Euler Characteristic Structure
and
Weight Homology

by

CHENG, Wing Kin

A Thesis Submitted to
The Hong Kong University of Science and Technology
in Partial Fulfillment of the Requirements for
the Degree of Master of Philosophy
in Mathematics

August 2004, Hong Kong

Copyright © by CHENG, Wing Kin 2004
Authorization

I hereby declare that I am the sole author of the thesis.

I authorize the Hong Kong University of Science and Technology to lend this thesis to other institutions or individuals for the purpose of scholarly research.

I further authorize the Hong Kong University of Science and Technology to reproduce the thesis by photocopying or by other means, in total or in part, at the request of other institutions or individuals for the purpose of scholarly research.

CHENG, Wing Kin
Euler Characteristic Structure
and
Weight Homology
by
CHENG, Wing Kin

This is to certify that I have examined the above MPhil thesis
and have found that it is complete and satisfactory in all respects,
and that any and all revisions required by
the thesis examination committee have been made.

[Signature]

Doctor YAN, Min
Supervisor

[Signature]

Doctor WANG, Xiao Ping
Acting Head of Department

Department of Mathematics
August 20, 2004
Acknowledgements

Glory to the heavenly Father who is the greatest invariant, which is a main theme in topology, in His created world. I am grateful for His mercy and grace on me that brings me through all difficulties, not to mention His fruitful supply at all times.

I am indebted to Dr. Yan, my supervisor, for his guidance and patience. His invaluable suggestions inspire me to greater efforts and sharpen my knowledge of topology as well as my ideas on how to present the subject. My thanks also go to the Department for offering me a position and assistance, both in teaching and learning.

I wish to thank my friends, especially some choir members in the University, for bringing me colour in the PG life. It is also my delight to acknowledge the help of Kelvin Yu in study as well as in faith.

Last but not least, I show my deep gratitude to my family and church for their love and tolerance.
Contents

Authorization Page ii
Signature Page iii
Acknowledgements iv
Table of Contents v
Abstract vii

1 Introduction 1
  1.1 Partially ordered set ........................................... 1
  1.2 Euler characteristic structure and weight homology .......... 4
  1.3 Main results .................................................. 5

2 Motivation 7
  2.1 Eulerian manifold ............................................... 7
  2.2 Eulerian stratification .......................................... 8
  2.3 Dehn-Sommerville equations for Eulerian stratified polyhedron . 12

3 Euler characteristic structure 14
3.1 Dimension function ..................................... 14
3.2 Relative Euler characteristic .............................. 15

4 Weight homology ............................................... 18
  4.1 Torsion .......................................................... 18
  4.2 Long exact sequence .......................................... 19
  4.3 Duality ............................................................ 20

5 Computation ...................................................... 22
  5.1 Examples ......................................................... 22
  5.2 Long exact sequence revisited .............................. 31

Bibliography ....................................................... 34
Euler Characteristic Structure
and
Weight Homology

by
CHENG, Wing Kin

Department of Mathematics
The Hong Kong University of Science and Technology

Abstract

In [2], Chen and Yan introduced the notion of Eulerian stratified spaces and found a correspondence between such spaces and partially ordered sets under some conditions. They further found that these conditions are highly related to Dehn-Sommerville equations after defining a boundary of weight on the poset. I pick some of those conditions and the boundary operator to define Euler characteristic structure equipped on and weight homology of the poset respectively. In this thesis, I study properties of the Euler characteristic structure as well as weight homology in detail.

In the Euler characteristic structure, I find all constraints of choosing $d$ and $\chi$ in the structure. These constraints even restrict the attention of certain posets. In the weight homology, I find several properties such as torsion, long exact sequence and duality. Moreover, several examples and an inductive algorithm in calculating the homology are included. Furthermore, after knowing the homology is a direct sum of $\mathbb{Z}_2$, I find the necessary and sufficient condition for the homology taking the maximal number of $\mathbb{Z}_2$. 
Chapter 1

Introduction

1.1 Partially ordered set

Definition A partially ordered set $P$ (or poset, for short) is a set together with a binary relation denoted $\leq$ (or $\leq_P$ when there is a possibility of confusion), satisfying the following three axioms:

1. For all $x \in P$, $x \leq x$. (reflexivity)

2. If $x \leq y$ and $y \leq x$, then $x = y$. (antisymmetry)

3. If $x \leq y$ and $y \leq z$, then $x \leq z$. (transitivity)

Strict relation $x < y$ means $x \leq y$ and $x \neq y$; while $x \geq y$(respectively $>$) means $y \leq x$(respectively $>$). Two elements $x$ and $y$ of $P$ are said to be comparable if $x \leq y$ or $y \leq x$; otherwise $x$ and $y$ are incomparable. For $x, y \in P$, $y$ covers $x$ if and only if $x < y$ and no element $z \in P$ satisfies $x < z < y$.

Definition Given a poset $P$, a subset $Q$ of $P$ is called a subposet of $P$ if $x \leq_Q y \iff x \leq_P y$ for any $x, y \in Q$. 
**Definition**  A chain is a poset in which any two elements are comparable. A subset $C$ of a poset $P$ is also called a chain if $C$ is a chain when regarded as a subposet of $P$.

**Definition**  An interval $[x, y] = \{z \in P : x \leq z \leq y\}$ is a subposet of $P$, defined whenever $x \leq y$. If every interval of $P$ is finite, then $P$ is called a locally finite poset.

**Definition**  Given a poset $P$, a dual poset $P^*$ of $P$ is the same set but reversing the order.

**Example 1.1.1.**

1. Let $n \in \mathbb{N}$. The set $\{1, 2, \cdots, n\}$ with its usual order forms an $n$-element chain.

2. Let $n \in \mathbb{N}$. Consider all subsets of $\{1, 2, \cdots, n\}$ and define $S \leq T$ if $S \subseteq T$ as sets. Denote this poset $B_n$ with the order by inclusion.

3. Let $n \in \mathbb{N}$. The set of all positive integral divisors of $n$ can be made into a poset $D_n$ by defining $i \leq j$ in $D_n$ if $j$ is divisible by $i$.

**Definition**  An integral function $d : P \to \mathbb{Z}$ is said to be a dimension function of $P$ if $d(x) < d(y)$ for any $x < y$.

**Remark**  $d$ can be regarded as generalization of rank function of poset.

**Definition**  Let $P$ be a locally finite poset, and let $Int(P)$ denote the set of intervals of $P$. The incidence algebra $I(P, \mathbb{Z})$ of $P$ over $\mathbb{Z}$ is the $\mathbb{Z}$-algebra of all functions $f : Int(P) \to \mathbb{Z}$ (with the usual structure of a vector space over $\mathbb{Z}$), Here multiplication is defined by

$$f * g(x, y) = \sum_{x \leq z \leq y} f(x, z)g(z, y).$$
**Remark**  The above sum is finite (and hence \( f \ast g \) is defined) since \( P \) is locally finite. Moreover \( I(P, \mathbb{Z}) \) is an associative \( \mathbb{Z} \)-algebra with (two-sided) identity, denoted by \( \delta \), defined by

\[
\delta(x, y) = \begin{cases} 
1, & \text{if } x = y \\
0, & \text{if } x \neq y.
\end{cases}
\]

**Definition**  An integral function \( \omega : P \to \mathbb{Z} \) is called a weight function of \( P \). There is no a priori restriction on the assignment of weight at any point. The set of \( \omega \) is denoted by \( \mathbb{Z}^P \).

**Definition**  For any \( f \in I(P, \mathbb{Z}) \) and \( \omega \in \mathbb{Z}^P \), \( f \circ \omega \) is defined as

\[
f \circ \omega(a) = \sum_{a \leq b} f(a, b) \omega(b).
\]

**Remark**  \( \mathbb{Z}^P \) is an \( I(P, \mathbb{Z}) \)-module, meaning that the followings are satisfied:

\[
(f_1 + f_2) \circ \omega = f_1 \circ \omega + f_2 \circ \omega,
\]

\[
f \circ (\omega_1 + \omega_2) = f \circ \omega_1 + f \circ \omega_2,
\]

\[
f_1 \circ (f_2 \circ \omega) = (f_1 \ast f_2) \circ \omega,
\]

\[
\delta \circ \omega = \omega.
\]
1.2 Euler characteristic structure and weight homology

**Definition** Let $P$ be a finite partially ordered set, $d : P \to \mathbb{N}$ be a dimension function and $\chi(a, b)$ be a collection of integers for $a \leq b \in P$ such that

$$\chi(a, b) \neq 0 \quad \text{whenever } b \text{ covers } a,$$

$$\chi \ast \chi = 2\chi$$

(1.1)

and

$$\chi(a, a) = 1 - (-1)^{d(a)} \quad \text{for any } a \in P.$$  

(1.3)

Call $(d, \chi)$ an Euler characteristic structure on $P$.

**Remark** The geometric meaning of the three conditions will be explained in Section 2.2 and Section 2.3.

**Definition** Given a finite poset $P$ possessing Euler characteristic structure $(d, \chi)$, the $n^{th}$ boundary weight $\partial_n \omega$ of a weight function $\omega$ is defined as

$$\partial_n \omega = [1 - (-1)^n] \omega + (-1)^n \chi \circ \omega.$$

**Remark** The motivation for defining the $n^{th}$ boundary weight operator $\partial_n$ this way will be explained in Section 2.3.

**Lemma 1.2.1.** Given an Euler characteristic structure $(d, \chi)$ on a poset $P$, $\partial_{n-1} \partial_n = 0$. 

4
\[ \partial_{n-1} \partial_n \omega = (1 - (-1)^{n-1}) \partial_n \omega + (-1)^{n-1} \chi \circ \partial_n \omega \]
\[ = (1 - (-1)^{n-1}) [(1 - (-1)^n) \omega + (-1)^n \chi \circ \omega] \]
\[ + (-1)^{n-1} \chi \circ [(1 - (-1)^n) \omega + (-1)^n \chi \circ \omega] \]
\[ = 2\chi \circ \omega - \chi \circ (\chi \circ \omega) \]
\[ = 2\chi \circ \omega - (\chi \ast \chi) \circ \omega \]
\[ = 2\chi \circ \omega - (2\chi) \circ \omega \]
\[ = 0. \]

The fourth equality holds because \( \mathbb{Z}^P \) is an \( I(P, \mathbb{Z}) \)-module while the fifth one follows from (1.2).

\[ \square \]

**Remark**  The proof actually shows that \( \partial_{n-1} \partial_n = 0 \) is equivalent to (1.2).

The above lemma leads to the following concept.

**Definition**  Given an Euler characteristic structure \((d, \chi)\) on a poset \( P \), the \textit{n}th \textit{weight homology} of \( P \) is

\[ H_n(P) = \frac{\ker \partial_n}{\text{im} \partial_{n+1}}. \]

**Remark**  The boundary operator \( \partial_n \) depends on the parity of \( n \) only. So does the weight homology \( H_n(P) \).

### 1.3 Main results

In this thesis, Chapter 2 is devoted to explaining how the Euler characteristic structure comes from geometry. In Chapter 3, I will first study poset \( P \) possessing Euler characteristic structure. It will rule out certain kind of posets.
Besides, the Euler characteristic structure restricts certain choices of \( d \) and \( \chi \). In Chapter 4 and 5, I study properties of the weight homology \( H_n(P) \), specifically,

1. \( H_n(P) \) is a direct sum of \( \mathbb{Z}_2 \). Furthermore, it has the maximal number of \( \mathbb{Z}_2 \) if and only if all \( \chi \)'s are even.

2. For \( P = Q \cup R \) such that there is no element \( q \in Q \) and \( r \in R \) satisfying \( q > r \), there is a long exact sequence relating \( H(P) \), \( H(Q) \) and \( H(R) \), namely,

\[
\cdots \rightarrow H_n(Q) \rightarrow H_n(P) \rightarrow H_n(R) \rightarrow H_{n-1}(Q) \rightarrow \cdots
\]

The long exact sequence also leads to an inductive algorithm counting the number of \( \mathbb{Z}_2 \) in the homologies.

3. If \( P \) possesses Euler characteristic structure \( (d, \chi) \), then the dual poset \( P^* \) also possesses another Euler characteristic structure \( (d^*, \chi^*) \) so that \( H_n(P^*) = H_n(P) \).
Chapter 2

Motivation

2.1 Eulerian manifold

**Definition** Let X be an n-dimensional compact polyhedron. For any triangulation Δ of X, define the f-vector $f(X, Δ) = (f_0, f_1, \ldots, f_n)$, where $f_i$ is the number of i-simplices of Δ.

**Definition** A neighborhood of a point $x$ inside a compact polyhedron X is homeomorphic to a cone $xL$ with vertex $x$ and base $L$. $L$ is itself a compact polyhedron unique up to PL-homeomorphism. $L$ is called the link of $x$ in X and is denoted by lk($x$, $X$).

**Definition** A locally compact polyhedron $M$ is called an n-dimensional PL-manifold with a closed subpolyhedron $\partial M$ as boundary if lk($x$, $M$) is PL-homeomorphic to the sphere $S^{n-1}$ for $x \in M - \partial M$ and is PL-homeomorphic to the disk $D^{n-1}$ for $x \in \partial M$.

**Definition** A locally compact polyhedron $M$ is called an n-dimensional
Eulerian manifold with a closed subpolyhedron \( \partial M \) as boundary if

\[
\chi(\text{lk}(x, M)) = \begin{cases} 
\chi(S^{n-1}) = 1 - (-1)^n & \text{for } x \in M - \partial M \\
\chi(D^{n-1}) = 1 & \text{for } x \in \partial M.
\end{cases}
\]

It is well known for a long time (see [6]) that for any triangulation \( \Delta \) of an \( n \)-dimensional compact Eulerian manifold without boundary \( M \), the \( f \)-vector satisfies Dehn-Sommerville equations,

\[
(1 - (-1)^{n-i}) f_i(M, \Delta) + \sum_{j>i} (-1)^{n-j-1} \binom{j+1}{i+1} f_j(M, \Delta) = 0, \quad 0 \leq i < n.
\]

Denote \( D(n) \) as the coefficient matrix in the above system. Then the system can be rewritten as \( D(n)f(M, \Delta) = 0 \).

In [1, 4], Chen and Yan generalized classical Dehn-Sommerville equations to

\[
D(n)f(M, \Delta) = f(\partial M, \partial \Delta)
\]  

(2.1)

for compact Eulerian manifolds with boundary \( M \) with triangulation \( \Delta \). In addition, based on \( \partial(\partial M) = \phi \) and \( \chi(\partial M) = (1 - (-1)^n)\chi(M) \), they further proved the following properties of \( D(n) \),

\[
D(n-1)D(n) = 0,
\]

(2.2)

\[
\chi D(n) = (1 - (-1)^n)\chi,
\]

(2.3)

where \( \chi \) is the Euler function on vectors: \( \chi(a_0, a_1, \cdots, a_n) = \sum (-1)^i a_i \).

### 2.2 Eulerian stratification

**Definition** A stratification of a compact polyhedron \( X \) indexed by a finite poset \( P \) is a decomposition \( X = \cup_{a \in P} X_a \) into union of compact subpolyhedra
such that for each \( a \in P \),

\[
X_a \cap X_b = \bigcup_{c \leq a, c \leq b} X_c.
\] (2.4)

\( X \) and \( X_a \) are called stratified polyhedron and closed stratum respectively.

The (pure) stratum is defined as

\[
X_a = X_a - \bigcup_{b < a} X_b.
\]

Then the condition (2.4) can be rephrased as

\[
X_a = \bigcup_{b \leq a} X_b.
\] (2.5)

Denote \( \chi(a) = \chi(X_a) \) and \( \overline{\chi}(a) = \chi(X_a) \). Then (2.5) implies that \( \chi \) and \( \overline{\chi} \) are related by Möbius inversion,

\[
\overline{\chi}(a) = \sum_{b \leq a} \chi(b), \quad \chi(a) = \sum_{b \leq a} \overline{\chi}(b) \mu(b, a),
\] (2.6)

where \( \mu \) is the Möbius function, defined on the pairs of \( P \) and characterized by

\[
\sum_{a \leq c \leq b} \mu(c, b) = \begin{cases} 
1 & \text{for } a = b \\
0 & \text{for } a < b.
\end{cases}
\]

In addition,

\[
\chi(X) = \sum_{a \in P} \chi(a).
\]

For each \( x \in X_a \), the link \( \mathrm{lk}(x, X) \) is a stratified polyhedron with strata \( \mathrm{lk}(x, X_b) \), indexed by \( b \) such that \( a \leq b \). Moreover, for fixed \( a < b \), \( \mathrm{lk}(x, X_b) \) is a stratified polyhedron with strata \( \mathrm{lk}(x, X_c) \), indexed by \( c \) such that \( a \leq c \leq b \).

**Definition** A stratified polyhedron \( X \) is called Eulerian if \( \chi(\mathrm{lk}(x, X_b)) \) is independent of the choice of \( x \in X_a \). In this case, \( \chi(a, b) = \chi(\mathrm{lk}(x, X_b)) \) is called the relative Euler characteristic of \( X_a \) in \( X_b \).
Observe that the notion of Eulerian stratification is purely topological. In fact, for fixed \( x \in X_a \), the system \( \{ \overline{\chi}(a, b) = \chi(\text{lk}(x, X_b)) : a \leq b \} \) and the system \( \{ \chi(a, b) : a \leq b \} \) are also related by Möbius inversion in a way similar to (2.6),

\[
\overline{\chi}(a, b) = \sum_{a \leq c \leq b} \chi(a, c), \quad \chi(a, b) = \sum_{a \leq c \leq b} \overline{\chi}(a, c) \mu(c, b).
\]

Therefore one system is independent of the choice of \( x \in X_a \) if and only if the other system is. Because the first system can be described in terms of the relative homology,

\[
\chi(\text{lk}(x, X_b)) = 1 - \sum (-1)^i \dim H_i(X_b, X_b - x),
\]

Eulerian stratification is a topological property.

Denote \( d(a) = \dim X_a \). For a point \( x \) in the interior of a \( d(a) \)-dimensional simplex of a triangulation of \( X_a \), \( \text{lk}(x, X_a) = S^{d(a)-1} \). Thus from the definition of Eulerian stratified polyhedron,

\[
\chi(\text{lk}(a, X_a)) = \chi(a, a) = \chi(\text{lk}(x, X_a)) = \chi(S^{d(a)-1}) = 1 - (-1)^{d(a)}
\]

for any \( y \in X_a \). This simply means that \( X_a \) is an Eulerian manifold without boundary. Hence Eulerian stratified polyhedra are obtained by gluing pieces of Eulerian manifolds together in "Eulerian fashion".

**Example 2.2.1.** Eulerian stratified polyhedron with 1-stratum is Eulerian manifold without boundary.

**Example 2.2.2.** Eulerian manifold with boundary \( M^n \) is an Eulerian 2-strata space with the boundary as the lower stratum. Moreover, \( \partial M \) is an Eulerian manifold without boundary and

\[
\chi(\text{lk}(x, M - \partial M)) = \chi(\text{lk}(x, M)) - \chi(\text{lk}(x, \partial M))
\]

\[
= 1 - [1 - (-1)^{n-1}]
\]

\[
= (-1)^{n-1}.
\]
Example 2.2.3. If $M^n$ is a boundaryless $PL$-manifold with $\chi(M) = 1 + (-1)^n$, then the cone $cM$ is an $(n+1)^{th}$-dimensional Eulerian manifold with boundary $M$. On the other hand, if $\chi(M) \neq 1 + (-1)^n$, then $cM$ is an Eulerian stratified polyhedron with 3 strata in the following way. The 3 strata is represented by a poset $P = \{0, 1, 2\}$ with the relation $0 < 2$ and $1 < 2$. The vertex $c$ of $cM$ is a lower stratum and is represented by 0 while the base $M$ of $cM$ is another lower stratum represented by 1. The upper stratum $cM - c - M$ is represented by 2. The relative Euler characteristics $\chi(0, 2)$ and $\chi(1, 2)$ are $\chi(M)$ and $(-1)^{n-1}$ respectively.

Example 2.2.4. If $X$ is Eulerian stratified polyhedron indexed by $P$ and $M$ is boundaryless Eulerian manifold, then $X \times M$ is an Eulerian stratified polyhedron indexed by $P$.

Remark Here is the reason for the requirement that $\chi(a, b) \neq 0$ in definition of Euler characteristic structure. In two strata case, $\chi(a, b) = 0$ implies that $X$ is an Eulerian manifold without boundary. Therefore $X$ is not stratified in natural topological sense (see [4]). In general, $\chi(a, b) = 0$ means that the Eulerian singular part of $X_b$ is empty. So $X_b$ should not be stratified further (see [3] and Example 2.2.3). In conclusion, the requirement of non-zero $\chi(a, b)$, when $b$ covers $a$, avoids topologically unnatural stratification.

Given an Eulerian stratified polyhedron, one has a corresponding poset, namely, the set of strata ordered by inclusion. Moreover, other information such as dimension of stratum and relative Euler characteristic can be passed to the Euler characteristic structure directly.

Conversely, given a poset possessing Euler characteristic structure in addition to a collection of integers $\chi(a)$ for $a \in P$, one can also construct an Eulerian stratified polyhedron in the light of the following realization theorem (see [2]).
Theorem 2.2.5. Suppose $P$ is a partially ordered set, and $d : P \to \mathbb{N}$ is a function such that $d(b) - 2 \geq d(a) \geq 1$ for $a < b$. Suppose $\chi(a)$ is a collection of integers for $a \in P$, and $\chi(a, b)$ is another collection of integers for $a \leq b$ in $P$. Then there exists an Eulerian stratified polyhedron $X$ indexed by $P$ with dimension function $d(a)$, Euler characteristic function $\chi(a)$, and relative Euler characteristic function $\chi(a, b)$ if and only if

$$
\chi(a, a) = 1 - (-1)^{d(a)},
$$

$$
\sum_{a \leq c \leq b} \chi(a, c)\chi(c, b) = 2\chi(a, b), \quad \text{for any fixed } a \leq b,
$$

$$
\sum_{a \leq b} \chi(a)\chi(a, b) = 0, \quad \text{for any fixed } b,
$$

(2.7)

are satisfied.

The first two equations are satisfied in the Euler characteristic structure. Extra condition is required in the theorem, namely, equation (2.7) and dimension gap. However, this can be easily accomplished.

Moreover, all of the three equations bear geometric meaning. The first one is explained in this section. The remaining two are consequences of Theorem 2.3.1 which is a generalization of Dehn-Sommerville equations to weighted $f$-vectors.

### 2.3 Dehn-Sommerville equations for Eulerian stratified polyhedron

**Definition** A triangulation $\Delta$ of a stratified polyhedron $X$ is called a *stratified triangulation* if each stratum $X_a$ is a union of the interior of some simplices in $\Delta$. 

12
Denote by $\triangle_a$ the collection of the simplices whose interiors are contained in $X_a$. Then the $a^{th}$ $f$-vector $f(X_a, \triangle_a)$ is given by

$$f_i(X_a, \triangle_a) = \text{number of } i\text{-dimensional simplices in } \triangle_a.$$

**Definition** Let $X$ be a stratified polyhedron indexed by $P$ and $\omega$ be a weight on $P$. For a stratified triangulation $\triangle$ of $X$, the $\omega$-weighted $f$-vector is

$$f(X, \triangle, \omega) = \sum f(X_a, \triangle_a) \omega(a).$$

The $\omega$-weighted Euler characteristic is

$$\chi(X, \omega) = \chi(f(X, \triangle, \omega)) = \sum \chi(a) \omega(a).$$

The dimension of the weight is

$$d(\omega) = \max_{\omega(a) \neq 0} d(a).$$

**Theorem 2.3.1.** Let $\omega$ be a weight on an $n$-dimensional Eulerian stratified polyhedron $X$. Then for any stratified triangulation $\triangle$,

$$D(n) f(X, \triangle, \omega) = f(X, \triangle, \partial_n \omega).$$

**Remark** The correspondence between $D(n)$ and $\partial_n$ gives a translation from two universal equations about $D(n)$, namely, (2.2) and (2.3), to the last two equations in theorem 2.2.5 (see [2]).

1. $\chi D(n) = (1 - (-1)^n) \chi$ is transliterated into $\sum_{a \leq b} \chi(a) \chi(a, b) = 0$, for any fixed $b$, which is exactly (2.7).

2. $D(n - 1)D(n) = 0$ is transliterated into $\chi \ast \chi = 2\chi$. 

13
Chapter 3

Euler characteristic structure

Before studying the weight homology, one should make sure that the object he studies is a right candidate for calculation. More precisely, one would ask whether any poset with the collection of $d$ and $\chi$ arbitrarily assigned possesses Euler characteristic structure.

Readers are reminded that $\chi(a, b) \neq 0$ whenever $b$ covers $a$ in $P$.

3.1 Dimension function

Proposition 3.1.1. If $b$ covers $a$ in $P$, then $d(a)$ and $d(b)$ have different parity.

Proof

$$2\chi(a, b) \overset{(1.2)}{=} \chi(a, a)\chi(a, b) + \chi(a, b)\chi(b, b)$$

$$\overset{(1.3)}{=} [2 - (-1)^{d(a)} - (-1)^{d(b)}]\chi(a, b).$$

$$0 = [(-1)^{d(a)} + (-1)^{d(b)}]\chi(a, b).$$

Since $\chi(a, b) \neq 0$, it follows that $(-1)^{d(a)} + (-1)^{d(b)} = 0$. \hfill \Box
Remark The condition of non-zero $\chi$ rejects some posets; for instance, $P = \{1, 2, 3, 4, 5\}$ with $1 < 2 < 3 < 5$ and $1 < 4 < 5$.

### 3.2 Relative Euler characteristic

Let $P = \{1 < 2 < \cdots < n\}$ be a finite chain possessing Euler characteristic structure $(d, \chi)$. Then

$$2\chi(i, i + k) = \chi \ast \chi(i, i + k)$$

$$= \chi(i, i)\chi(i, i + k) + \chi(i, i + k)\chi(i + k, i + k) + \sum_{j = i + 1}^{i + k - 1} \chi(i, j)\chi(j, i + k)$$

$$= [2 - (-1)^{d(i)} - (-1)^{d(i + k)}]\chi(i, i + k) + \sum_{j = i + 1}^{i + k - 1} \chi(i, j)\chi(j, i + k).$$

Combining $\chi(i, i + k)$ together, one gets

$$\sum_{j = i + 1}^{i + k - 1} \chi(i, j)\chi(j, i + k) = \begin{cases} (-1)^{d(i)}2\chi(i, i + k) & \text{if } k \text{ is even} \\ 0 & \text{if } k \text{ is odd}. \end{cases} \quad \text{(eq k)}$$

By repeatedly substituting (eq even k) to the right side of (eq even k) whenever one encounters $\chi(a, b)$ with $b - a$ is even and $b - a < k$, one eventually gets

$$\chi(i, i + k) = \sum_{i + k \text{ is odd}} c_{a_0 \cdots a_{2n}} \chi(a_0, a_1)\chi(a_1, a_2) \cdots \chi(a_{2n - 1}, a_{2n}), \quad \text{(eq even k)}$$

where $c_{a_0 \cdots a_{2n}}$ is a constant discussed in the following lemma.

**Lemma 3.2.1.**

$$c_{a_0 \cdots a_{2n}} = \begin{cases} \frac{(-1)^{d(a_0)}}{2} & n = 1 \\ (-1)^{d(a_0) \frac{1 \cdot 3 \cdot 5 \cdots (2n - 3)}{2^{n-1}} n^2} & n \geq 2. \end{cases}$$
Proof In order to simplify notations in the proof, replace $c_{a_0 \cdots a_{2n}}$ by $\frac{(-1)^{d(a_n)}}{2} d_n$.

Then

$$d_n = \frac{1}{4} [d_1 d_{n-1} + d_2 d_{n-2} + \cdots + d_{n-1} d_1].$$

Let $f(x) = \sum_{n=1}^{\infty} d_n x^n$. Then

$$[f(x)]^2 = \sum_{n=2}^{\infty} 4d_n x^n = 4f(x) - 4d_1 x$$

and hence

$$f(x) = \frac{4 \pm \sqrt{16 - 16d_1 x}}{2} = 2 \pm 2\sqrt{1 - d_1 x}.$$

However, $d_1 = f'(0)$ from the power series takes $f(x)$ to be $2 - 2\sqrt{1 - d_1 x}$. By generalized Binomial Theorem,

$$f(x) = -2 \sum_{n=1}^{\infty} \left(\frac{1}{2}\right) \left(\frac{1}{n}\right) (-d_1)^n x^n$$

and for $n \geq 2$,

$$d_n = -2 \cdot \frac{\frac{1}{2}(-\frac{1}{2})(-\frac{3}{2}) \cdots (-\frac{2n-3}{2})}{n!} \cdot (-d_1)^n = \frac{1 \cdot 3 \cdot 5 \cdots (2n - 3)}{2^{n-1} \cdot n!} (d_1)^n.$$

From (eq even $k$)*, it is easy to see that $d_1 = 1$. Hence the proof is complete.

By repeatedly substituting (eq even $k$) to the left side of (eq odd $k$) in the similar way, I claim that all the terms cancel out.

When $j - i$ is even,

$$\chi(i, j) \chi(j, i + k) = \sum_{i=a_0 < a_1 < a_2 < \cdots < a_{2n} = j; \quad a_r - a_{r-1} \text{ is odd}}^{a_0 = a_2 \cdots = a_{2n-1} \text{ is odd}} c_{a_0 \cdots a_{2n}} \chi(a_0, a_1) \chi(a_1, a_2) \cdots \chi(a_{2n-1}, a_{2n}) \chi(a_{2n}, i + k).$$
When \( j - i \) is odd, then \( i + k - j \) is even, so that

\[
\chi(i, j)\chi(j, i + k) = \sum_{j=b_0<b_1<\cdots<b_{2m}=i+k: b_2-b_{2m-1} \text{ is odd}} c_{b_0\cdots b_{2m}} \chi(b_0, b_1)\chi(b_1, b_2) \cdots \chi(b_{2m-1}, b_{2m}).
\]

Consider the left side of (eq odd \( k \)). Each term can be broken into the sum of building blocks by the either one of the previous two equations. Pick one building block, say, \( \chi(i, i + 2k_0 + 1)\chi(i + 2k_0 + 1, i + 2k_1) \cdots \chi(i + 2k_m, i + k) \). By the previous two equations, this building block must come from two terms only, namely, \( \chi(i, i + 2k_0 + 1)\chi(i + 2k_0 + 1, i + k) \) and \( \chi(i, i + 2k_m)\chi(i + 2k_m, i + k) \). Moreover, their coefficients must be of opposite pair by Lemma 3.2.1. Therefore the sum is 0. The following proposition summarizes the discussion.

**Proposition 3.2.2.** An Euler characteristic structure on \( P = \{1 < 2 < \cdots < n\} \) is in \( 1 - 1 \) correspondence with the collections \( \{\chi(a, b)\} \) such that

1. \( \chi(i, i + 2k + 1) \) is free such that right side of (eq \( 2m \))\(^*\), \( 2m < 2k + 1 \), is an integer;

2. \( \chi(i, i + 2k) \) is determined by \( \chi(j, j + 2m + 1) \)'s via (eq \( 2k \))\(^*\).

**Remark** In particular, \( \chi(i, i + 1) \) and \( \chi(i + 1, i + 2) \) cannot be both odd.

Note that any poset can be viewed as union of subposets, where each subposet is a chain. Therefore the proposition serves for general poset as well as chains.
Chapter 4

Weight homology

This chapter is devoted to the discussion on the properties of weight homology. Firstly there is an elegant proof of the main Theorem 4.1.1 about torsion. Next, a long exact sequence is included and it will be further used for developing an inductive algorithm for calculating homologies. Lastly, there is a discussion about homologies of dual poset.

4.1 Torsion

Theorem 4.1.1. For any poset $P$ possessing Euler characteristic structure $(d, \chi)$, $H_n(P)$ is a direct sum of $\mathbb{Z}_2$.

Proof Observe that

$$\partial_n + \partial_{n+1} = 2I.$$

If $\omega \in \ker \partial_n$, then $2\omega = \partial_{n+1}\omega$ and hence $2\omega \in \text{im} \partial_{n+1}$. Therefore $H_n(P)$ is a direct sum of $\mathbb{Z}_2$. \qed

18
4.2 Long exact sequence

**Theorem 4.2.1.** Let $P = Q \cup R$ be a poset possessing Euler characteristic structure $(d, \chi)$, where $Q$ and $R$ are two subposets such that there is no $q > r$ for any $q \in Q$ and $r \in R$. Then there is a long exact sequence of weight homology:

$$\cdots \rightarrow H_n(Q) \rightarrow H_n(P) \rightarrow H_n(R) \rightarrow H_{n-1}(Q) \rightarrow \cdots$$

**Proof** Let $i_n : \mathbb{Z}_n^Q \rightarrow \mathbb{Z}_n^P$ be the inclusion map and $p_n : \mathbb{Z}_n^P \rightarrow \mathbb{Z}_n^R$ be the projection map. Consider the following diagram.

\[
\begin{array}{cccc}
0 & 0 & 0 & 0 \\
\downarrow & \downarrow & \downarrow & \downarrow \\
\cdots & \mathbb{Z}_n^Q & \mathbb{Z}_n^P & \mathbb{Z}_n^R \\
\iota_{n+1} & \iota_n & \iota_{n-1} & \\
\cdots & \mathbb{Z}_n^Q & \mathbb{Z}_n^P & \mathbb{Z}_n^R \\
p_{n+1} & p_n & p_{n-1} & \\
\cdots & \mathbb{Z}_n^Q & \mathbb{Z}_n^P & \mathbb{Z}_n^R \\
0 & 0 & 0 & 0
\end{array}
\]

The only task is to check that $i_*$ and $p_*$ are chain maps. For any $a \in P$,

\[
i_n \partial_{n+1}^Q \omega(a) = i_n \left( (1 - (-1)^{n+1}) \omega(a) + (-1)^{n+1} \sum_{a \leq b} \chi(a, b) \omega(b) \right)
\]

\[
= \begin{cases} 
(1 - (-1)^{n+1}) \omega(a) + (-1)^{n+1} \sum_{a \leq b} \chi(a, b) \omega(b) & \text{if } a \in Q \\
0 & \text{if } a \in R.
\end{cases}
\]
\[ \partial_{n+1}^P \omega(a) \]
\[ = (1 - (-1)^{n+1}) \iota_{n+1} \omega(a) + (-1)^{n+1} \sum_{a \leq b} \chi(a, b) \iota_{n+1} \omega(b) \]
\[ = \begin{cases} 
(1 - (-1)^{n+1}) \omega(a) + (-1)^{n+1} \sum_{a \leq b} \chi(a, b) \omega(b) & \text{if } a \in Q \\
0 & \text{if } a \in R.
\end{cases} \]

Therefore upper rectangles are commutative. Next, for any \( a \in R, \)
\[ \partial_{n+1}^R p_{n+1} \omega(a) = \partial_{n+1}^R \omega(a) \]
\[ = (1 - (-1)^{n+1}) \omega(a) + (-1)^{n+1} \sum_{a \leq b} \chi(a, b) \omega(b). \]
\[ p_n \partial_{n+1}^P \omega(a) = p_n \left( (1 - (-1)^{n+1}) \omega(a) + (-1)^{n+1} \sum_{a \leq b} \chi(a, b) \omega(b) \right) \]
\[ = (1 - (-1)^{n+1}) \omega(a) + (-1)^{n+1} \sum_{a \leq b} \chi(a, b) \omega(b). \]

Therefore lower rectangles are commutative. Moreover, each column is a short exact sequence and each row is a chain complex. Hence there is a long exact sequence of weight homology:
\[ \cdots \to H_n(Q) \to H_n(P) \to H_n(R) \to H_{n-1}(Q) \to \cdots \]
\[ \square \]

### 4.3 Duality

**Theorem 4.3.1.** Let \( P \) be a poset with Euler characteristic structure \((d, \chi)\) and \( P^* \) be the dual poset of \( P \). For any \( a < c \) in \( P \), take \( d^*(a) \) and \( \chi^*(c, a) \) to be \( 2N - d(a) \), where \( N \) is a sufficiently large natural number, and \( \chi(a, c) \) respectively. Then \((d^*, \chi^*)\) is an Euler characteristic structure on \( P^* \) and \( H_n(P^*) = H_n(P) \).
Proof The first task is to check (1.2) and (1.3) hold.

\[ \chi^*(c, a) = \sum_{c \leq b \leq a} \chi^*(c, b) \chi^*(b, a) \]
\[ = \sum_{c \geq b \geq a} \chi(b, c) \chi(a, b) \]
\[ = \chi^*(a, c) \]
\[ = 2 \chi(a, c) \]
\[ = 2 \chi^*(c, a). \]

\[ \chi^*(a, a) = \chi(a, a) \]
\[ = 1 - (-1)^{d(a)} \]
\[ = 1 - (-1)^{d^*(a)}. \]

In addition, if \( b \) covers \( a \) in \( P \), then \( a \) covers \( b \) in \( P^* \) and \( \chi^*(b, a) = \chi(a, b) \neq 0 \). Therefore \((d^*, \chi^*)\) is an Euler characteristic structure on \( P' \).

Consider the chain complex

\[ \cdots \to \mathbb{Z}^{P}_{n+1} \to \mathbb{Z}^{P}_{n} \to \mathbb{Z}^{P}_{n-1} \to \cdots \]

which will give \( H_n(P) \). Taking the dual will get

\[ \cdots \leftarrow (\mathbb{Z}^{P}_{n+1})^* \leftarrow (\mathbb{Z}^{P}_{n})^* \leftarrow (\mathbb{Z}^{P}_{n-1})^* \leftarrow \cdots \]

as well as \( H^n(P) \), which is exactly \( H_{n-1}(P^*) \). By Universal Coefficient Theorem,

\[ H^n(P) = \text{Hom}(H_n(P), \mathbb{Z}) \oplus \text{Ext}(H_{n-1}(P), \mathbb{Z}). \]

Recall that \( \text{Hom}(\mathbb{Z}_2, \mathbb{Z}) = 0 \) and \( \text{Ext}(\mathbb{Z}_2, \mathbb{Z}) = \mathbb{Z}_2 \). By Theorem 4.1.1 and finiteness of \( P \),

\[ H^n(P) = H_{n-1}(P). \]

\[ \square \]
Chapter 5

Computation

Some simple examples, especially chains, are included in Section 5.1. The last section is devoted to developing an inductive algorithm by a long exact sequence. It also involves a discussion on the necessary and sufficient condition for two homologies taking maximal number of $\mathbb{Z}_2$.

5.1 Examples

Example 5.1.1. If $P = \{0\}$, then $\partial_{2n}(\omega(0)) = [1 - (-1)^{d(0)}] \omega(0)$ and $\partial_{2n+1}(\omega(0)) = [1 + (-1)^{d(0)}] \omega(0)$.

$$
(\partial_{2n}(\omega(0)), \partial_{2n+1}(\omega(0))) =
\begin{cases}
(2\omega(0), 0) & \text{if } d(0) \text{ is odd} \\
(0, 2\omega(0)) & \text{if } d(0) \text{ is even}.
\end{cases}
$$

$$
(H_{2n}(P), H_{2n+1}(P)) =
\begin{cases}
(0, \mathbb{Z}_2) & \text{if } d(0) \text{ is odd} \\
(\mathbb{Z}_2, 0) & \text{if } d(0) \text{ is even}.
\end{cases}
$$
In other words,
\[
H_n(P) = \begin{cases} 
\mathbb{Z}_2 & \text{if } d(0) \text{ and } n \text{ have the same parity} \\
0 & \text{otherwise.} 
\end{cases}
\]

**Example 5.1.2.** \( P = \{1 > 0\} \). Then \( d(0) \) and \( d(1) \) have different parity by Prop 3.1.1.

\[
\partial_n \begin{pmatrix} 
\omega(1) \\
\omega(0) 
\end{pmatrix} = \begin{pmatrix} 
1 - (-1)^{n+d(1)} & 0 \\
(-1)^n \chi(0,1) & 1 - (-1)^{n+d(0)} 
\end{pmatrix} \begin{pmatrix} 
\omega(1) \\
\omega(0) 
\end{pmatrix}.
\]

For odd \( d(0) \), the matrix representation of \( \mathbb{Z}_{2n+1}^P \to \mathbb{Z}_{2n}^P \) is \( \begin{pmatrix} 2 & 0 \\ -\chi(0,1) & 0 \end{pmatrix} \),

while that of \( \mathbb{Z}_{2n}^P \to \mathbb{Z}_{2n-1}^P \) is \( \begin{pmatrix} 0 & 0 \\ \chi(0,1) & 2 \end{pmatrix} \).

\( \ker \partial_{2n+1} = \{ (0, b) : b \in \mathbb{Z} \} \) and \( \text{im } \partial_{2n+2} = \{ (0, \chi(0,1)c + 2d) : c, d \in \mathbb{Z} \} \).

Therefore
\[
H_{2n+1}(P) = \begin{cases} 
0 & \text{if } \chi(0,1) \text{ is odd} \\
\mathbb{Z}_2 & \text{if } \chi(0,1) \text{ is even}. 
\end{cases}
\]

\( \ker \partial_{2n} = \{ (a, b) : \chi(0,1)a + 2b = 0; a, b \in \mathbb{Z} \} \) and \( \text{im } \partial_{2n+1} = \{ (2c, -\chi(0,1)c) : c \in \mathbb{Z} \} \). Therefore
\[
H_{2n}(P) = \begin{cases} 
0 & \text{if } \chi(0,1) \text{ is odd} \\
\mathbb{Z}_2 & \text{if } \chi(0,1) \text{ is even}. 
\end{cases}
\]

Similarly, for even \( d(0) \), the same result is obtained:

\[
H_*(P) = \begin{cases} 
0 & \text{if } \chi(0,1) \text{ is odd} \\
\mathbb{Z}_2 & \text{if } \chi(0,1) \text{ is even}. 
\end{cases}
\]

**Example 5.1.3.** \( P = \{0, 1, \cdots, m\} \) with \( i < m \) for every \( i \).
For odd $d(0)$, the matrix representation of $\mathbb{Z}_{2n+1}^P \rightarrow \mathbb{Z}_{2n}^P$ is
\[
\begin{pmatrix}
0 \\
-\chi(m-1,m) & 2 \\
-\chi(m-2,m) & 2 & 0 \\
& \ddots & \ddots \\
-\chi(0,m) & & 0 & 2
\end{pmatrix}
\]

and that of $\mathbb{Z}_{2n}^P \rightarrow \mathbb{Z}_{2n-1}^P$ is
\[
\begin{pmatrix}
2 \\
\chi(m-1,m) & 0 \\
\chi(m-2,m) & 0 & 0 \\
& \ddots & \ddots \\
\chi(0,m) & 0 & 0 & 0
\end{pmatrix}
\]
with respect to the ordered basis $\{\omega(m), \omega(m-1), \cdots, \omega(1), \omega(0)\}$.

From
\[
\mathbb{Q}\ker \partial_{2n+1} = \left\{ \lambda \begin{pmatrix}
1 \\
\frac{\chi(m-1,m)}{2} \\
\frac{\chi(m-2,m)}{2} \\
& \ddots \\
\frac{\chi(0,m)}{2}
\end{pmatrix} \right\}
\]

it is easy to see that
\[
H_{2n+1}(P) = \begin{cases} 
\mathbb{Z}_2 & \text{if all } \chi \text{'s are even} \\
0 & \text{otherwise.}
\end{cases}
\]
Similarly, \( \ker \partial_{2n} = \{(0, x_1, x_2, \cdots x_m) : x_i \in \mathbb{Z}, 1 \leq i \leq m \} \) gives

\[
H_{2n}(P) = \bigoplus_{k \text{ copies}} \mathbb{Z}_2,
\]

where \( k \) is the number of even \( \chi(i, m), 0 \leq i \leq m - 1 \). The two homologies will be interchanged if \( d(0) \) is even.

**Example 5.1.4.** \( P = \{0, 1, \cdots, m\} \) with \( 0 < i \) for every \( i \).

For odd \( d(0) \), the matrix representation of \( \partial_{2n+1} \) is

\[
\begin{pmatrix}
0 & & & & \\
& 0 & & & \\
& & 0 & & \\
& & & 0 & \\
& & & & \ddots \\
& & & & & 0 \\
0 & & & & \\
\end{pmatrix}
\]

and that of \( \partial_{2n} \) is

\[
\begin{pmatrix}
2 & & & & \\
& 2 & & & \\
& & 2 & & \\
& & & 2 & \\
& & & & \ddots \\
& & & & & 2 \\
0 & & & & \\
\end{pmatrix}
\]

with respect to the ordered basis \( \{\omega(m), \omega(m-1), \cdots, \omega(1), \omega(0)\} \). Then

\[
\mathbb{Q}\ker \partial_{2n+1} = \left\{ \lambda_1 \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \\ \frac{\chi(0, m)}{2} \end{pmatrix} + \lambda_2 \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \\ \frac{\chi(m, 0-1)}{2} \end{pmatrix} + \cdots + \lambda_m \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ \frac{\chi(0, 1)}{2} \end{pmatrix} \right\}
\]
Therefore $H_{2n+1}(P) = \bigoplus_{i=1}^{k} \mathbb{Z}_2$, where $k$ is the number of even $\chi(0, i), 1 \leq i \leq m$. On the other hand, $\ker \partial_{2n} = \{(0, \cdots, 0, x) : x \in \mathbb{Z}\}$ gives

$$H_{2n}(P) = \begin{cases} \mathbb{Z}_2 & \text{if all } \chi\text{'s are even} \\ 0 & \text{otherwise}. \end{cases}$$

The two homologies will be interchanged if $d(0)$ is even.

**Example 5.1.5.** $P = \{2 > 1 > 0\}$ and $d(0)$ is odd. With respect to the natural order in $P$,

$$\partial_{2n} = \begin{pmatrix} 2 \\ \chi(1, 2) & 0 \\ \chi(0, 2) & \chi(0, 1) & 2 \end{pmatrix}, \quad \partial_{2n-1} = \begin{pmatrix} 0 \\ -\chi(1, 2) & 2 \\ -\chi(0, 2) & -\chi(0, 1) & 0 \end{pmatrix}.$$ 

$$\mathbb{Q}\text{-ker } \partial_{2n} = \left\{ \lambda \left(0, 1, \frac{-\chi(0, 1)}{2} \right) \right\}.$$ 

$$\text{im } \partial_{2n+1} = \mathbb{Z}(0, -\chi(1, 2), -\chi(0, 2)) + \mathbb{Z}(0, 2, -\chi(0, 1)).$$ 

$$\mathbb{Q}\text{-ker } \partial_{2n-1} = \left\{ \lambda_1 \left(1, \frac{\chi(1, 2)}{2}, \frac{-\chi(0, 2)}{2} \right) + \lambda_2(0, 0, 1) \right\}.$$ 

$$\text{im } \partial_{2n} = \mathbb{Z}(2, \chi(1, 2), \chi(0, 2)) + \mathbb{Z}(0, 0, \chi(0, 1) + 2).$$

Note that $(0, -\chi(1, 2), -\chi(0, 2))$ and $(0, 2, -\chi(0, 1))$ are $\mathbb{Z}$-linearly dependent.

<table>
<thead>
<tr>
<th>$\chi(0, 1)$</th>
<th>$\chi(1, 2)$</th>
<th>$H_{2n}$</th>
<th>$H_{2n-1}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>even</td>
<td>even</td>
<td>$\mathbb{Z}_2$</td>
<td>$\mathbb{Z}_2 \oplus \mathbb{Z}_2$</td>
</tr>
<tr>
<td>even</td>
<td>odd</td>
<td>0</td>
<td>$\mathbb{Z}_2$</td>
</tr>
<tr>
<td>odd</td>
<td>$4N$</td>
<td>0</td>
<td>$\mathbb{Z}_2$</td>
</tr>
<tr>
<td></td>
<td>2-odd</td>
<td></td>
<td>0</td>
</tr>
</tbody>
</table>

**Example 5.1.6.** $P = \{3 > 2 > 1 > 0\}$ and $d(0)$ is odd. With respect to the
natural order in $P$, 
\[
\partial_{2n} = \begin{pmatrix} 0 & 2 \\
\chi(2,3) & \chi(1,3) & \chi(1,2) & 0 \\
\chi(1,3) & \chi(0,3) & \chi(0,2) & \chi(0,1) & 2 \\
\chi(0,3) & \chi(0,2) & \chi(0,1) & 0 \\
\end{pmatrix},
\]
\[
\partial_{2n-1} = \begin{pmatrix} 2 \\
\chi(2,3) & 0 \\
\chi(1,3) & \chi(1,2) & 2 \\
\chi(0,3) & \chi(0,2) & \chi(0,1) & 0 \\
\end{pmatrix}.
\]

$\mathbb{Q}$-ker $\partial_{2n} = \left\{ \lambda_1 \begin{pmatrix} 1, -\chi(2,3) \\
\frac{-\chi(1,3)}{2}, -\chi(0,3) \\
\end{pmatrix}, \lambda_2 \begin{pmatrix} 0, 0, 1, -\chi(0,1) \\
\frac{1}{2} \end{pmatrix} \right\}.$

im $\partial_{2n+1} = \mathbb{Z}(2, -\chi(2,3), -\chi(1,3), -\chi(0,3)) + \mathbb{Z}_2(0, 0, -\chi(1,2), -\chi(0,2)) + \mathbb{Z}(0, 0, 2, -\chi(0,1))}.$

Note that $(0, 0, -\chi(1,2), -\chi(0,2))$ and $(0, 0, 2, -\chi(0,1))$ are $\mathbb{Z}$-linearly dependent. The choice of basis of im $\partial_{2n+1}$ depends on some parity conditions of $\chi$'s.

$\mathbb{Q}$-ker $\partial_{2n-1} = \left\{ \lambda_1 \begin{pmatrix} 0, 1, \chi(1,2) \\
\frac{1}{2}, \frac{\chi(0,2)}{2} \\
\end{pmatrix}, \lambda_2 \begin{pmatrix} 0, 0, 0, 1 \\
\end{pmatrix} \right\}$

The choice of basis of im $\partial_{2n}$ also depends on some parity conditions of $\chi$'s.
The following discussion is about these parity conditions.

Case 1: $\chi(2,3)$ is odd.
\[ \text{im } \partial_{\zeta n} = \mathbb{Z} \begin{pmatrix} 0 \\ \chi(2, 3) \\ \chi(1, 3) \\ \chi(0, 3) \end{pmatrix} + \mathbb{Z} \begin{pmatrix} 0 \\ 2 \\ \chi(1, 2) \\ \chi(0, 2) \end{pmatrix} + \mathbb{Z} \begin{pmatrix} 0 \\ 0 \\ 0 \\ \chi(0, 1) \end{pmatrix} + \mathbb{Z} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 2 \end{pmatrix} \]

\[ = \mathbb{Z} \begin{pmatrix} 0 \\ \frac{1}{\chi(1, 2)} \\ \chi(0, 3) - \frac{\chi(2, 3) - 1}{2} \cdot \chi(0, 2) \end{pmatrix} + \mathbb{Z} \begin{pmatrix} 0 \\ 2 \\ \chi(1, 2) \\ \chi(0, 2) \end{pmatrix} + \mathbb{Z} \begin{pmatrix} 0 \\ 0 \\ 0 \\ \chi(0, 1) \end{pmatrix} + \mathbb{Z} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 2 \end{pmatrix} \]

\[ = \mathbb{Z} \begin{pmatrix} 0 \\ \frac{1}{\chi(1, 2)} \\ \chi(0, 3) - \frac{\chi(2, 3) - 1}{2} \cdot \chi(0, 2) \end{pmatrix} + \mathbb{Z} \begin{pmatrix} 0 \\ 0 \\ 0 \\ \chi(0, 2) - 2 \left( \chi(0, 3) - \frac{\chi(2, 3) - 1}{2} \cdot \chi(0, 2) \right) \end{pmatrix} + \mathbb{Z} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 2 \end{pmatrix} \]
Note that the second term can be further reduced to \( \mathbb{Z} \begin{pmatrix} 0 \\ 0 \\ 0 \\ \chi(0,2) \end{pmatrix} \).

Therefore a basis can be chosen as

\[
\begin{pmatrix}
0 \\
1 \\
\frac{\chi(1,2)}{2} \\
\chi(0,3) - \frac{\chi(2,3)-1}{2} \cdot \chi(0,2)
\end{pmatrix},
\begin{pmatrix}
0 \\
0 \\
0 \\
\delta
\end{pmatrix}
\]

where

\[
\delta = \begin{cases} 
2 & \text{if both } \chi(0,1) \text{ and } \chi(0,2) \text{ are even} \\
1 & \text{otherwise.}
\end{cases}
\]

\(\mathbb{Q}\)-ker \(\partial_{2n-1}\) can be modified into the following way:

\[
\begin{pmatrix}
0 \\
1 \\
\frac{\chi(1,2)}{2} \\
\chi(0,3) - \frac{\chi(2,3)-1}{2} \cdot \chi(0,2)
\end{pmatrix},
\begin{pmatrix}
0 \\
0 \\
0 \\
1
\end{pmatrix}
\]

Case 2: \(\chi(2,3)\) is even.

One of the basis vector can be chosen as

\[
\begin{pmatrix}
0 \\
2 \\
\chi(1,2) \\
\chi(0,2)
\end{pmatrix}
\]

while the remaining
one is in the form of
\[
\begin{pmatrix}
0 \\
0 \\
0 \\
\delta
\end{pmatrix},
\]
where

\[
\delta = \begin{cases} 
2 & \text{if both } \chi(0, 1) \text{ and } \chi(0, 3) - \frac{\chi(2, 3)}{2} \cdot \chi(0, 2) \text{ are even} \\
1 & \text{otherwise.}
\end{cases}
\]

<table>
<thead>
<tr>
<th>$\chi(0, 1)$</th>
<th>$\chi(1, 2)$</th>
<th>$\chi(2, 3)$</th>
<th>$\chi(0, 3)$</th>
<th>$H_{2n}$</th>
<th>$H_{2n-1}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>even</td>
<td>even</td>
<td>even</td>
<td>even</td>
<td>$\mathbb{Z}_2 \oplus \mathbb{Z}_2$</td>
<td>$\mathbb{Z}_2 \oplus \mathbb{Z}_2$</td>
</tr>
<tr>
<td>odd 4N</td>
<td>even</td>
<td>even</td>
<td>Z2</td>
<td>0</td>
<td>Z2</td>
</tr>
<tr>
<td>even 4N</td>
<td>odd 2-odd</td>
<td>even</td>
<td>0</td>
<td>Z2</td>
<td>A</td>
</tr>
<tr>
<td>even</td>
<td>even odd</td>
<td>even</td>
<td>Z2</td>
<td>Z2</td>
<td></td>
</tr>
<tr>
<td>even</td>
<td>even even</td>
<td>odd</td>
<td>Z2</td>
<td>Z2</td>
<td></td>
</tr>
<tr>
<td>odd</td>
<td>even odd</td>
<td>even</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>odd</td>
<td>even even</td>
<td>odd</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>even</td>
<td>odd even</td>
<td>odd</td>
<td>Z2</td>
<td>Z2</td>
<td>0</td>
</tr>
<tr>
<td>even</td>
<td>even odd</td>
<td>odd</td>
<td>Z2</td>
<td>Z2</td>
<td>0</td>
</tr>
<tr>
<td>odd</td>
<td>even odd</td>
<td>odd</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

\[
A = \begin{cases} 
0 & \text{if } \chi(0, 1) \text{ and } \chi(2, 3) \text{ have the form } 2 \cdot \text{odd} \\
\mathbb{Z}_2 & \text{otherwise.}
\end{cases}
\]

\[
B = \begin{cases} 
\mathbb{Z}_2 & \text{if } \chi(0, 1) \text{ and } \chi(2, 3) \text{ have the form } 2 \cdot \text{odd} \\
0 & \text{otherwise.}
\end{cases}
\]
The previous two examples demonstrate the complexity in computing homologies of simple posets by direct calculation. So a feasible and systematic algorithm is desired in order to work out homologies of more complicated poset.

5.2 Long exact sequence revisited

Long exact sequence is a powerful tool in calculating homologies. The one mentioned in Theorem 4.2.1 is in no exception. Let \( c \) be a point in \( P \) such that there is no \( b \in P \) with \( b > c \). Take \( R \) and \( Q \) to be \{c\} and \( P - R \) respectively. Suppose \( n \) and \( d(c) \) have the same parity, then by Example 5.1.1, \( H_n(\{c\}) = \mathbb{Z}_2 \) and \( H_{n-1}(\{c\}) = 0 \). So the long exact sequence becomes

\[
\begin{array}{cccccc}
0 & \longrightarrow & H_n(Q) & \longrightarrow & H_n(P) & \xrightarrow{\epsilon_n} \mathbb{Z}_2 \\
& \searrow & \downarrow & & & \\
& & H_{n-1}(Q) & \longrightarrow & H_{n-1}(P) & \longrightarrow 0,
\end{array}
\]

where \( \epsilon_n \) sends \( [\omega] \) to \( \omega(c) \) modulo 2. There are two possibilities for \( \epsilon_n \):

1. \( \epsilon_n = 0 \). Then \( \partial_n \) is injective by exactness and the long exact sequence can be viewed as

\[
0 \rightarrow H_n(Q) \rightarrow H_n(P) \rightarrow 0
\]

and

\[
0 \rightarrow \mathbb{Z}_2 \rightarrow H_{n-1}(Q) \rightarrow H_{n-1}(P) \rightarrow 0.
\]

Hence \( H_n(P) = H_n(Q) \) and \( H_{n-1}(P) \cong H_{n-1}(Q)/\mathbb{Z}_2 \).

2. \( \epsilon_n \) is onto. Then \( \partial_n = 0 \) by exactness and the long exact sequence becomes

\[
0 \rightarrow H_n(Q) \rightarrow H_n(P) \rightarrow \mathbb{Z}_2 \rightarrow 0
\]

and

\[
0 \rightarrow H_{n-1}(Q) \rightarrow H_{n-1}(P) \rightarrow 0.
\]

Hence \( H_n(P) \cong H_n(Q) \oplus \mathbb{Z}_2 \) and \( H_{n-1}(P) = H_{n-1}(Q) \).
Remark The result marked by (†) and (‡) is based on Theorem 4.1.1, which shows that homology must be a direct summand of $\mathbb{Z}_2$.

To check whether $e_n$ is onto, it is equivalent to find a cycle $\omega$ in $H_n(P)$ such that $\omega(c)$ is odd. For any $a \in P$, let a weight

$$\omega_a(p) = \begin{cases} 1 & \text{if } p = a \\ 0 & \text{otherwise.} \end{cases}$$

By $\partial_{n+1}\omega_c(c) = [1 - (-1)^{n+1+d(c)}] \omega_c(c) = 2$ and $\partial_{n+1}\omega_c$ being a cycle in $H_n(P)$ (this follows by $\partial_n \partial_{n+1} = 0$), the checking reduces to find a cycle $\omega$ such that $\omega(c) = 1$.

**Algorithm 1.** If there exists a cycle $\omega \in H_n(P)$ such that $\omega(c) = 1$, then $H_n(P) = H_n(Q) \oplus \mathbb{Z}_2$ and $H_{n-1}(P) = H_{n-1}(Q)$; otherwise, $H_n(P) = H_n(Q)$ and $H_{n-1}(P) = H_{n-1}(Q)/\mathbb{Z}_2$.

**Lemma 5.2.1.** $\text{im } \partial_n \cap \text{im } \partial_{n+1} = \{0\}$.

**Proof** Since $\partial_n + \partial_{n+1} = 2I$, $\ker \partial_n \cap \ker \partial_{n+1} = \{0\}$. In addition, as $\text{im } \partial_{n+1}$ being a subset of $\ker \partial_n$, the result follows. \qed

**Lemma 5.2.2.** The existence of a cycle $\omega$ in $H_n(P)$ such that $\omega(c) = 1$ is equivalent to the existence of a collection of integers $\{k_a : a \in Q\}$ such that $\partial_n^P \omega_c = \sum_{a \in Q} k_a \partial_n^P \omega_a$.

**Proof** Suppose $\partial_n^P \omega_c$ can be written as $\sum_{a \in Q} k_a \partial_n^P \omega_a$. Consider

$$\left(\partial_n^P \omega_c - \sum_{a \in Q} k_a \partial_n^P \omega_a\right) + \left(\partial_{n+1}^P \omega_c - \sum_{a \in Q} k_a \partial_{n+1}^P \omega_a\right) = 2 \left(\omega_c - \sum_{a \in Q} k_a \omega_a\right).$$

The first term in the left side is 0 while the second term lies in $\ker \partial_n^P$. So

$$\left(\omega_c - \sum_{a \in Q} k_a \omega_a\right) = \frac{1}{2} \left(\partial_{n+1}^P \omega_c - \sum_{a \in Q} k_a \partial_{n+1}^P \omega_a\right) \in \ker \partial_n^P.$$

32
In addition,

\[ \frac{1}{2} \left( \partial_{n+1}^P \omega_c - \sum_{a \in Q} k_a \partial_{n+1}^P \omega_a \right) (c) = 1. \]

Conversely, as a weight that \( \omega(c) = 1 \), \( 2 \omega \) can be written as \( 2 \omega_c - 2 \sum_{a \in Q} k_a \omega_a \).

In addition, \( \omega \in H_n(P) \) implies that \( \partial_n^P \omega = 0 \). Therefore

\[ 2 \omega = 2 \omega_c - 2 \sum_{a \in Q} k_a \omega_a \]

\[ \partial_{n+1}^P \omega = (\partial_n^P + \partial_{n+1}^P) \omega_c - \sum_{a \in Q} k_a (\partial_n^P + \partial_{n+1}^P) \omega_a \]

\[ \partial_{n+1}^P \omega - \partial_{n+1}^P \omega_c + \sum_{a \in Q} k_a \partial_{n+1}^P \omega_a = \partial_n^P \omega_c - \sum_{a \in Q} k_a \partial_n^P \omega_a. \]

By Lemma 5.2.1, both sides equal 0 in the last equation. \( \Box \)

**Theorem 5.2.3.** \( H_\ast(P) \) has maximal number of \( \mathbb{Z}_2 \) if and only if all \( \chi \)'s are even.

**Proof** The proof proceeds with induction in the light of Algorithm 1. In other words, one needs to show that, assuming all \( \chi(a, b) \) is even for any \( a < b \) and \( b \) distinct from \( c \), \( H_\ast(P) \) has maximal number of \( \mathbb{Z}_2 \) if and only if each \( \chi(a, c) \) is even for any \( a < c \).

If \( \chi(a, c) \) is even for any \( a < c \), take \( \omega = \frac{1}{2} \partial_{n-1}^P \omega_c \). Note that \( \partial_{n-1}^P \omega_c(c) = 2 \).

By Lemma 5.2.2 in addition to Algorithm 1, \( H_\ast(P) \) has maximal number of \( \mathbb{Z}_2 \) if and only if \( \partial_n^P \omega_c = \sum_{a \in Q} k_a \partial_n^P \omega_a \) for some integers \( k_a \). For any \( q \in Q \),

\[ \partial_n^P \omega_a(q) = \begin{cases} 
1 - (-1)^{n+d(a)} & \text{if } q = a \\
(-1)^n \chi(q, a) & \text{if } q < a \\
0 & \text{otherwise.} \end{cases} \]

By assumption, \( \partial_n^P \omega_a \) is a weight consisting of even entries. So if \( H_\ast(P) \) has maximal number of \( \mathbb{Z}_2 \), then \( \partial_n^P \omega_c \) is also a weight consisting of even entries. Hence \( \chi(a, c) \) is even for any \( a < c \), where \( a \in Q \). \( \Box \)
Bibliography


