n-2}| + \cdots + |F_{k+1} - F_k| \\
\leq \frac{|t_k|}{k} \left(1 + \frac{1}{2} + \cdots + \frac{1}{2^{n-1}}\right) \\
\leq \frac{2|t_k|}{k}
\]

(h) $\{F_k\}_{k=1}^{\infty}$ is now a Cauchy sequence hence has a limit $F$ defined on $D$.

Then $|F - F_k| = \lim_{n \to \infty} |F_{k+n} - F_k| \leq \frac{2|t_k|}{k}$ on $D$.

(i) By (h), for every $x \in D\setminus E_k$, we get
\[
\left|\frac{F(x + t_k, y) - F(x, y)}{t_k} - P_k(x, y)\right| \\
\leq \left|\frac{F(x + t_k, y) - F(x, y) - (F_k(x + t_k, y) - F_k(x, y))}{t_k}\right| \\
+ \left|\frac{F_k(x + t_k, y) - F_k(x, y)}{t_k} - P_k(x, y)\right| \\
\leq \frac{4|t_k|}{kt_k} + \frac{1}{k} = \frac{5}{k}
\]

Let $\phi : D \to \mathbb{R}$ be a measurable function.

(j) By Lusin’s theorem, there exists a continuous $g$ on $D$ so that $g = \phi$ on $D\setminus B_j$ for some set $B_j$ with measure less than $2^{-j}$. Then by Weierstrass approximation theorem, we have $|P_{k_j} - g| < 1/j$ on $D$. Thus we have $|P_{k_j} - \phi| < 1/j$ on $D\setminus B_j$.

(k) By (i) and (j), writing $s_j = t_{k_j}$, we have
\[
\left|\frac{F(x + s_j, y) - F(x, y)}{s_j} - \phi(x, y)\right| < \frac{6}{j}
\]
for $(x, y) \in D\setminus (E_{k_j} \cup B_j)$.
(l) Set \( A = \bigcap_{r=1}^{\infty} \bigcup_{j=r}^{\infty} (E_{k_j} \cup B_j) \). As

\[
\mu \left( \bigcup_{j=r}^{\infty} (E_{k_j} \cup B_j) \right) \leq \sum_{j=r}^{\infty} \mu(E_{k_j} \cup B_j) < \sum_{j=r}^{\infty} 2^{-(j-1)} = 2^{-(r-2)},
\]

we see that \( \mu(A) = 0 \).

(m) By (k) and (l), for \( x \in D \setminus A \),

\[
\lim_{j \to \infty} \frac{F(x + s_j, y) - F(x, y)}{s_j} = \phi(x, y)
\]

and the proof is completed.

Following exactly the same argument we can prove the same result with

\[
\lim_{j \to \infty} \frac{F(x + h_{n_j}, y) - F(x, y)}{h_{n_j}} = \phi(x, y)
\]

replaced by

\[
\lim_{j \to \infty} \frac{F(x, y + h_{n_j}) - F(x, y)}{h_{n_j}} = \phi(x, y).
\]

Next we are going to prove the theorem at the very beginning.

Proof of theorem.

Define

\[
R_k = \left\{ F \in C(D) : \left| \sum_{s_n} \frac{F(x+s_n,y) - F(x,y)}{s_n} - P_k(x,y) \right| < \frac{1}{k} \text{ on some } D \setminus E \text{ with } \mu(E) < 2^{-k}, \text{ for some } n \geq k. \right\}
\]

\[
S_k = \left\{ F \in C(D) : \left| \sum_{t_n} \frac{F(x,y+t_n) - F(x,y)}{t_n} - P_k(x,y) \right| < \frac{1}{k} \text{ on some } D \setminus E \text{ with } \mu(E) < 2^{-k}, \text{ for some } n \geq k. \right\}
\]

By (b), the set of functions satisfying \( F_x = P_k \) and \( F_y = P_k \) almost everywhere is dense in \( C(D) \). Together with (e), we have \( R_k \) and \( S_k \) also dense. To see \( R_k \) is open, pick \( F \in R_k \).

Let \( r = \frac{1}{k} - \sup_{(x,y) \in D} \left| \sum_{s_n} \frac{F(x+s_n,y) - F(x,y)}{s_n} - P_k(x,y) \right| \). Choose \( \epsilon = \frac{rn}{2} \). If
\[ |G - F| < \epsilon \text{ on } D, \]

\[ \leq \frac{|G(x + s_n, y) - G(x, y)|}{s_n} \left| \frac{P_k(x, y) - P_k(x, y)}{s_n} \right| + \frac{|G(x + s_n, y) - F(x + s_n, y) + F(x, y) - G(x, y)|}{s_n} \]

\[ < \frac{1}{k}. \]

This shows \( R_k \) is open. Similarly \( S_k \) is open. Hence \( R_k \cap S_k \) is open and dense. Applying the Baire Category Theorem we get \( A = \bigcap_{k=1}^{\infty} (R_k \cap S_k) \) is dense hence nonempty. Now pick \( F \in A \) and \( \phi_1, \phi_2 : D \to \mathbb{R} \) be measurable functions.

As in (j), apply Weierstrass and Lusin’s theorem to choose \( k_j \) so that

\[ |\phi_1(x, y) - P_{k_j}(x, y)| < 2^{-j} \text{ on } D \setminus B_j \text{ with } \mu(B_j) < 2^{-j}. \]

When \( j = 1 \), as \( F \in R_{k_1} \), we have \( n_1 > k_1 \) such that if \( t_1 = s_{n_1}, \)

\[ \left| \frac{F(x+t_1, y) - F(x, y)}{t_1} - P_{k_1}(x, y) \right| < \frac{1}{k_1} \text{ on } D \setminus (E_{k_1} \cup B_1). \]

Inductively, after we get \( t_{j-1} \), as \( F \in A_{k_j} \), we have \( n_j > k_j \) such that if \( t_j = s_{n_j}, \)

\[ \left| \frac{F(x+t_j, y) - F(x, y)}{t_j} - P_{k_j}(x, y) \right| < \frac{1}{k_j}. \]

Here without loss of generality, we can assume \( k_j \geq n_{j-1} \) so that \( n_j > n_{j-1} \). Hence

\[ \left| \frac{F(x+t_j, y) - F(x, y)}{t_j} - \phi(x, y) \right| < \frac{2}{j} \text{ on } D \setminus (E_{k_j} \cup B_j). \]

Finally, we can proceed as in step (j) and (m) to show that \( F \) satisfies

\[ \lim_{j \to \infty} \frac{F(x + s_{n_j}, y) - F(x, y)}{s_{n_j}} = \phi_1(x, y) \]

almost everywhere on \( D \).

Similarly, we can show

\[ \lim_{j \to \infty} \frac{F(x, y + t_{m_j}) - F(x, y)}{t_{m_j}} = \phi_2(x, y) \]

almost everywhere on \( D \). This completes the proof of the theorem.

Let \( D_k = [-k, k] \times [-k, k] \). If \( F \in C(D_k), G \in C(\mathbb{R}^2) \) and

\[ |F(x, y) - G(x, y)| < \epsilon \text{ on } D_k, \]

we can extend \( F \) to be continuous on \( \mathbb{R}^2 \) and

\[ |F - G| < \epsilon \text{ on } \mathbb{R}^2. \]

To do this, we can simply apply Tietze extension theorem.
or construct directly as follow. Let \( r(y) = F(k, y) - G(k, y) \) for \( y \in [-k, k] \). Note that \(|r(y)| < \epsilon\) and is continuous. Define

\[
F(x, y) = G(x, y) + r(y) \quad \text{on} \quad [k, \infty) \times [-k, k]
\]

and on \([k, \infty) \times [k, \infty)\) we define

\[
F(x, y) = G(x, y) + r(k)
\]

We define similarly \(F\) in the remaining regions. It is not difficult to see \(F\) is the desired one.

With help of this, we can see that

\[
R_k = \left\{ F \in C(\mathbb{R}^2) : \left| \frac{F(x + s_n, y) - F(x, y)}{s_n} - P_k(x, y) \right| < \frac{1}{k} \text{ on some set } D \setminus E \text{ with } \mu(E) < 2^{-k}, \text{ for some } n \geq k. \right\}
\]

is dense. Imitate the argument in (6) of section 2.3 we can have the same conclusion with \(D = [a, b] \times [a, b]\) replaced by \(\mathbb{R}^2\).

In above if \(s_n = t_n\), given \(\phi_1\) and \(\phi_2\) measurable on \(D\), can we have one \(s_{n_j}\) works for them? It seems we cannot ensure this. But we can do so if we change the statement a bit.

**Theorem.** Given \(D = [a, b] \times [a, b] \subseteq \mathbb{R}^2\) and \(\{h_n\}_{n=1}^\infty\) be any sequence of nonzero numbers converging to 0. There exists a continuous \(F : D \to \mathbb{R}\) such that for \(\phi_1, \phi_2 : I = [a, b] \to \mathbb{R}\) are any measurable functions, there is a subsequence \(\{h_{n_j}\}_{j=1}^\infty\) such that

\[
\lim_{j \to \infty} \frac{F(x + h_{n_j}, y) - F(x, y)}{h_{n_j}} = \phi_1(x)
\]

and

\[
\lim_{j \to \infty} \frac{F(x, y + h_{n_j}) - F(x, y)}{h_{n_j}} = \phi_2(y)
\]

for almost every \((x, y) \in D\).

Note that one subsequence works for two measurable functions, therefore this result cannot follow directly from the previous one.

**Proof.**
(a) Given \( G \in C(D) \) and \( \epsilon > 0 \). Divide \( D \) into squares \( S_k \) with length small enough so that for \( (x_1, y_1), (x_2, y_2) \in S_k \), \( |G(x_1, y_1) - G(x_2, y_2)| < \epsilon/2 \).
By lemma 3.1, we can define \( H \) on each square, where \( H \) is continuous on \( D \). Furthermore, for \( (x, y) \in D \), \( (x, y) \in S_k \) for some \( k \), hence

\[
|H(x, y) - G(x, y)| < |H(x, y) - H(a, a)| + |G(a, a) - G(x, y)| < \epsilon,
\]

by the choice of squares and the comment under the lemma 3.1. Note that \( H' = 0 \) almost everywhere on each square, thus almost everywhere on \( D \).

(b) Given \( Q, G_1 \in C(D) \), \( Q_x, Q_y \) exists and \( \epsilon > 0 \). Let \( G = G_1 - Q \) in (a) to get \( H \), and \( G_2 = H + Q \). Then \( |G_2 - G_1| = |H - G| < \epsilon \) and \( (G_2)_x = Q_x, (G_2)_y = Q_y \) almost everywhere on \( D \). In particular, for polynomial \( P_1(x), P_2(y) \), letting \( Q(x, y) = \int P_1(x)dx + \int P_2(y)dy \), we have \( (G_2)_x(x, y) = Q_x(x, y) = P_1(x) \), and \( (G_2)_y(x, y) = Q_y(x, y) = P_2(y) \) almost everywhere on \( D \).

(c) The collection of the functions in the form \( P_1(x) + P_2(y) \), where \( P_1(x), P_2(y) \) are polynomials with rational coefficient, is countable. Denote this collection by \( \{Q_k\}_{k=1}^\infty \) and let \( Q_k(x, y) = p_k(x) + q_k(y) \). Also let \( F_1 = 0, E_1 = \phi, n_1 = 1 \) and \( t_1 = h_1 \).

(d) By (b), there exists \( F_k \in C(D) \) such that \( (F_k)_x = p_k \) and \( (F_k)_y = q_k \) almost everywhere and \( |F_k - F_{k-1}| < \frac{t_{k-1}}{k-1} \).

(e) Use Egoroff’s theorem on \( f_n(x, y) = h_n^{-1}[F_k(x + h_n, y) - F_k(x, y)] \) and \( g_n(x, y) = h_n^{-1}[F_k(x, y + h_n) - F_k(x, y)] \) to get \( E \) and \( E' \) with \( \mu(E) < 1/2^{k+1} \) and \( \mu(E') < 1/2^{k+1} \), so that \( f_n(x, y) \) converges to \( p_k(x) \) on \( D \setminus E \) and \( g_n(x, y) \) converges to \( q_k(y) \) on \( D \setminus E' \) on \( D \setminus E' \) uniformly. From this we get \( t_k = h_{n_k} < t_{k-1}/2, E_k = E \cup E' (\mu(E_k) < 1/2^k) \) such that

\[
\left| \frac{F_k(x + t_k, y) - F_k(x, y)}{t_k} - p_k(x) \right| < \frac{1}{k}
\]
and

\[
\left| \frac{F_k(x, y + t_k) - F_k(x, y)}{t_k} - q_k(y) \right| < \frac{1}{k}
\]
on \( D \setminus E_k \).

(f) This defines \( F_k, E_k, n_k \) and \( t_k \) for all \( k \in \mathbb{N} \).
(g) For $n, k \in \mathbb{N}$, we have
\[
|F_{k+n} - F_k| \\
\leq |F_{k+n} - F_{k+n-1}| + |F_{k+n-1} - F_{k+n-2}| + \cdots + |F_{k+1} - F_k| \\
< \sum_{i=k}^{k+n-1} \frac{|t_i|}{i} \\
< \frac{|t_k|}{k} \left(1 + \frac{1}{2} + \cdots + \frac{1}{2^{n-1}}\right) \\
< \frac{2|t_k|}{k}
\]

(h) \{F_k\}_{k=1}^{\infty} is now a Cauchy sequence hence has a limit $F$ defined on $D$. Then $|F - F_k| = \lim_{n \to \infty} |F_{k+n} - F_k| \leq \frac{2|t_k|}{k}$ on $D$.

(i) By (e) and (h), for every $x \in D \setminus E_k$, we get
\[
\left|\frac{F(x + t_k, y) - F(x, y)}{t_k} - p_k(x)\right| \\
< \left|\frac{F(x + t_k, y) - F(x, y) - (F_k(x + t_k, y) - F_k(x, y))}{t_k}\right| \\
+ \left|\frac{F_k(x + t_k, y) - F_k(x, y)}{t_k} - p_k(x)\right| \\
< \frac{4t_k}{kt_k} + \frac{1}{k} = \frac{5}{k}
\]

Similarly,
\[
\left|\frac{F(x, y + t_k) - F(x, y)}{t_k} - q_k(y)\right| < \frac{5}{k} \text{ on } D \setminus E_k.
\]

Let $\phi_1, \phi_2 : [a, b] \to \mathbb{R}$ be measurable functions.

(j) For each $j \in \mathbb{N}$, choose polynomials $p(x)$ and $q(x)$ with rational coefficients such that $Q_{k_j}(x, y) = p(x) + q(y)$, where $k_j > k_{j-1}$ and
\[
|\phi_1(x) - p(x)| < j^{-1} \text{ on } D \setminus A_j \text{ with } \mu(A_j) < 1/2^{k+1} \\
|\phi_2(y) - q(y)| < j^{-1} \text{ on } D \setminus A'_j \text{ with } \mu(A'_j) < 1/2^{k+1}.
\]
Let $B_j = A_j \cup A'_j$. Then $\mu(B_j) < 2^{-j}$.
(k) Writing $s_j = t_{kj}$, we have for each $j$,

$$\left| \frac{F(x + s_j, y) - F(x, y)}{s_j} - \phi_1(x) \right| < \left| \frac{F(x + s_j, y) - F(x, y)}{s_j} - p(x) \right| + |\phi_1(x) - p(x)|$$

$$< \frac{5}{k_j} + \frac{1}{j} \leq \frac{6}{j} \quad \text{on } D \setminus (E_{kj} \cup B_j).$$

(l) Set $A = \bigcap_{r=1}^{\infty} \bigcup_{j=r}^{\infty} (E_{kj} \cup B_j)$. As

$$\mu(\bigcup_{j=r}^{\infty} (E_{kj} \cup B_j)) \leq \sum_{j=r}^{\infty} \mu(E_{kj} \cup B_j) < \sum_{j=r}^{\infty} 2^{-(j-1)} = 2^{-(r-2)},$$

we see that $\mu(A) = 0$.

(m) For $(x, y) \in D \setminus A$,

$$\lim_{j \to \infty} \frac{F(x + s_j, y) - F(x, y)}{s_j} = \phi_1(x),$$

$$\lim_{j \to \infty} \frac{F(x, y + s_j) - F(x, y)}{s_j} = \phi_2(y).$$

This finishes the proof of our assertion.

This result seems to be related to the one variable result. Let $F$ be a function above and $\phi_r(x) = r$ for $r \in \mathbb{R}$. To imply the one variable result, there must be a fixed $y_0 \in [a, b]$ such that $F(x, y_0)$ is an one variable Marcinkiewicz function. For each $r$, define

$$V_r = \left\{ y \in [a, b] : \lim_{j \to \infty} \frac{F(x + s_j, r, y) - F(x, y)}{s_{j,r}} = \phi_r(x) \text{ for some subsequence } s_{j,r} \text{ and almost every } x \in [a, b] \right\}.$$

Clearly, $\lambda(V_r) = b - a$. However $\bigcap_{r \in \mathbb{R}} V_r$ may be empty, since it is an uncountable intersection. Hence $y_0$ may not exist.

The main point is that we cannot make the original problem to be a special case of this result is the uncontrollable zero measure set $A$. As $A$ depends on $\phi$, we cannot even ensure that there exists a point in $[a, b] \times [a, b]$ such that the limit expressions hold on this point for all $\phi$ with different subsequences. So can we set some controls on the set $A$? The following is a result in this direction.
Theorem. Given $D = [a, b] \times [c, d], \{h_n\}_{n=1}^\infty$ which is nonzero and converging to 0, $R = \{r_n\}_{n=1}^\infty \subset [c, d]$ and $S = \{s_n\}_{n=1}^\infty \subset [a, b]$ are two countable sets. Then there exists continuous $F$ on $D$ such that given any measurable functions $\{\phi_{r_n}\}_{n=1}^\infty : [a, b] \to \mathbb{R}$ and $\{\phi_{s_n}\}_{n=1}^\infty : [c, d] \to \mathbb{R}$, there is a subsequence $\{h_{n_j}\}_{j=1}^\infty$ such that

$$\lim_{j \to \infty} \frac{F(x + h_{n_j}, r_i) - F(x, r_i)}{h_{n_j}} = \phi_{r_i}(x)$$

for $r_i \in R$, almost every $x \in [a, b]$, and

$$\lim_{j \to \infty} \frac{F(s_i, y + h_{n_j}) - F(s_i, y)}{h_{n_j}} = \phi_{s_i}(y)$$

for $s_i \in S$, almost every $y \in [c, d]$.

To prove this we show the following first.

Lemma 1 Given $D = [0, 1] \times [0, 1]$. Once a function $G$ is defined on the boundary of $D$, which is continuous along boundary, we can extend $G$ to be continuous on the entire $D$.

Proof For $(x, y) \in (0, 1) \times (0, 1)$ and $y - x = c \geq 0$, there is $t \in (0, 1)$ such that $(x, y) = (0, c) + t(1 - c, 1 - c)$. Define

$$G(x, y) = G(0, c) + t[G(c, 1) - G(0, c)].$$

For $(x, y) \in (0, 1) \times (0, 1)$ and $y - x = c \leq 0$, there is $t \in (0, 1)$ such that $(x, y) = (-c, 0) + t(1 + c, 1 + c)$. Define

$$G(x, y) = G(-c, 0) + t[G(1, -c) - G(-c, 0)].$$

This defines $G$ on the whole $D$. For $y - x = c > 0$, write $(x, y) = (0, c) + t(1 - c, 1 - c)$, and choose $r$ and $s$ small enough so that $(x + r, y + s) = (0, c + s - r) + t_0(1 - c - s + r, 1 - c - s + r)$. Then a little computation leads to

$$t_0 = \frac{(1 - c)t + r}{1 - c - s + r}.$$

Therefore, $t_0$ converges to $t$ as $(r, s)$ converges to $(0, 0)$. Since $G(c + s + r, 1)$ converges to $G(c, 1)$ and $G(0, c + r + s)$ converges to $G(0, c)$, we conclude that $G(x + r, y + s)$ converges to $G(x, y)$. Similarly we have the same conclusion for $y - x = c < 0$. As for $y = x$, $G$ can be defined in both way above, we show $G(x, y)$ is continuous on this line, hence on the entire $D$. 

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Remark: Note that this $G$ satisfies

$$\min_{(x,y)\in \partial D} G(x,y) \leq G(x,y) \leq \max_{(x,y)\in \partial D} G(x,y).$$

Also if $D = [a,b] \times [c,d]$ and $G$ is defined continuously on $\partial D$. Then $H(s,t) = G(a + (b - a)s, c + (d - c)t)$ is defined continuously on the boundary of $[0,1] \times [0,1]$. Apply the above to extend $H$ to be continuous on $[0,1] \times [0,1]$. Correspondingly, $G$ is extended to be continuous on $D$.

Lemma 2 Given $S$ and $R$ two finite sets on $[a,b]$ and $[c,d]$ respectively, $G, H \in C(D)$ and $\epsilon > 0$. There exists $F$ such that $|F - H| < \epsilon$ on $D$, and $F_x = G$ for $y \in R$ and almost every $x \in [a,b]$, $F_y = G$ for $x \in S$ and almost every $y \in [c,d]$.

Proof Divide $D$ into small rectangles $I_k$ such that for $(x_1, y_1), (x_2, y_2) \in I_k$, $|H(x_1, y_1) - H(x_2, y_2)| < \frac{\epsilon}{16}$.

Define $F_1(x, y) = \frac{H(x, y)}{2}$ for those $y$ which is the vertices of $I_k$ and not in $R$.

For $r_1 \in R$, define $F_1(x, r_1) = f_{r_1}(x)$ where $f_{r_1}'(x) = G(x, r_1)$ for almost every $x \in [a,b]$ and $|f_{r_1}(x) - \frac{H(x, r_1)}{2}| < \frac{\epsilon}{32}$.

Let $T$ be the union of $R$ and the $y$-coordinates of $I_k$. Denote $T = \{q_1, q_2, ..., q_n\}$ with $q_{i+1} > g_i$. For each $x \in U$, the union of $S$ and $x$-coordinates of $I_k$, define $F_1(x, y)$ be the Cantor function with endpoints $F_1(x, q_i)$ and $F_1(x, q_{i+1})$.

Now $F_1$ is defined on the boundary of rectangles with vertices on $U \times T$, hence by lemma 1, we can extend $F_1$ to be continuous on $D$ with $|F_1 - \frac{H}{2}| < \frac{\epsilon}{2}$. To see this, pick $(x, y) \in D$ and hence this $(x, y)$ is on one of those rectangles. Let $(x', y')$ be a point on the boundary of that rectangle. Now as $|F_1 - \frac{H}{2}| < \frac{\epsilon}{32}$ on the boundary, by the remark under the lemma 1, $|F_1(x, y) - F_1(x', y')| < \frac{\epsilon}{16}$. Also since $|\frac{H(x, y)}{2} - \frac{H(x', y')}{2}| < \frac{\epsilon}{16}, |F_1(x, y) - \frac{H(x, y)}{2}| < \frac{\epsilon}{2}$.

Moreover, this $F_1$ satisfies $(F_1)_x = G$ for $y \in R$ and almost every $x \in [a,b]$, $(F_1)_y = 0$ for $x \in S$ and almost every $y \in [c,d]$. Here $(F_1)_x$ and $(F_1)_y$ denote the partial derivatives of $F_1$ along $x$ and $y$ direction.

Define $F_2$ in the same so that $|F_2 - \frac{H}{2}| < \frac{\epsilon}{2}$ on $D$, $(F_2)_x = 0$ for $y \in R$ and almost every $x \in [a,b]$, $(F_2)_y = G$ for $x \in S$ and almost every $y \in [c,d]$.

Finally $F = F_1 + F_2$ satisfies the required property.

Lemma 3 Given a finite real set $R = \{r_1, r_2, ..., r_n\} \subset \mathbb{R}$ and $S = \{s_1, s_2, ..., s_n\} \subset [a,b]$, $\epsilon > 0$ and a measurable $\phi : [a, b] \to \mathbb{R}$. There exists a continuous function $g$ and a measurable set $B$ with measure less than $\epsilon$ such that $g = \phi$ on $\mathbb{R}\setminus B$ and $g(s_i) = r_i$ for $i = 1, 2, ..., n$. 27
Proof: Apply Lusin’s theorem to get a continuous \( f \) such that \( f = \phi \) on \([a, b] \setminus A\) with measure less than \( \epsilon/2 \). Choose \( n \) open intervals with each interval containing only one \( s_i \), and total length less than \( \epsilon/2 \). Define \( g = f \) on the complement of the intervals, \( g(s_i) = r_i \), and otherwise linear. This \( g \) is the desired one.

With these three lemmas in hand, we are ready to prove the theorem above.

Proof of theorem. First denote \( \{P_k\}_{k=1}^{\infty} \) be the collection of polynomial in \( x \) and \( y \), with \( P_1 = 0 \). Define \( F_1 = 0 \), \( E_{1,1} = \phi = W_{1,1} \), \( R_k = \{r_1, r_2, \ldots, r_k\} \subset R \) and \( S_k = \{s_1, s_2, \ldots, s_k\} \subset S \), \( t_1 = h_1 \).

By lemma 2, there exists \( F_k \in C(D) \) such that \( (F_k)_x = P_k \) for \( y \in R_n \) and for almost every \( x \in [a, b] \), \( (F_k)_y = P_k \) for \( x \in S_n \) and for almost every \( y \in [c, d] \), and \( |F_k - F_{k-1}| < \frac{|t_{k-1}|}{2} \) on \( D \).

As \( R_k \) and \( S_k \) are finite, by Egoroff’s theorem, there exists \( t_k = h_{n_k} \) and \( |t_k| < |t_{k-1}|/2 \) such that for \( r_i \in R_k \),

\[
\frac{F_k(x + t_k, r_i) - F_k(x, r_i)}{t_k} - P_k(x, r_i) < \frac{1}{k}
\]
on \( [a, b] \setminus E_{k,i} \) with \( \lambda(E_{k,i}) < 2^{-k} \), \( i = 1, 2, ..., k \), and for \( s_i \in S_k \),

\[
\frac{F_k(s_i, y + t_k) - F_k(s_i, y)}{t_k} - P_k(s_i, y) < \frac{1}{k}
\]
on \( [c, d] \setminus W_{k,i} \) with \( \lambda(W_{k,i}) < 2^{-k} \), \( i = 1, 2, ..., k \).

This defines \( F_k, E_{k,i}, W_{k,i}, t_k \) for \( k \in \mathbb{N} \). Now as before we have \( |F_{k+n} - F_k| < 2|t_k|/k \) on \( D \), and hence there exists \( F \in C(D) \) such that \( |F - F_k| \leq 2|t_k|/k \) on \( D \). This gives for \( r_i \in R_k \),

\[
\frac{F(x + t_k, r_i) - F(x, r_i)}{t_k} - P_k(x, r_i) < \frac{5}{k}
\]
on \( [a, b] \setminus E_{k,i} \) with \( \lambda(E_{k,i}) < 2^{-k} \), \( i = 1, 2, ..., k \), and for \( s_i \in S_k \),

\[
\frac{F(s_i, y + t_k) - F(s_i, y)}{t_k} - P_k(s_i, y) < \frac{5}{k}
\]
on \( [c, d] \setminus W_{k,i} \) with \( \lambda(W_{k,i}) < 2^{-k} \), \( i = 1, 2, ..., k \).

Now pick measurable functions \( \{\phi_{r_n}\}_{n=1}^{\infty} : [a, b] \rightarrow \mathbb{R} \) and \( \{\phi_{s_n}\}_{n=1}^{\infty} : [c, d] \rightarrow \mathbb{R} \).

For a fixed \( j \in \mathbb{N} \), apply Lusin’s theorem to get continuous \( g_{r_i} \) such that \( g_{r_i} = \phi_{r_i} \) on \([a, b] \setminus U_{j,i}\) with \( \lambda(U_{j,i}) < 2^{-j} \), for \( i = 1, 2, ..., j \). Define
\( G_j(x, r_i) = g_{r_i}(x) \) for \( i = 1, 2, \ldots, j \). Apply lemma 3 to get \( g_{r_i} = \phi_{r_i} \) on 
\([c, d]\setminus V_{j,i} \) for \( i \leq j \) with \( g_{r_i}(r_i) = G_j(s_i, r_i) \) and \( \lambda(V_{j,i}) < 2^{-j} \). Define 
\( G_j(s_i, y) = g_{r_i}(y) \). Now apply lemma 1 to extend \( G_j \) to be continuous on \( D \).

By Weierstrass’s theorem, there exists \( P_{k_j} \) such that \( |P_{k_j} - G_j| < 1/j \) on \( D \). Hence

\[
\left| \frac{F(x + t_{k_j}, r_i) - F(x, r_i)}{t_{k_j}} - \phi_{r_i}(x) \right| < \frac{6}{j}
\]

for almost every \( x \in [a, b]\setminus(E_{k_j,i} \cup U_{j,i}) \) and \( i = 1, 2, \ldots, j \), and

\[
\left| \frac{F(s_i, y + t_{k_j}) - F(s_i, y)}{t_{k_j}} - \phi_{s_i}(y) \right| < \frac{6}{j}
\]

for almost every \( y \in [c, d]\setminus(W_{k_j,i} \cup V_{j,i}) \) and \( i = 1, 2, \ldots, j \).

Now fix \( i \in \mathbb{N} \), define \( A_i = \bigcap_{r=1}^{\infty} \bigcup_{j=r}^{\infty} (E_{k_j,i} \cup U_{j,i}) \). As before, \( \lambda(A_i) = 0 \). Finally for \( x \in [a, b]\setminus A_i \),

\[
\lim_{j \to \infty} \frac{F(x + t_{k_j}, r_i) - F(x, r_i)}{t_{k_j}} = \phi_{r_i}(x).
\]

Similarly we conclude that there exists \( B_i \) with \( \lambda(B_i) = 0 \) such that for \( y \in [c, d]\setminus B_i \),

\[
\lim_{j \to \infty} \frac{F(s_i, y + t_{k_j}) - F(s_i, y)}{t_{k_j}} = \phi_{s_i}(y).
\]

This holds for all \( i \in \mathbb{N} \) and completes the proof.

Obviously, this result implies the original problem immediately. The original problem does not talk more about the subsequence \( \{h_{n_j}\}_{j=1}^{\infty} \), namely: For a function \( H \) satisfying the properties in the problem and a given measurable \( \mu \) on \( [a, b] \), there is a corresponding subsequence \( \{h_{n_j}\}_{j=1}^{\infty} \). But is there any other subsequence, which does not include infinitely many terms of \( \{h_{n_j}\}_{j=1}^{\infty} \), and serves for the same \( \Phi \)? The answer is \textit{affirmative} by the above result. To see this, set \( \phi_{r_1} = \Phi, \phi_{r_2} = C \) and \( \phi_{r_i} = 0 \) for \( i \geq 3 \). Varying \( C \in \mathbb{R} \) gives subsequence \( t_{C,j} \) such that

\[
\lim_{j \to \infty} \frac{F(x + t_{C,j}, r_1) - F(x, r_1)}{t_{C,j}} = \Phi(x) \quad \text{for almost every } x \in [a, b],
\]

where

\[
\lim_{j \to \infty} \frac{F(x + t_{C,j}, r_2) - F(x, r_2)}{t_{C,j}} = C \quad \text{for almost every } x \in [a, b].
\]

Obviously if \( C \neq C' \), and \( n \) is large enough, \( t_{C,j} \neq t_{C',k} \) for \( k, j \geq n \). Let \( G(x) = F(x, r_1) \). Then \( G \) is a function which satisfies the property of the
original problem. The above reasoning shows for any fixed measurable $\Phi$, on 
$[a,b]$, there are uncountably many subsequences $t_{C,j}$ such that

$$\lim_{j \to \infty} \frac{G(x + t_{C,j}) - G(x)}{t_{C,j}} = \Phi(x) \text{ for almost every } x \in [a,b].$$

Note that functions in the collection $L = \{ F(x, r_i) \}_{i=1}^{\infty}$ have different graphs.
To be precise, pick $f, g \in L$, there does not exists $C \in \mathbb{R}$ such that $f = g + C$.
Otherwise,

$$\lim_{j \to \infty} \frac{f(x + h_{n_j}) - f(x)}{h_{n_j}} = \lim_{j \to \infty} \frac{g(x + h_{n_j}) - g(x)}{h_{n_j}},$$

a contradiction to the result. The same conclusion holds for the collection

$\{ F(s_i, y) \}_{i=1}^{\infty}$.

Writing $(\phi_{r_1}, \phi_{r_2}, ..., \phi_{r_i}, ...)$ and letting $\phi_{r_i}(x) = b_i \in \mathbb{R}$ and $b_i$ to be varying
in the real line, we see that for any element in $\mathbb{R}^{\infty}$ (here $\infty$ is the countably
infinite), there is a subsequence $h_{n_j}$ mapped by $F$ to this element. Hence the
cardinality of $\mathbb{R}^{\infty}$ is the same as that of $\mathbb{R}$.

A question may be raised right after the result of the original problem: Let $F$
be the function in the problem. After we get a measurable $\phi$, we get the
corresponding $\{h_{n_j}\}_{j=1}^{\infty}$. But is there any function with “other shapes”
satisfying the problem and gives $\phi$ by putting the same subsequence? The
answer is affirmative. This can be seen easily by setting $\phi_{r_i} = \phi$ for $i \in \mathbb{N}$ in
the above. Then there are countably infinitely many functions with other
shapes which maps the same sequence to $\phi$. 

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Bibliography
