TIME DEPENDENT EFFECTS OF REINFORCED
CONCRETE SUBJECTED TO TORSIONAL
LOADING

By

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鄭百翔
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Glory be to the Father, and to the Son, and to the Holy Spirit. As it was in the beginning, is now, and ever shall be, world without end. Amen.
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ABSTRACT

The main objective of this research is to present a study into the time dependent behaviour of plain concrete and reinforced concrete members subjected to pure torsional loading.

In this research project, the time dependent deformation of the plain concrete and for both uncracked and cracked reinforced concrete have been investigated. Three major elements are included in this research: 1) The analytical investigation, 2) The experimental study and 3) The computer simulations.

Analytical models for the time dependent effect on reinforced concrete members under pure torque is developed for both cases of uncracked and cracked concrete with aided of the experimental investigation and previous study on uniaxial load. While short tests for determining the strength and elastic behaviour and plain and reinforced concrete are required for supporting the information of basic material properties. According to the results of the experiment, a mathematical model is developed for predicting the time dependent behaviour.

At the same time, the developed analytical predictions are compared with measurements from test data and computer modeling so that the validation of the analytical models can be verified.

The analytical model and analysis show good correlation with experimental results obtained in this study.
Chapter One

Introduction

1.1 Background

Concrete and steel are the two most commonly used structural materials in the world. They sometimes complement one another and compete with one another so that structures of similar type and function can be built using either of these materials. However, seldom engineers are familiar with the properties of the concrete even though most of the structures are made by reinforced concrete. This is because concrete is not an isotropic and heterogeneous material as it is made of water, cement, sand and aggregate with different mechanical and chemical behaviour. Thus, studying the behaviour of concrete is one of the popular topics in the field of civil and structural engineering research in the world.

Torsion is a significant factor to consider in the design of many types of reinforced concrete structures such as skew bridges, horizontal curve beams, beams supporting cantilever slabs...etc. Engineers generally ignore torsion and this has caused many cases of torsional distress and failure.

In most structural members, creep and shrinkage of concrete cause an increase in deformation and redistribution of stresses, but does not adversely affect strength. In some situations, however, the deformations caused by creep and shrinkage can lead to an increase in the internal actions in the structure and therefore, a reduction in strength carrying capacity. It also has significant effects on the design of the concrete structure, particularly in the serviceability design of reinforced concrete as it is highly affected by the properties of creep. A slender column carrying a torsion load is such an example.

Normally, when a concrete column is subjected to a load, its response is immediate and time-dependent. Under sustained load this deformation of the member gradually increases with time, and may eventually be many times greater than its instantaneous value. Consider a slender column subjected to a pure torque, T applied at a constant value, the instantaneous deflections of the column under the torque, T is the angle of twist at initial, \( \theta_i \). After a long time, the angle increases and reaches a stable value.
1.2 Objective

The objective of this research is to investigate the time dependent deformation of the plain concrete for both uncracked and cracked reinforced concrete. The research is basically composed of three major components. They are the analytical investigation, experimental study and computer simulations. In the analytical part, the major task conducted the formulation of the relationship of the cross section of column under pure torque. Analytical predictions are compared with measurements from test data and computer modeling.

In this study, an analytical model for the time dependent effect on reinforced concrete column under pure torque is developed for both cases of uncracked and cracked concrete, through the experimental investigation and previous study on uniaxial load. At the same time, short tests for finding the strength and elastic behaviour and plain and reinforced concrete are required. Based on the results of the experiment, a mathematical model is developed for predicting the time dependent behaviour.
1.3 Thesis Outline

This thesis has seven chapters; Chapters 2 presents the literature review, which includes brief reviews on the theory of torsion, composite materials properties and creep of concrete are summarized. In chapter 3, the behaviour aspect of structural concrete beams for both with longitudinal bars only and with longitudinal and stirrups in pure torsion are discussed. The derivation theoretical equations for post-cracking rigidity are shown. Detail on the time dependent analysis of reinforced concrete columns under uniaxial load is presented in chapter 4. Some simple numerical examples are illustrated to provide a clear picture for the explanation. The analytical methods for solving the rate of twist of uncracked reinforced concrete and cracked concrete are discussed in chapter 5. Numerical examples are also provided for illustrate the approaches. In chapter 6, the experimental work and results are shown, comparisons for experimental results and theoretical prediction are also given. The conclusions of the work of this research project are shown in chapter 7.
Chapter Two

Literature Review

2.1 Introduction

The topics of creep in different areas are commonly discussed. There are various journals, periodicals and related information accessible for referring the creep of concrete and its structural effects. The topics can be material related, which discuss creep in the point of view of material properties. Also, they can be structural related, which focus on the effects of creep on structure members.

Creep and shrinkage can cause deformation in reinforced concrete structures. The deformation gradually increase with time and eventually may be several times greater that the instantaneous values. In order to accurately predict these deformations, several models are provided for the creep and shrinkage characteristics of concrete. Analytical methods are also available for the time-dependent analysis of reinforced concrete cross-section and members. The effective modulus method and the age-adjusted effective modulus method, rate of creep method are introduced for analysis. In the subsequent sections in this chapter, a review of the detail in creep including the material behaviour of concrete and the structure deformation due to creep of concrete is presented.
2.2 Creep and Shrinkage of Concrete

The stress and strain in a reinforced concrete structure are subjected to change for a long period of time with constant load, during which creep and shrinkage of concrete develop gradually. This inelastic deformation was first observed and reported some eighty years ago. In this section, a general description of the elastic and inelastic deformation that characterizes concrete behaviour is made.

2.2.1 Deformation of Concrete

Creep is the increase in strain with time due to sustained load. Initial deformation due to load is the elastic strain, while the additional strain due to the same sustained load is creep strain. Creep is generally not measured directly but is determined only by deducting elastic strain and shrinkage strain from the total deformation. Although, shrinkage and creep are not completely independent phenomena, it can be assumed that superposition of strains is valid, hence,

\[ e(t) = \varepsilon_e(t) + \varepsilon_c(t) + \varepsilon_{sh}(t) \quad (2.1) \]

Where \( \varepsilon_e(t) \) = elastic strain or instantaneous strain

\( \varepsilon_c(t) \) = creep strain

\( \varepsilon_{sh}(t) \) = shrinkage strain

These relative values illustrate that stress-strain relationships for short-term loading become dominant on the behaviour of a structure.

Fig 2.1 is a graphical representation of equation 2.1 and illustrates the increase in creep strain with time and as is also the case of shrinkage, the rate of creep decreases with time. This representation is for linear creep where the applied stress to the concrete is generally less the a value equal to about 50 %.

As in shrinkage, creep increases the deflection of beams and slabs and cause loss of prestressing force in prestressed concrete elements. In addition, the initial eccentricity of a reinforced concrete column along its length increases with time due to creep deflect, resulting in the transfer of the compressive load from the concrete to the steel in the concrete section. Once the steel yields, additional load has to be carried by the concrete.
Consequently, the resisting capacity of the column is reduced and the curvature of the column increases further.

The prediction of the time-dependent behaviour of a concrete member requires the accurate prediction of each of these strain components at critical locations. This requires knowledge of stress history, in addition to accurate data for the material properties.

Fig 2.1 Concrete strain components under sustained stress.

2.2.1a Elastic strain

The instantaneous strain $\varepsilon_i(t)$, which occurs immediately on the application of stress and depends on:

1) The magnitude of the stress
2) The rate at which the stress is applied
3) The age of concrete

It can be found that, part of the instantaneous strain is elastic (recoverable), while part is inelastic (irrecoverable). It is reasonable to assume that the instantaneous strain in concrete at service loads is entirely elastic. It can be shown graphically in Fig 2.2. It is therefore expressing the $\varepsilon_i(t)$ as:

$$\varepsilon_i(t) = \frac{\sigma(t)}{E_i(t)} \quad (2.2)$$
Fig 2.2 Typical stress Vs instantaneous strain curve for concrete in compression.

The value of $E_c(t)$ increases with time and is also dependent on the rate of loading. The faster the load is applied, then the larger is the value of $E_c(t)$. According to Pauw variations of the elastic modulus with may be expressed as:

$$E_c(t) = \rho^{1.5} 0.043 \sqrt{f_c(t)} \quad (in \quad MPa) \quad (2.3)$$

This equation is widely used in many building codes [5-6]. Where $\rho$ is density of concrete, about 2400 kg/m$^3$ and $f_c(t)$ is the mean compressive strength in MPa.

Rüsch et al. considers that the influence of aggregate type on the modulus of elasticity should not be ignored and he proposed that:

$$E_c(t) = 4.3 \beta_a \rho^3 \sqrt{f_c(t)} \quad (2.4)$$

where $\beta_a$ is a coefficient that depends on the type of aggregate. Table 2.1 shows the comparison between equations 2.3 and 2.4.
<table>
<thead>
<tr>
<th>$f_r(t)$ MPa</th>
<th>$E_r(t)$ MPa</th>
<th>Sandstone ($\rho = 2280$ kg/m$^3$)</th>
<th>Limestone ($\rho = 2300$ kg/m$^3$)</th>
<th>Quartzite ($\rho = 2300$ kg/m$^3$)</th>
<th>Basalt ($\rho = 2400$ kg/m$^3$)</th>
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<tr>
<td></td>
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<td>Eq 2.3</td>
<td>Eq 2.3</td>
<td>Eq 2.3</td>
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<tr>
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<td>18130</td>
<td>16930</td>
<td>18370</td>
<td>21950</td>
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<td>24160</td>
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<td>29610</td>
<td>23470</td>
<td>30000</td>
<td>30440</td>
<td>30000</td>
</tr>
</tbody>
</table>

Table 2.1 Elastic modulus of concrete – Pauw Vs Rüsch et al.

To determine $E_r(t)$ at any particular time, the variation of compressive strength with time $f_r(t)$ needs to be known. According to ACI committee 209 recommendations,

$$f_r(t) = \frac{\tau}{\alpha + \beta_t} f_r(t)$$

Where $\alpha$ and $\beta$ depend on the cement type and curing conditions. For normal Type I cement, $\alpha = 4$ and $\beta = 0.85$ for moist cured concrete (and $\alpha = 1$ and $\beta = 0.95$ for steam cured concrete). For rapid hardening Type III cement, the corresponding values are $\alpha = 2.3$ and $\beta = 0.92$ (moist cured) and $\alpha = 0.7$ and $\beta = 0.98$ (steam cured).

With the elastic modulus increasing gradually, the elastic strain decreases gradually. However it is usual to assume that the instantaneous strain remains constant with time.

To date the discussion has only considered the elastic modulus in compression, however the elastic modulus in tension is usually slightly higher than in compression. However, in most of the cases, both values are usually assumed to be equal. The strength of concrete in tension $f_r(t)$ is of the order of 10 percent of the compressive strength.

Empirical formulae for the tensile strength of concrete usually take the form.

$$f_r(t) = k_1 \sqrt{f_r(t)}$$

and

$$f_r(t) = k_2 \sqrt[3]{f_r(t)}$$
Values for upper and lower 5 percentile limits and mean values for $k_1$ and $k_2$ are given in Table 2.2 for both concentric flexural tensile strength in MPa.

<table>
<thead>
<tr>
<th>Type of Strength</th>
<th>$k_1$</th>
<th></th>
<th>$k_2$</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Lower 5 % limit</td>
<td>Mean</td>
<td>Upper 5 % limit</td>
<td>Lower 5 % limit</td>
</tr>
<tr>
<td>Concentric Tensile Strength</td>
<td>0.4</td>
<td>0.5</td>
<td>0.7</td>
<td>0.2</td>
</tr>
<tr>
<td>Flexural Tensile Strength</td>
<td>0.6</td>
<td>0.85</td>
<td>1.1</td>
<td>0.35</td>
</tr>
</tbody>
</table>

Table 2.2 Tensile strength coefficients, $k_1$ and $k_2$.

2.2.1b Creep strain

The second term of equation 2.1 represents the creep strain. In a loaded specimen that is in hygral equilibrium with the ambient medium, the time dependent deformation is known as basic creep. The additional creep, which occurs in a drying specimen is sensibly known as drying creep. In general creep occurs at its greatest rate in the first month after application of load. The creep will tend to a definite value only after several years. In general, creep is assumed to approach a limiting value as the time after loading approaches infinity (often represented by 30 years).

The magnitude of creep and its rate of development are influenced by many factors including the properties of the concrete mix, environmental and loading conditions.

1) Properties of the concrete mix.
   An increase in concrete strength causes a decrease in creep. Creep is also reduced by an increase in the aggregate to cement ratio, and increases in the maximum aggregate size as well as by the use of stiffer aggregate type. Creep by depends on the water-cement ratio and the cement type in so far as these influence concrete strength. Moreover, the magnitude of creep also depends on the degree of hydration at first loading. Creep decreases as the age at which first loading occurs increase.

2) Environmental conditions
   Creep increases as the environmental humidity decreases and creep is greater in smaller members than in larger ones.
A rise in temperature also increases creep. The deformability of the cement paste is increased by a temperature rise and drying is accelerated. The dependence of creep on temperature is more pronounced at relatively elevated temperatures above ambient values but is insignificant for temperature variations between 0°C and 20°C. However, creep in concrete at a mean temperature 40°C is 20% higher than that at 20°C.

3) Loading conditions

When the sustained stress is less than about one half of the compressive strength of concrete, the creep strain is approximately proportional to the stress level and is known as linear creep. At higher stress levels creep increases at faster rate and becomes non-linear with respect to stress, as shown in Fig 2.3. This non-linear behaviour of creep at high stress levels is thought to be related to an increase in microcracking.

Fig 2.3 Influence of load intensity and duration on strain.

2.2.1c Shrinkage Strain

Drying shrinkage is the decrease in the volume of a concrete element when it loses moisture by evaporation. The opposite phenomenon, i.e. volume increase through water absorption, is termed swelling. When the change in volume by shrinkage or by swelling is restrained, stresses develop. In reinforced concrete structures, the restraint may be caused by the reinforcing steel, by the supports or by the difference in volume change of various parts of the structure. Stresses caused by shrinkage are generally
reduced by the effect of creep of concrete. Thus the effects of these two simultaneous phenomena must be considered in stress analysis. The amount of free shrinkage and an expression for its variation with time is needed. Shrinkage starts to develop at time $t$, when moist curing stops.

Shrinkage is affected by all the factors which affect the drying of concrete, in particular the water content and the water content and the water cement ratio of the mix, the size and shape of the member and the ambient relative humidity, type of cement and amount of reinforcement. Shrinkage increases when the water-cement ratio increases, the relative humidity decreases and the ratio of the exposed surface area to volume increases. Temperature rises accelerate drying and therefore increase shrinkage.

The volume and type of aggregate also affect shrinkage. Aggregate provides restraint to shrinkage of the cement paste, so that an increase in aggregate content reduces shrinkage. Shrinkage is also smaller when stiffer aggregates are used, i.e. aggregate with greater elastic moduli.

Reinforced concrete shrinks less than plain concrete, the relative difference is a function of the reinforcement percentage. Rapid-hardening cement shrinks somewhat more than other type, while shrinkage-compensating cements minimize or eliminate shrinkage cracking when used with restraining reinforcement.
2.3 Methods for Predicting the Creep Coefficient and Shrinkage

The ratio of creep strain at time, \( t \), to instantaneous elastic strain in a specimen subjected to constant sustained stress, is defined as the creep coefficient, i.e.

\[
\phi(t, \tau) = \frac{\varepsilon_c(t, \tau)}{\varepsilon_e(\tau)} \quad (2.7)
\]

The creep strain depends on the age of the concrete at the time of first loading, as does the creep coefficient. This is because both the creep and the instantaneous strain components are proportional to stress. The creep coefficient \( \phi(t, \tau) \) of equation (2.7) is a pure time function and is independent of the applied stress. As time approaches infinity, the creep coefficient is assumed to approach a final value \( \phi(\infty, \tau) \), where

\[
\phi(\tau) = \phi(\infty, \tau) = \frac{\varepsilon_c(\tau)}{\varepsilon_e(\tau)} \quad (2.8)
\]

The final creep coefficient is a useful measure of the capacity of concrete to creep. However, the most accurate means for predicting the final creep coefficient and shrinkage strain is to extrapolate from short-term test results. Creep can be measured over a relatively short period in comparison-unloaded specimens. Some methods of prediction are summarized in the following sections.

2.3.1 Prediction from Short-term Tests

Numerous expressions have been proposed for the development of creep an shrinkage with time. Exponential and hyperbolic functions approach a limiting value as time approaches infinity has been used to model the development of both creep and shrinkage. Logarithmic and power expressions that increase indefinitely have also been proposed to model creep.

Such expressions can be used to provide estimates of long-term deformations from short-term tests.

The power functions of the general form

\[
\phi(t, \tau) = \alpha(t - \tau)^{\beta} \quad (2.9)
\]
In the above equation, \( (t, \tau) \) is the duration of load or the time since the load was first applied. The straight line of best fit through the short-term measurements of \( \log \phi(t, \tau) \) Vs \( \log(t, \tau) \) enables the constants \( \alpha \) and \( \beta \) to be determined.

Bazant and Osman proposed the double power expressions with the agreement of considerable experimental data of the form:

\[
\phi(t, \tau) = \alpha (t - \tau)^\beta \tau^\gamma \tag{2.10}
\]

Ross proposed a hyperbolic equation of the following expression:

\[
\phi(t, \tau) = \frac{(t - \tau)}{\alpha + \beta(t - \tau)} \tag{2.11}
\]

Meyers at al. further developed the hyperbolic power expression to the following form:

\[
\phi(t, \tau) = \frac{(t - \tau)^\gamma}{\alpha + \beta(t - \tau)^\gamma} \tag{2.12}
\]

The ACI suggests the use of equation (2.12) with \( \beta = 10 \) and \( \gamma = 0.6 \). When \( (t - \tau) = 28 \) days, the relationship between the 28 day and final creep coefficient obtained from equation (2.12) as

\[
\phi(28 + \tau, \tau) = \frac{\phi'(\tau)}{2.35} \tag{2.13}
\]

The final creep coefficient can be obtained by substituting equation (2.13) to equation (2.12).

Neville at al. claimed that equation obtained by the ACI method mentioned above overestimates creep and have proposed the following function for predicting creep after several years:

\[
\phi(t, \tau) = [2.51 \ln(t - \tau) - 6.19]^{0.379} \phi(28 + \tau, \tau) \tag{2.14}
\]

Where \( \phi(28 + \tau, \tau) \) is the creep coefficient measured 28 days after first loading, i.e. when \( t = 28 + \tau \). A comparison of predictions made using the above equations can be made in Fig 2.4.
Fig 2.4 Long-term creep predictions from 28 days tests.

Meyers et al. have suggested the shrinkage-time relationship in the following form.

\[ \varepsilon_{sh}(t) = \frac{t}{\alpha + t} \varepsilon_{sh}^f \quad (2.15) \]

\( t \) is time (in days) measured from the start of drying, \( \varepsilon_{sh}^f \) is the final shrinkage strain and \( \alpha \) is constant taken to be 35 for moist cured concrete and 55 for steam cured concrete.
Neville et al. have proposed the following expression for moist cured concrete and claim a slight improvement in predictions at later ages:

\[ \varepsilon_{sh}(t) = \alpha \varepsilon_{sh}(28)^{\beta} \quad (2.16) \]

Where

\[ \alpha = \left[ 1.53 \ln(t) - 4.17 \right]^2 \]

and

\[ \beta = \frac{100}{2.90 + 29.2 \ln(t)} \]

A comparison of predictions equation (2.15) and equation (2.16) and experimental data is made in Fig 2.5.

Fig 2.5 Comparison of shrinkage predictions.

2.3.2 The ACI 209 Method (1992)

ACI Committee 209 uses a hyperbolic function to represent the creep-time relationship, it shows in the following:

\[ \phi(t, \tau) = \frac{(t - \tau)^{0.6}}{10 + (t - \tau)^{0.6}} \phi'((\tau) \quad (2.17) \]
where $\tau$ is the age of the concrete at first loading, $(t - \tau)$ is the duration of loading, $\phi'(\tau)$ is the final creep coefficient and is expressed as

$$\phi'(\tau) = 2.35\gamma_1\gamma_2\gamma_3\gamma_4\gamma_5\gamma_6 \quad (2.18)$$

$\gamma_1$ to $\gamma_6$ are empirical correction factors which account for many of the parameters, which affect the magnitude of creep. Detail of the values is detailed.

The shrinkage strain at time $t$ measured from the start of drying is given by,

For moist cured concrete:

$$\varepsilon_{sh}(t) = \frac{t}{35 + t} \varepsilon_{sh}(t) \quad (2.19a)$$

For steam cured concrete:

$$\varepsilon_{sh}(t) = \frac{t}{55 + t} \varepsilon_{sh}(t) \quad (2.19b)$$

where $\varepsilon_{sh}(t)$ is the final shrinkage at time infinity and is given by

$$\varepsilon_{sh} = 780\gamma_2\gamma_3\gamma_4\gamma_5\gamma_6\gamma_7\gamma_8 \quad (2.20)$$

### 2.3.3 The CEB-FIP Method, 1978

Rüscher and Jungwirth proposed that the creep strain at time $t$ caused by a constant sustained stress $\sigma_0$ applied at time $\tau$ is assumed to be

$$\varepsilon_c(t, \tau) = \frac{\sigma_0}{E_{c28}} \phi(t, \tau) \quad (2.21)$$

where $E_{c28}$ is the longitudinal modulus of deformation at 28 days. The creep coefficient $\phi(t, \tau)$ is thus taken as the ratio creep strain at time $t$ to the instantaneous elastic strain at age 28 days.

The creep coefficient is divided into a reversible delayed elastic component and an irreversible flow component. The flow component is further sub-divided into an initial component and a subsequent flow component, so that
\[ \phi'(t, \tau) = \phi_d \beta_d (t - \tau) + \beta_a(\tau) + \phi_f [\beta_f (t) + \beta_f (\tau)] \] (2.22)

\( \phi_d \) is final delayed elastic creep coefficient (i.e. The ratio of the final delayed elastic strain and the instantaneous strain at 28 days) and is taken to be 0.4. \( \beta_d (t - \tau) \) is a function describing the development of the delayed elastic strain with time and can be taken from Fig 2.6.

![Graph showing development of delayed elastic strain with time.](attachment:image.jpg)

Fig 2.6 Development of delayed elastic strain with time (CEB-FIP, 1978).

The mean shrinkage according to the CEB-FIP method that occurs within the time interval, \( t_a \) to \( t \) is given by

\[ \varepsilon_{sh}(t; t_a) = \varepsilon_{sh}(t_a) [\beta_{sh}(t) - \beta_{sh}(t_a)] \] (2.23)

### 2.3.4 British Standard – Structural Use of Concrete, BS 8110 – Part 2

The method contained in BS 8110 is in fact based on CEP-FIP, 1970 recommendations.

The static modulus of elasticity of 28 days for normal weight concrete is given by:

\[ E_{e, 28} = K_a + 0.2 f_{cu}(28) \] (2.24)
Where $f_{cu}(28)$ is the 28 day cube strength, in MPa and the constant $K_a$ depends on the stiffness of the aggregate and may be taken as 20 GPa for normal weight concrete. For lightweight concrete, the value of $E_{c,28}$ obtained from equation (2.24) should be multiplied by $(\rho/2400)^2$, where $\rho$ is the density of lightweight concrete kg/m$^3$. The elastic modulus at any time $t$ may be derived from $E_{c,28}$ using equation (2.25).

$$E_c(t) = E_{c,28} \left[ 0.4 + 0.6 \frac{f_{cu}(t)}{f_{cu}(28)} \right]$$

(2.25)

The final creep strain may be predicted from

$$\varepsilon_{cr} = \frac{\sigma}{E_c(t)} \phi'$$

(2.26)

where $E_c(t)$ is the elastic modulus at time of loading and $\phi'$ is the final creep coefficient which may be taken Fig 2.7.

![Fig 2.7 Final creep coefficient, BS8110.](image)

For concrete of normal workability, without reducing admixtures, the 30 years and 6 months shrinkage strains may be estimated from Fig 2.8.
Fig 2.8 Shrinkage of normal weight concrete. BS 8110.
2.4 Introduction to Composite Materials Properties

A composite is a material, which is composed of two or more distinct phases. Thus a composite is heterogeneous. The simplest treatment of the elastic behaviour of aligned long-fiber composites is based on the premise that the material can be treated as if it were composed of parallel slabs of the two constituents bonded together, with relative slabs of the thickness in proportion to the volume fractions of matrix and fiber. This is illustrated in Fig (2.9). The two slabs are constrained to have the same lengths parallel to the bonded interface. Thus if a stress is applied in direction of fiber alignment (the 1-direction), both constituents exhibit the same strain in this direction, $\varepsilon_1$. This 'equal strain' condition is valid for loading along the fiber axis if there is no interfacial sliding.

It is a simple matter to derive the Young's modulus of the composite, $E_1$. The axial strain in the fiber and matrix must correspond to the ratio between the stress and the Young's modulus for each of the two components, so that:

$$\varepsilon_1 = \varepsilon_{1f} = \frac{\sigma_{1f}}{E_f} = \varepsilon_{1m} = \frac{\sigma_{1m}}{E_m}$$

(2.27)

Hence, for a composite in which the fibers are much stiffer than the matrix ($E_f \gg E_m$), the reinforcement is subject to much higher stresses ($\sigma_{1f} \gg \sigma_{1m}$) than the matrix and there is a redistribution of the load. The overall stress $\sigma_1$ can be expressed in terms of the two contributions being made

$$\sigma_1 = (1 - f)\sigma_{1m} + f\sigma_{1f}$$

(2.28)

The Young's modulus of the composite can now be written

$$E_1 = \frac{\sigma_1}{\varepsilon_1} = \frac{[(1 - f)\sigma_{1m} + f\sigma_{1f}]}{(\sigma_{1f}/E_f)} = E_f \left[\frac{(1 - f)\sigma_{1m}}{\sigma_{1f}} + f\right]$$

Using the ratio between the stresses in the components given by equation (2.27), this simplifies to

$$E_1 = (1 - f)E_m + fE_f$$

(2.29)
Fig 2.9 Schematic illustration of (a) a composite containing a volume fraction \( f \) of aligned, continuous fibers, and (b) a representation of this as bonded slabs of matrix and fiber material. (c) On applying a stress \( \sigma_1 \) parallel to the fiber axis, the two slabs experience the same axial strain \( \varepsilon_1 \).

Prediction of the transverse of a stiffness of a composite from the elastic properties of the constituents is far more difficult than the axial value. The conventional approach is to assume that the system can again be represented by 'slab model' depicted in Fig (2.9). In the fiber composite shown Fig (2.9a), both 2- and 3- directions are transverse to the fibers. An obvious problem with slab model is that the two transverse directions are not identical; direction 3 is equivalent to the axial direction. In reality, the matrix is subjected to an effective stress intermediate between the full-applied stress operating on the matrix when it is normal to the plane of the slab interface and the reduced value calculated in the parallel model equation for a stress axis parallel to this interface. Before considering this any further, the limiting case of the 'equal stress' model will be examined. Thus,

\[
\sigma_2 = \sigma_{2f} = \varepsilon_{2f} E_f = \varepsilon_{2m} E_m \tag{2.30}
\]

so that the component strains can be expressed in terms of the applied stress. The overall net strain can be written as

\[
\varepsilon_2 = f \varepsilon_{2f} + (1-f) \varepsilon_{2m} \tag{2.31}
\]

from which the composite modulus is given by
Time Dependent Effects of Reinforced Concrete Subjected to Torsional Loading

\[ E_2 = \frac{\sigma_2}{\varepsilon_2} = \frac{\sigma_{2f}}{f \varepsilon_{2f} + (1 - f) \varepsilon_{2m}} \]

Substituting expressions for \( \varepsilon_{2f} \) and \( \varepsilon_{2m} \) derived from equation (2.30) gives

\[ E_2 = \left[ \frac{f}{E_f} + \frac{(1 - f)}{E_m} \right]^{-1} \quad (2.32) \]

It is instructive to consider the true nature of the stress and strain distributions during this type of loading when the 'slab' reinforcement is replaced by fibers, close to them and in line along the loading direction, are subjected to a high stress similar to that carried by the reinforcement – as depicted in Fig (2.10b). The regions of the matrix 'parallel' with the fibers, i.e. adjacent laterally, are constrained to have the same strain as the reinforcement and carry a low stress as illustrated by Fig (2.9c).

Fig 2.10 Schematic showing (a) the slab model and (b) the 'equal stress' assumption during transverse stressing.

The shear modules of composites can be predicted in a similar way to the axial and transverse stiffnesses, using the slab model. This is done by evaluating the net shear strain induced when a shear stress is applied to the composite, in terms of the individual displacement contributions from the two constituents. It is important to understand the nomenclature convention, which used. A shear stress designated \( \tau_{ij} \) \((i \neq j)\) refers to a stress acting in the \( i \)-direction on the plane with a normal in the \( j \)-direction. Similarly,
a shear strain \( \gamma_{ij} \) is a rotation towards the \( i \) - direction of the \( j \) - axis. The shear modulus \( G_{ij} \) is the ratio of \( \tau_{ij} \) to \( \gamma_{ij} \). As the composite body is not rotating, the condition \( \tau_{ij} = \tau_{ji} \) must hold. In addition, \( G_{ij} = G_{ji} \) so that \( \gamma_{ij} = \gamma_{ji} \). Since the 2- and 3- directions are equivalent in the aligned fiber composite, it follows that there are two shear modules, because \( G_{12} = G_{21} = G_{13} = G_{31} \neq G_{23} = G_{32} \).

There are also two shear modules for the slab model (Fig 2.11)), but these are unlikely to correspond closely with the values for the fiber composite. The stresses \( \tau_{12} \) and \( \tau_{21} \) are assumed to operate equally within both of the constituents. The derivation is similar to the equal stress treatment leading to equation (2.32) for transverse stiffness

\[
\tau_{12} = \gamma_{12f}G_f = \gamma_{12m}G_m
\]

where \( \gamma_{12f} \) and \( \gamma_{12m} \) are the individual shear strains in the two constituents. The total shear strain is found by summing the two contributions to the total shear displacement in \( 1 \) – direction

\[
\gamma_{12} = \frac{u_{1f} + u_{1m}}{f + (1 - f)} = f\gamma_{12f} + (1 - f)\gamma_{12m}
\]

\[
\therefore G_{12} = \frac{\tau_{12}}{\gamma_{12}} = \frac{\tau_{12f}}{f\gamma_{12f} + (1 - f)\gamma_{12m}} = \left[ \frac{f}{G_f} + \frac{(1 - f)}{G_m} \right]^{-1}
\]

\[
i.e. \quad G_{12} = \left[ \frac{f}{G_f} + \frac{(1 - f)}{G_m} \right]^{-1} \quad (2.33)
\]

The other shear modulus shown by the slab model, \( G_{13} = G_{31} \) in Fig 2.11, corresponds to an equal shear strain condition and is analogous to the axial tensile modulus case. It is readily shown that

\[
G_{13} = fG_f + (1 - f)G_m \quad (2.34)
\]

which is similar to equation (2.29). It may be noted that neither the equal stress condition nor the equal strain condition are close to the situation during shearing of the fiber composite, in which the strain partitions unevenly within the matrix. Therefore neither of the above equations is expected to be very reliable, particularly the equal strain expression.
Fig 2.11 Schematic illustration of how the shear modules are defined for a real fiber composite and for the slab model representation, indicating how stress and strain partition between the two constituents in each case.
2.5 Saint-Venant Torsion Theory

To solve the general equations of elasticity together with the given boundary conditions, the direct method of solution may not always be possible. For the solution of many problems, the inverse method and the semi-inverse method have been found to be useful. In the inverse method, any functions satisfying the differential equations are examined to see what boundary conditions these functions will satisfy. In this way useful solutions may be obtained. In the semi-inverse method, first proposed by Saint-Venant, simplifying assumptions are made regarding the stress components or the displacements so that the differential equations are simplified to such an extent that they may be solved without too much mathematical difficulty. These simplifying assumptions will evidently limit the generality of the resulting solution, but they can usually be made in such a way that the required solution can still be obtained. For example, in the case of torsion of a prismatic bar, which we shall discuss, we shall assume the displacements \( u, v, w \) to be of certain form, thus reducing the governing equations to one differential equation. Owing to these assumptions, we shall be able to obtain not the solution for non-prismatic bars under torsion but only that for bars of constant cross section. The semi-inverse method has proved to be one of the most useful methods in solving elasticity problems.

Suppose that a prismatic bar, of length \( L \), is fixed at one end in the \( x-y \) plane, while the other end is acted upon by a couple whose moment lies along the \( z \)-axis (Fig. 2.12).

Fig 2.12 A prismatic bar subjected to torsional load.

We assume that the fixed end is prevented from rotating but that both ends are allowed to extend or contract in the \( z \) direction. Under the action of the bar will be twisted and the generations of the cylinder will be deformed into helical curves. The
amount of rotation at any section will depend on the distance of the section from the fixed end. Since the deformation is small, it is reasonable to assume that the angle of twist $\alpha$ is proportional to the distance of the section from the fixed end. Thus

$$\alpha = \theta z \quad (2.35)$$

where $\theta$ is the angle of twist per unit length. We shall assume that the angle of twist $\alpha$ is small. Consider a section of the bar, which is at a distance $z$ from the fixed end. A point $P$ which has its coordinates $x$, $y$, $z$ is displaced to $P'$ $(x + u, y + v, z + w)$ after deformation. The projection of $P'$ on the $x$-$y$ plane, $P_1'$, is shown in Fig. 2.13. Assume that on the $x$-$y$ plane $P$ is rotated to $P_1'$ through the angle of twist $\alpha$ with $OP \equiv OP_1' = r$.

![Fig 2.13 Deformation of point P to P'.](image)

If $\alpha$ is small, we have $\cos \alpha \equiv 1$ and $\sin \alpha \equiv \alpha$. Thus,

$$u = r \cos(\beta + \alpha) - r \cos \beta = r \cos \alpha \cos \beta - r \sin \alpha \sin \beta - r \cos \beta \equiv -y\alpha$$

$$v = r \sin(\beta + \alpha) - r \sin \beta = r \sin \alpha \cos \beta + r \cos \alpha \sin \beta - r \sin \beta \equiv x\alpha$$

Combining with equation (2.35), we obtain

$$u = -\theta yz \quad v = \theta xz \quad (2.36)$$

Which shall be our assumed from $u$ and $v$. For the present we shall make no assumptions about $w$, except that $w$ is a function of $x$, $y$ only and is independent of $z$. Thus, we may write
\[ w = \partial \Psi(x, y) \quad (2.37) \]

Where \( \Psi(x, y) \) is some function of \( x \) and \( y \). Since \( w \) defines the warping of the end surfaces. \( \Psi \) may be called a warping function. Our object is to determine whether or not the assumed displacement, together with some yet unknown function \( \Psi \), will result in a given boundary conditions. The boundary conditions in this case are that there should be only pure torsional moments on the two ends and no forces acting on the lateral surface of the bar.

The above values of the displacements give

\[ \varepsilon_x = \varepsilon_y = \varepsilon_z = \gamma_{xy} = 0 \quad \gamma_{yx} = \theta \left( \frac{\partial \Psi}{\partial y} + x \right) \quad \gamma_{zx} = \theta \left( \frac{\partial \Psi}{\partial x} - y \right) \quad (2.38) \]

And

\[ \sigma_x = \sigma_y = \sigma_z = \tau_{xy} = 0 \quad \tau_{yx} = G \theta \left( \frac{\partial \Psi}{\partial y} + x \right) \quad \tau_{zx} = G \theta \left( \frac{\partial \Psi}{\partial x} - y \right) \quad (2.39) \]

A substitution of these values in the equilibrium equations shows that these equations will be satisfied if \( \Psi(x, y) \) satisfies the equation

\[ \nabla^2 \Psi = \frac{\partial^2 \Psi}{\partial x^2} + \frac{\partial^2 \Psi}{\partial y^2} = 0 \quad (2.40) \]

Throughout the cross section of he bar, where \( \nabla^2 = (\partial^2 / \partial x^2) + (\partial^2 / \partial y^2) \) is the Laplace operator.

Let us now investigate the boundary conditions. Since \( \bar{X} = \bar{Y} = \bar{Z} = 0 \) on the lateral surface of the bar, the last equation becomes

\[ \left( \frac{\partial \Psi}{\partial x} - y \right) l + \left( \frac{\partial \Psi}{\partial y} + x \right) m = 0 \quad \text{on } S \quad (2.41) \]

where \( S \) is the boundary of the cross section the bar. The other two equations are identically zero because of (2.39) and because on the lateral surface \( n = \cos N z = 0 \).

On the other two bounding surfaces, viz., the ends of the bar defined by the planes \( z = 0 \) and \( z = L \), we must show that the distribution of stresses given (2.39) is equivalent.
to a torsional couple and that there is no resultant force. The resultant force in the x direction is given by

\[ \iint_{R} \tau_{x} \, dx \, dy = G\theta \iint_{R} \left( \frac{\partial \Psi}{\partial x} - y \right) \, dx \, dy \]  

(2.42)

And this can be written as

\[ G\theta \iint_{R} \left[ \frac{\partial}{\partial x} \left( x \left( \frac{\partial \Psi}{\partial x} - y \right) \right) + \frac{\partial}{\partial y} \left( x \left( \frac{\partial \Psi}{\partial y} + x \right) \right) \right] \, dx \, dy \]  

(2.43)

Since

\[
\frac{\partial}{\partial x} \left[ x \left( \frac{\partial \Psi}{\partial x} - y \right) \right] + \frac{\partial}{\partial y} \left[ x \left( \frac{\partial \Psi}{\partial y} + x \right) \right] = \left( \frac{\partial \Psi}{\partial x} - y \right) + x \frac{\partial^{2} \Psi}{\partial x^{2}} + x \frac{\partial^{2} \Psi}{\partial y^{2}}
\]

\[
= \left( \frac{\partial \Psi}{\partial x} - y \right) + x \left( \frac{\partial^{2} \Psi}{\partial x^{2}} + \frac{\partial^{2} \Psi}{\partial y^{2}} \right)
\]

(2.37)

Where the last step is achieved because \( \left( \frac{\partial^{2} \Psi}{\partial x^{2}} \right) + \left( \frac{\partial^{2} \Psi}{\partial y^{2}} \right) = 0 \) according to equation (2.37).

Now if \( f \) is any function of \( x \) and \( y \), we have (Fig 2.14)

\[
\iint_{R} \frac{\partial f}{\partial x} \, dx \, dy = \int \left( \int \frac{\partial f}{\partial x} \, dx \right) \, dy = \int (f_{1} - f_{2}) \, dy
\]

where \( f_{1} \) is the value of \( f \) on the right side part of the boundary and \( f_{2} \) is the values on the left side. The integration with respect to \( y \) now has to be carried out on the boundary curve from \( y = y_{A} \) to \( y = y_{B} \). If we integrate \( f \) along the curve following a counterclockwise direction, \( dy \) is positive on the right side of the boundary and is negative on the left side. As a result both \( f_{1} \, dy \) and \( f_{2} \, dy \) become positive, and we may write

\[
\iint_{R} \frac{\partial f}{\partial x} \, dx \, dy = \frac{1}{2} \int f \, dy
\]  

(2.44)

Similarly,
\[ \int f \frac{\partial f}{\partial y} \, dx \, dy = -\int f \, dx \]  \hspace{1cm} (2.45)

Fig 2.14 Boundary for \( f_1 \) and \( f_2 \).

Using equations (2.43) and (2.44), the expression (2.45) becomes

\[ \oint_s \left[ x \left( \frac{\partial \Psi}{\partial x} - y \right) \, dy - x \left( \frac{\partial \Psi}{\partial y} + x \right) \, dx \right] = \oint_s \left[ \frac{\partial \Psi}{\partial x} - y \right] \frac{dy}{ds} \left( \frac{\partial \Psi}{\partial y} + x \right) \frac{dx}{ds} \, ds \]  \hspace{1cm} (2.46)

Since an outward normal \( N \) is defined as positive and the positive direction of \( s \) is counterclockwise, we have from Fig. (2.14b)

\[ l = \cos N_x = \frac{dx}{dN} = \frac{dy}{ds} \]
\[ m = \cos N_y = \frac{dy}{dN} = \cos(\pi - \beta) = -\cos \beta = -\frac{dx}{ds} \]  \hspace{1cm} (2.47)

Expression (2.46) then becomes

\[ \oint_s \left[ \left( \frac{\partial \Psi}{\partial x} - y \right) l + \left( \frac{\partial \Psi}{\partial y} + x \right) m \right] ds = 0 \]
The last step is obtained because \( \int ((d\Psi / dx) - y) + \int (d\Psi / dy) + x \) is equal to zero on \( S \) according to (2.41). Thus, we have proved that

\[
\int_{R} \tau_{s} \, dxdy = 0
\]

In a similar way, we can also show that the \( y \) component of the resultant force is zero, i.e.,

\[
\int_{R} \tau_{y} \, dxdy = 0
\]

so that the resultant force acting on the ends of the cylinder vanishes.

The resultant torsional moment \( T \) on the ends of the bar due to the assumed stress distribution is

\[
T = \int_{R} \left( x \tau_{xy} - y \tau_{s} \right) \, dxdy = G \theta \int_{R} \left( x^2 + y^2 + x \frac{\partial \Psi}{\partial y} - y \frac{\partial \Psi}{\partial x} \right) \, dxdy
\]

(2.48)

The integral appearing in (2.48) depends on the torsion function \( \Psi \) and hence on the cross section \( R \) of the bar. Letting

\[
J = \int_{R} \left( x^2 + y^2 + x \frac{\partial \Psi}{\partial y} - y \frac{\partial \Psi}{\partial x} \right) \, dxdy
\]

(2.49)

We have

\[
T = GJ \theta
\]

(2.50)

Where \( J \) is called torsional constant. Equation (2.50) shows that the torsional moment is proportional to the angle of twist per unit length, so that the product \( GJ \) provides a measure of the rigidity of a bar subjected to torsion and is called the torsional rigidity of the bar.

**Torsion of rectangular Bars**

Let the sides of the rectangular cross section be \( 2a \) and \( 2b \) and let origin be at the center of the rectangular with the coordinate axes parallel to its sides (Fig 2.15). Our equations are, as before,
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\[ \frac{\partial^2 \Psi}{\partial x^2} + \frac{\partial^2 \Psi}{\partial y^2} = 0 \]

Fig 2.15 Layout for a rectangular section.

Over the whole rectangle and on the boundary.

\[ \left( \frac{\partial \Psi}{\partial x} - y \right) + \left( \frac{\partial \Psi}{\partial y} + x \right) m = 0 \]

Now, on the boundary lines \( x = \pm a \) or AB and CD, we have \( l = \pm 1 \) and \( m = 0 \), and on the boundary lines BC and AD we have \( l = 0 \) and \( m = \pm 1 \). The boundary condition (2.41) may be rewritten as

\[ \frac{\partial \Psi}{\partial x} = y \quad \text{on} \quad x = \pm a \]
\[ \frac{\partial \Psi}{\partial y} = -x \quad \text{on} \quad y = \pm b \] \hspace{1cm} (2.51)

These boundary conditions can be transformed into more convenient forms if we introduce a new function \( \Psi_1 \) such that

\[ \Psi = xy - \Psi_1 \] \hspace{1cm} (2.52)

It is easy to verify that, in terms of \( \Psi_1 \), our governing equation is

\[ \frac{\partial^2 \Psi_1}{\partial x^2} + \frac{\partial^2 \Psi_1}{\partial y^2} = 0 \] \hspace{1cm} (2.53)
Over the whole rectangle and the boundary conditions become

\[
\frac{\partial \Psi_1}{\partial x} = 0 \quad \text{on} \quad x = \pm a \quad (2.54)
\]

\[
\frac{\partial \Psi_1}{\partial y} = 2x \quad \text{on} \quad y = \pm b \quad (2.55)
\]

We shall assume that the solution of equation (2.53) can be expressed in the form of an infinite series

\[
\Psi_1 = \sum_{n=0}^{\infty} X_n(x)Y_n(y) \quad (2.56)
\]

Where each term of the series satisfies the differential equation \( X_n(x) \) and \( Y_n(x) \) are, respectively, functions of \( x \) alone and \( y \) alone. It is obvious that if the solution \( \Psi_1 \) cannot be expressed in the form of (2.56), we shall not be able to solve for the functions \( X_n(x) \) and \( Y_n(x) \) have these functions satisfy the boundary conditions.

Substituting \( X_n(x), \ Y_n(x) \) in equation (2.53) and denoting the derivatives by primes, we obtain

\[
X_n'(x)Y_n(y) + X_n(x)Y_n'(y) = 0 \quad \text{or} \quad \frac{X_n'(x)}{X_n(x)} = -\frac{Y_n'(y)}{Y_n(y)}
\]

Since the expression on the left-hand side the above equation is a function of \( x \) alone and the one on the right-hand side depends only on \( y \), the equality can be fulfilled only if the expression on either side is equal to a constant; call it \( K_n \). We are thus led to a pair of ordinary differential equations.

\[
\frac{d^2X_n}{dx^2} + k_nX_n = 0 \quad \text{and} \quad \frac{d^2Y_n}{dy^2} + k_nY_n = 0
\]

These differential equations can be easily solved the well-known methods of solving linear ordinary differential equations with constant coefficients. Their solutions are
\[ X_n = c_1 \sin k_n x + c_2 \cos k_n x \quad (2.57) \]
\[ Y_n = c_3 \sinh k_n y + c_4 \cosh k_n y \quad (2.58) \]

Now let us examine the boundary condition (2.55). First we observe that

\[ \frac{\partial \Psi_1}{\partial y} = \sum_{n=0}^{\infty} X_n(x) Y_n'(y) = 2x \]

Must have the same value for \( y = +b \) or \(-b \). This condition can be satisfied if \( Y_n'(y) \) are symmetric functions in \( y \). Second, for \( y = \pm b \), we have

\[ \sum_{n=0}^{\infty} Y_n'(b) X_n(x) = 2x \]

This condition is satisfied if \( X_n(x) \) are antisymmetric functions in \( x \). From these considerations, we find that \( c_2 = c_4 = 0 \). Now the condition (2.54) is satisfied if \( X_n(\pm a) = 0 \) or

\[ c_1 k_n \cos k_n a = 0 \]

From which we find

\[ k_n = \frac{(2n+1)\pi}{2a} \]

Since \( c_1 \) and \( c_3 \) are arbitrary constants, we may write \( \Psi_1 \) in the following form,

\[ \Psi_1 = \sum_{n=0}^{\infty} A_n \sin k_n x \sinh k_n y \quad (2.59) \]

Where \( k_n = \frac{(2n+1)\pi}{2a} \) and the constant \( A_n \) are to be determined so as to satisfy the boundary condition (2.55).
Differentiating $\Psi_1$ with respect to $y$, and substituting $y = \pm b$, we have from (2.55) that

$$2x = \sum_{n=0} A_n k_n \cosh k_n b \sin k_n x = \sum_{n=0} B_n \sin k_n x \quad (2.60)$$

Where to simplify the writing we have introduced the symbol

$$B_n = A_n k_n \cosh k_n b$$

The coefficients $A_n$ can now be determined by utilizing the scheme used in expanding functions in Fourier series. If we multiply both sides of equation (2.60) by $\sin (2m+1)\pi x/2a$ and integrate term by term with respect to $x$, recalling that

$$\sin k_n x \sin k_m x = \frac{1}{2} \left[ \cos (k_n - k_m)x - \cos (k_n + k_m)x \right]$$

$$= \frac{1}{2} \left[ \frac{\cos (n - m)\pi x}{a} - \frac{\cos (n + m + 1)\pi x}{a} \right]$$

$$\sin^2 k_m x = \frac{1}{2} \left[ 1 - \cos \left( \frac{(2m + 1)\pi x}{2} \right) \right]$$

we find

$$\int_a^0 \sin k_n x \sin k_m x dx = 0 \quad \text{if} \quad m \neq n$$

$$= a \quad \text{if} \quad m = n$$

and

$$\int_a^0 2x \sin k_m x dx = \int_a^0 B_n \sin^2 k_m x dx$$

Upon evaluating the integrals in the above expression, we obtain

$$B_m = \frac{16 (-1)^n a}{\pi^3 (2m + 1)^2} \quad \text{or} \quad A_n = \frac{32 (-1)^n a^2}{\pi^3 (2n + 1)^3 \cosh k_n b}$$

So that the solution is
\[ \Psi = xy - \Psi_1 \]
\[ = xy - \frac{32a^2}{\pi^3} \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^3} \frac{1}{\cosh k_n b \sinh k_n y} \]  
\[ (2.61) \]

The torsional constant \( J \) can be evaluated from formula (2.49) as follows:

\[ J = \int_{y=-b}^{y=a} \int_{x=-a}^{x=a} \left( x^2 + y^2 + x \frac{\partial \Psi}{\partial y} - y \frac{\partial \Psi}{\partial x} \right) dx dy \]
\[ = \frac{8a^3b}{3} \left[ 1 + \frac{96}{\pi^4} \sum_{n=0}^{\infty} \frac{1}{(2n+1)^4} - \frac{384a}{\pi^3b} \sum_{n=0}^{\infty} \frac{\tanh k_n b}{(2n+1)^5} \right] \]

Since
\[ \sum_{n=0}^{\infty} \frac{1}{(2n+1)^4} = \frac{\pi^4}{96} \]

We have the formula

\[ J = 16a^3b \left[ \frac{1}{3} - \frac{64a}{\pi^3b} \sum_{n=0}^{\infty} \frac{\tanh k_n b}{(2n+1)^5} \right] = \kappa a^3b \]  
\[ (2.62) \]

For various \( b/a \) ratios, the corresponding values of \( \kappa \) are given in Table (2.3). The series in equation (2.62) can be written as

\[ \sum_{n=0}^{\infty} \frac{\tan k_n b}{(2n+1)^5} \]

We note that \( \sum_{n=1}^{\infty} \frac{\tanh k_n b}{(2n+1)^5} \) is less than \( \sum_{n=1}^{\infty} \frac{1}{(2n+1)^5} = 0.0046 \), while \( \tanh \left( \frac{\pi b}{2a} \right) \)
\[ \geq 0.917 \text{ if } b \geq a. \] Thus, the first term of the series gives the value of the sum to within 1/2 per cent, and for practical purposes, we can use the approximate formula
\[ J = 16a^3 b \left( \frac{1}{3} - \frac{64}{\pi^2} \frac{a}{b} \tanh \frac{\pi b}{2a} \right) \] (2.63)

With some calculation we find that the shearing stresses are given by the following formulas:

\[ \tau_{x} = \frac{T}{J} \left( \frac{\partial \Psi}{\partial x} - y \right) = -\frac{16Ta}{J\pi^2} \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^2} \frac{\sinh k_n y}{\cosh k_n b} \cos k_n x \]
\[ \tau_{y} = \frac{T}{J} \left( \frac{\partial \Psi}{\partial y} + x \right) = \frac{T}{J} \left[ 2x - \frac{16a}{\pi^2} \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^2} \frac{\cosh k_n y}{\cosh k_n b} \sin k_n x \right] \] (2.64)

Assuming that \( b > a \), it can be shown that the maximum shearing stress is at the midpoints of the long sides \( x = \pm a \) of the rectangle. Substituting \( x = a, y = 0 \) in (2.64), we find \( \tau_{x} = 0 \) and

\[ \tau_{\text{max}} = \tau_{y} = \frac{2Ta}{J} \left[ 1 - \frac{8}{\pi^2} \sum_{n=0}^{\infty} \frac{1}{(2n+1)^2} \frac{1}{\cosh k_n b} \right] = \kappa_1 \frac{Ta}{J} \] (2.65)

The infinite series on the right side, which we denote by \( \kappa_1 / 2 \), converges very rapidly when \( b > a \), and there is no difficulty in calculating \( \tau_{\text{max}} \) with sufficient accuracy for any particular value of \( b/a \). For various \( b/a \) ratios, the corresponding values of \( \kappa_1 \) are included in Table (2.3). Substituting the values of \( J \) from (2.62) into (2.65), we have

\[ \tau_{\text{max}} = \kappa_2 \frac{T}{a^2 b} \] (2.66)

Where \( \kappa_2 \) is another numerical factor, several values of which are given in Table (2.3).
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Table 2.3 St Venant’s coefficients for rectangular sections.
2.6 Membrane Analogy

From the example worked in the previous section, it becomes evident that a rigorous solution of the torsion problem for a bar with more complicated cross-sectional shape is likely to be very difficult. In developing approximate formulas for the torsional constants of many engineering sections, the so-called membrane analogy has proved very valuable. The membrane analogy is based on the mathematical analogy between the torsion problem and the behaviour of a stretched elastic membrane subjected to a uniform lateral pressure.

![Fig 2.16 The contour lines of the surface of φ.](image)

![Fig 2.17 A thin homogeneous membrane.](image)

Let a thin homogenous membrane (Fig 2.17) be stretched with uniform tension and fixed at its edge, which is a given curve in the x, y plane. When the membrane is subjected to a uniform lateral pressure \( p \), it will undergo a small displacement \( z \), where \( z \) is a function of \( x \) and \( y \). Consider the equilibrium of an infinitesimal element ABCD of
the membrane after deformation. Let \( F \) be the uniform tension per unit length of the membrane. On the side AD, \( F \) is inclined at an angle \( \beta \) with respect to the axis. Since the deformation is small, \( \beta \equiv \partial z / \partial x \). The deflection \( z \) varies from point to point; therefore on the side BC, \( F \) is now inclined at an angle \( \beta + (\partial \beta / \partial x) dx \equiv (\partial z / \partial x) + (\partial^2 z / \partial x^2) dx \).

Similarly, on the sides AB and CD, the tensile forces are inclined at the angles \( \partial z / \partial y \) and \((\partial^2 z / \partial y^2) dy \), respectively. Summing up the \( z \) components of the forces acting on the four sides, we have

\[
-(Fdy) \frac{\partial z}{\partial x} + (Fdy) \left( \frac{\partial z}{\partial x} + \frac{\partial^2 z}{\partial x^2} dx \right) - (Fdx) \frac{\partial z}{\partial y} + (Fdx) \left( \frac{\partial z}{\partial y} + \frac{\partial^2 z}{\partial y^2} dy \right) + p dx dy = 0
\]

From which

\[
\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = -\frac{P}{F} \quad \text{in} \quad R \quad (2.67)
\]

On the boundary, the deflection of the membrane is zero. The boundary condition is therefore

\[
z = 0 \quad \text{on} \quad S \quad (2.68)
\]

Now let us return to our torsion problem. The governing differential equation is

\[
\frac{\partial^2 \Psi}{\partial x^2} + \frac{\partial^2 \Psi}{\partial y^2} = 0 \quad \text{in} \quad R
\]

And the boundary condition is

\[
\left( \frac{\partial \Psi}{\partial x} - y \right) + \left( \frac{\partial \Psi}{\partial y} + x \right) m = 0 \quad \text{on} \quad S
\]

Comparing these relations with (2.67) and (2.68), we find that apparently they are not analogous. However, they can be reduced to an analogous from if we introduce a new function \( \psi(x, y) \) such that
\[
\begin{align*}
\frac{\partial \Psi}{\partial x} &= \frac{\partial \Psi}{\partial y} + y, & \frac{\partial \Psi}{\partial y} &= -\frac{\partial \Psi}{\partial x} - x \\
\frac{\partial^2 \Psi}{\partial x^2} &= \frac{\partial^2 \Psi}{\partial x \partial y}, & \frac{\partial^2 \Psi}{\partial y^2} &= -\frac{\partial^2 \Psi}{\partial x \partial y}
\end{align*}
\]

(2.69)

From (2.69) we have

\[
\frac{\partial^2 \Psi}{\partial x^2} + \frac{\partial^2 \Psi}{\partial y^2} = \frac{\partial^2 \Psi}{\partial x \partial y} - \frac{\partial^2 \Psi}{\partial x \partial y} = 0
\]

The differential equation is identically satisfied, since

That is, if \( \psi \) is defined as in (2.69) the equilibrium equations are satisfied identically.

In terms of \( \psi \), the shearing stresses \( \tau_{xz} \) and \( \tau_{yz} \) are

\[
\begin{align*}
\tau_{xz} &= \frac{T}{J} \left( \frac{\partial \Psi}{\partial x} - y \right) = \frac{T}{J} \frac{\partial \Psi}{\partial y}, & \tau_{yz} &= \frac{T}{J} \left( \frac{\partial \Psi}{\partial y} + x \right) = -\frac{T}{J} \frac{\partial \Psi}{\partial x}
\end{align*}
\]

(2.70)

If \( \psi \) is obtained, we can compute the shearing stresses by a simple differentiation. \( \psi \) is therefore the stress function, and the solution of \( \psi \) is equivalent to the solution of the stresses. In such a case, the compatibility equation must be used. With the stress system

\[
\begin{align*}
\sigma_x = \sigma_y = \sigma_z = \tau_{xy} = 0
\end{align*}
\]

\[
\tau_{xz} = \frac{T}{J} \frac{\partial \psi}{\partial y}, & \tau_{yz} = -\frac{T}{J} \frac{\partial \psi}{\partial x}
\]

The corresponding strain components are

\[
\begin{align*}
\varepsilon_x = \varepsilon_y = \varepsilon_z = \gamma_{xy} = 0
\end{align*}
\]

\[
\begin{align*}
\gamma_{xz} &= \frac{T}{GJ} \frac{\partial \psi}{\partial y}, & \gamma_{yz} &= -\frac{T}{GJ} \frac{\partial \psi}{\partial x}
\end{align*}
\]
Substituting into the compatibility equations, we find that the first three and the last equations are identical satisfied. The fourth and the fifth equations become

\[
\frac{\partial}{\partial x} \left( -\frac{\partial \gamma_{yx}}{\partial x} + \frac{\partial \gamma_{xx}}{\partial y} \right) = 0 \quad \frac{\partial}{\partial y} \left( \frac{\partial \gamma_{yx}}{\partial x} - \frac{\partial \gamma_{xx}}{\partial y} \right) = 0
\]

Integrating, we have

\[
-\frac{\partial \gamma_{yx}}{\partial x} + \frac{\partial \gamma_{xx}}{\partial y} = \text{cons} \tan t = c_1
\]

This constant can be determined by substituting into the above equation

\[
\gamma_{yx} = \frac{\tau_{yx}}{G} = \frac{T}{GJ} \left( \frac{\partial \Psi}{\partial x} - y \right) \quad \gamma_{yx} = \frac{\tau_{yx}}{G} = \frac{T}{GJ} \left( \frac{\partial \Psi}{\partial x} + x \right)
\]

From which we find

\[
\frac{T}{GJ} \left( -\frac{\partial^3 \Psi}{\partial x \partial y} - 1 + \frac{\partial^2 \Psi}{\partial x \partial y} - 1 \right) = c_1
\]

Or \( c_1 = -2T / GJ \). With this value of \( c_1 \), in terms of \( \Psi \), the above compatibility equation becomes

\[
\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = -2 \quad \text{in} \quad R \quad (2.71)
\]

Which is the differential equation that \( \psi \) must satisfy. It may be pointed out that equation (2.71) is directly obtainable by differentiating equations (2.69) and then eliminating \( \Psi \) between these equations. However, this will conceal the fact that (2.71) is actually the compatibility equation.
In terms of $\psi$, the boundary condition becomes

$$\frac{\partial \psi}{\partial y} l - \frac{\partial \psi}{\partial x} m = 0 \quad \text{on} \quad S$$

We have already shown that

$$l = \frac{dx}{dN} = \frac{dy}{ds} \quad \quad m = \frac{dy}{dN} = -\frac{dx}{ds}$$

The boundary condition can therefore be written as

$$\frac{\partial \psi}{\partial y} \frac{dy}{ds} + \frac{\partial \psi}{\partial x} \frac{dx}{ds} = \frac{d\psi}{ds} = 0$$

or

$$\psi = \text{constant} = c_2 \quad \text{on} \quad S \quad (2.72)$$

We note that, in computing the stresses, only the derivatives of $\psi$ are of interest and the value of the constant $c_2$ in (2.72) is irrelevant to the problem. For that reason, we may let $c_2 = 0$. Thus, the torsion problem is reduced to finding the function $\psi$ such that

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = -2 \quad \text{in} \quad R \quad (2.71)$$

$$\psi = 0 \quad \text{on} \quad S \quad (2.72)$$

Comparing these equations with the membrane equations, we see that they are exactly analogous if $p/F$ is taken to be 2 and if the shape of the boundary of the membrane is the same as the cross section of the bar.

The membrane analogy is useful in the experimental determination of the stress function.

The membrane analogy not only is useful in actually determining the stress but also provides a visual picture of the stress distribution. Fig. (2.18) represents such a
membrane with the contour lines of the deflection surface plotted. Consider any point B on the membrane. Along the contour line, the deflection is constant, i.e.,

From the analogy, we have

$$\frac{\partial \psi}{\partial s} = 0$$

![Diagram of Membrane with Contour Lines](image)

Fig 2.18 Membrane with the contour lines of the deflection surface.

Since

$$\frac{\partial \psi}{\partial s} = \frac{\partial \psi}{\partial y} \frac{dy}{ds} + \frac{\partial \psi}{\partial x} \frac{dx}{ds} = \frac{J}{T} \left( \frac{\tau_{xy}}{ds} - \tau_{yz} \frac{dx}{ds} \right) = \frac{J}{T} \left( \frac{\tau_{xy}}{dN} + \tau_{yz} \frac{dx}{dN} \right) = \frac{J}{T} \tau_{\psi}$$

it follows that the component of the shearing stress normal to the contour line is zero. In other words, the shearing stress at a point B in the twisted bar is the direction of the tangent to the contour line through this point. The magnitude of the resultant shearing stress $\tau$ at B can now be found from the following formula:

$$\tau = \tau_{\psi} \frac{dy}{ds} + \tau_{\alpha} \frac{dx}{ds} = -\frac{T}{J} \left( \frac{\partial \psi}{\partial x} \frac{dx}{dN} + \frac{\partial \psi}{\partial y} \frac{dy}{dN} \right) = -\frac{T}{J} \frac{d\psi}{dN}$$

Thus the slope of the membrane normal to the contour line gives the magnitude of the shearing stress at B, and therefore the maximum shearing stress occurs at the points where the contour lines are closest to each other. From the surface of the membrane, it
can be seen that the maximum slope occurs on the boundary. It can therefore be concluded that the maximum shearing stresses also occur on the boundary of the bar.

We shall now proceed to derive the expression of the torsional constant $J$ in terms of $\psi$. From formula (2.49),

$$J = \iint_R \left( x^2 + y^2 + x \frac{\partial \psi}{\partial y} - y \frac{\partial \psi}{\partial x} \right) dx dy$$

$$= -\iint_R \left( x \frac{\partial \psi}{\partial x} + y \frac{\partial \psi}{\partial y} \right) dx dy$$

$$= -\iint_R \left[ \frac{\partial}{\partial x} (x\psi) + \frac{\partial}{\partial y} (y\psi) - 2\psi \right] dx dy$$

$$= -\oint_S x\psi dy + \oint_S y\psi dx + \iint_R 2\psi dx dy$$

$$= 2 \iint_R \psi dx dy \quad (2.73)$$
2.7 Torsion with Thin-Walled Hollow Sections

In terms of the stress function \( \psi \), we have shown that on the boundary \( \psi \) must be a constant. When we have a solid section, we may let the constant be zero. When the section is bounded by two closed curves, as in Fig 2.19, we shall again lose no generality by assuming that \( \psi \) vanishes on the outer boundary \( S_1 \) but we cannot assume it also vanishes on the inner boundary \( S_2 \), although we know that it has a constant value there. Because of this unknown constant, we need one additional equation may be obtained from the condition that the displacements must be single-valued.

![Figure 2.19 The shear distribution for a singly hollow thin tube.](image)

From previous equations, we have

\[
\tau_{\alpha} = G\theta \left( \frac{\partial \psi}{\partial x} - y \right) = G \left( \frac{\partial w}{\partial x} - \theta y \right) \quad \tau_{\gamma} = G\theta \left( \frac{\partial \psi}{\partial y} + x \right) = G \left( \frac{\partial w}{\partial y} + \theta x \right)
\]

Let us now calculate the integral \( \oint \tau ds \) along the inner boundary.

\[
\oint_{S_2} \tau ds = \oint_{S_1} \left( \tau_{\alpha} \frac{dx}{ds} + \tau_{\gamma} \frac{dy}{ds} \right) ds
\]

\[
= G\oint_{S_1} \left( \frac{\partial w}{\partial s} + \frac{\partial w}{\partial s} \right) + G\theta \oint_{S_2} (x dy - y dx)
\]

From the condition that \( w \) is a single-valued function and the integration is taken round a closed curve, the first integral vanishes. The second integral is equal to twice the area enclosed by \( S_2 \). Hence

\[
\oint_{S_2} \tau ds = 2G\theta A_2 \quad (2.74)
\]
Where \( A_2 \) is the area enclosed by \( S_2 \).

Let us now return to the membrane analogy. If we replace the membrane inside \( S_2 \) by a weightless flat plate (Fig. 2.19), the equation of equilibrium of the plate is

\[
\oint_{S_2} F \frac{\partial z}{\partial n} \, ds = pA_2 \tag{2.75}
\]

Where \( F \), \( z \) are surface tension and deflection of the membrane as defined. Since

\[
\frac{\partial z}{\partial n} = \frac{p}{2F} \frac{\partial \psi}{\partial n} = \frac{p}{2F} \frac{\tau}{G\theta}
\]

We have from equation (2.75)

\[
\oint_{S_2} F \frac{p}{2F} \frac{\tau}{G\theta} \, ds = pA_2 \quad \oint_{S_2} \tau \, ds = 2G\theta A_2
\]

we obtained as equation (2.74). Thus if we have a hollow section, we may consider the membrane as stretched over the outer boundary and a weightless flat plate on the inner boundary.

In Fig 2.19, \( BB' \) and \( CC' \) are the levels of the outer and inner boundary, \( BC \) and \( B'C' \) are the cross section of the membrane stretched between these boundaries. If the wall is thin, \( BC \) and \( B'C' \) become approximately straight lines and the variation in slope of the membrane is negligible. This is equivalent to assuming that the shearing stresses are uniform across the thickness of the wall. If we denote by \( h \) the constant value of \( \psi \) on \( S_2 \), from the membrane analogy, \( h \) is the difference in levels of the two boundaries. Let \( t \) be the variable thickness of the wall. The shearing stress at any point is given by the slope of the membrane, or

\[
\tau = \frac{Th}{Jt} \tag{2.76}
\]

The formula for \( J \) must now be modified. In the derivation of equations, the positive normal \( N \) has been taken outward from the cross section. With respect to the inner boundary, the same sign convention must be used, i.e., the positive direction is inward. On \( S_1 \), we have \( \psi = 0 \), and on \( S_2 \) we have \( \psi = h \). Therefore

\[
J = h \oint_{S_2} (xdy - ydx) + 2\iint_R \psi dxdy \tag{2.77}
\]
Where $R$ denotes the area bounded between $S_1$ and $S_2$, or $A_1$. Because the section is thin, the value of $\psi$ in the second integral can be replaced by its average value taken over $S_1$ and $S_2$, namely, $h/2$. We have, therefore,

Where $A$ is the area enclosed by the mean line of the boundaries. Substituting into

$$J = 2h \left(A_2 + \frac{1}{2} A_1 \right) = 2Ah$$

equation (2.76), we have

$$\tau = \frac{T}{2At} \quad (2.78)$$

The angle of twist $\theta$ may be calculated from equation (2.74),

$$\oint_{S} \tau ds = \frac{T}{2A} \oint_{S} \frac{ds}{t} = 2G\theta A$$

from which we find

$$\theta = \frac{T}{4A^2G} \oint_{S} \frac{ds}{t} \quad (2.79)$$

Where $S$ is mean line of boundaries. Equations (2.78) and (2.79) were first obtained by Bredt and are known as Bredt's formulas.

If the cross section of the tubular member has more than two boundaries (Fig 2.20), the portions of membrane inside the inner boundaries can be again replaced by weightless flat plates.
Fig 2.20 The shear distribution for a doubly hollow thin tube.

Assuming that the thickness of the wall is small, we have

$$\tau_1 = \frac{Th_1}{Jt_1}$$
$$\tau_2 = \frac{Th_2}{Jt_2}$$
$$\tau_3 = \frac{T}{J} \frac{h_1 - h_2}{t_3} = \frac{\tau_1 t_1 - \tau_2 t_2}{t_3}$$

(2.80)

Where $h_1$ and $h_2$ are the levels of the inner boundaries $CC'$ and $DD'$ Equation (2.77) becomes

$$J = 2 \iint \psi dx dy + \sum_i 2h_i A_i = 2h_1 A_1 + 2h_2 A_2$$

Where $A_i$ is the area enclosed by the boundary $S_1$ and $A_1, A_2$ are the areas enclosed by the mid-section curves $S_1$ and $S_2$. Therefore

$$T = 2\tau_1 t_1 A_1 + 2\tau_2 t_2 A_2$$

(2.81)

Assume that the thickness $t_1, t_2, t_3$ are constant. Let $s_1, s_2$ and $s_3$ be the lengths of the mid-section curves. By applying Equation (2.74) first over $A_1$ and then over $A_2$, we obtain
\[ \tau_1 s_1 + \tau_3 s_3 = 2G\theta A_1 \quad \tau_2 s_2 - \tau_3 s_3 = 2G\theta A_2 \]  \hspace{1cm} (2.82)

\( \tau_1, \tau_2, \tau_3, \) and \( \theta \) can be calculated by solving equations (2.81) and (2.82) simultaneously.

From Equation (2.80), we note that, in a tubular branch of the cross section, \( \tau t \) is a constant. When several tubular members meet, as at point H, we have

\[ \tau_1 f_1 = \tau_2 f_2 + \tau_3 f_3 \]  \hspace{1cm} (2.83)

This suggests a hydro-dynamical analogy, viz., and the quantity \( q = \tau t \) is analogous to the quantity of ideal liquid circulating in a channel having the same shape as the tubular bar. Equation (2.83) then states that the amount of incoming liquid must be equal to the amount of liquid flowing out. The quantity \( q \) is therefore called shear flow.
2.8 Conformal Mapping

In complex variable theory, it can map the points, curves and regions from one complex plane to another. In the problems of solving boundary value, it is normally beneficial to map a region, one to one, onto a unit circle so as to simplify the boundary conditions.

If it is mapping from one to another, then for each point in the region of definition in one plane there will be one and only one corresponding point in the other plane. Therefore, a point in the $\zeta$-plane can be mapped into a point in the $z$-plane by the mapping function $z = f(\zeta)$ (Fig 2.21).

![Schematic representation of mapping function.](image)

If the mapping is that the angles between curves are preserved from the $z$-plane to the corresponding curves in the $\zeta$-plane, the mapping is called a conformal map. For any point in a given region, a mapping function is analytic and its first derivatives are not zero, it can map the given region onto another complex plane conformally. From the point of view of the theory of elasticity, it is possible to apply any mapping function that maps one to one in an effort to simplify the boundary conditions of a given elasticity problem. By using a conformal mapping function $z = f(\zeta)$, so that analytic function of $z$ in the $z$-plane becomes another analytic function of $\zeta$, in the $\zeta$-plane.

When considering the elasticity problem, the boundary conditions can be simplified by mapping the square boundaries $\Gamma_1$ and $\Gamma_2$ onto circular rings of radii $\rho_1$ and $\rho_2$, respectively, the boundary conditions on the circular ring can be expressed in terms of one variable only (Fig 2.2).
Fig 2.22 Schematic representation for mapping concentric squares onto an annulus.

By using the Schwarz-Christoffel transformation, the conformal mapping function
that maps the boundaries $\Gamma_1$ and $\Gamma_2$ onto circular rings can be developed. The general
form of this transformation can map the upper half plane, $y \geq 0$, onto the interior of an n-
sided polygon.

The idea of the Schwarz-Christoffel Transformation is mainly clarified by
considering a point 'a' on the real axis in the $z$-plane. The argument of function $(z-a)$ is $\pi$
if $z < a$, but arg. $(z - a) = 0$ if $z > a$. Therefore, the argument of the function decreases
abruptly by $\pi$ at the point 'a' as $z$ moves along the real axis.

Then, consider a more complicated function,

$$F(z) = (z-a)^{-\alpha_1} (z-b)^{-\alpha_2} (z-c)^{-\alpha_3} \ldots (z-k)^{-\alpha_n}$$  (2.84)

where, $a, b, c, \ldots k$ are fixed points along the real axis and $\alpha_1, \alpha_2, \alpha_3, \ldots \alpha_n$ are constants.
As a variable point 'z' moves along the real axis arg. $F(z)$ remains unchanged as long as
$z < a$, but arg. $F(z)$ suddenly increases by $\alpha^\frac{\pi}{2}$ as 'z' passes through point 'a'. Between
any two consecutive points, arg. $F(z)$ again changes simultaneously.

If any $n$-sided polygon with regard to the slope of its sides is examined, it is
obvious that the slope changes only at the vertices. Hence, if mapping the upper half
plane onto the interior of an $n$-sided polygon, $F(z)$ is considered as the derivative of the
required mapping function $z = \omega(\zeta)$, since it possesses the necessary conditions on the
derivative of the mapping function, i.e., $\frac{d\omega(\zeta)}{dz} = k$ except at vertices of the polygon.
Hence, \( \frac{d\omega(\zeta)}{dz} = AF(z) \), where A is a complex constant, or \( \omega(\zeta) = A\int F(z)dz + B \), where \( F(z) \) is defined by equation (2.84) and where

\[
\sum \alpha_k = 2 \quad (2.85).
\]

The exterior angles of the polygon are defined by the constants \( \alpha, \beta, \ldots \) etc. The positions of the vertices are defined by the constants \( a, b, c, \ldots \) etc. Two polygons are always equiangular and similar only if they are triangles. For polygons with more than three sides (n-3) constants \( a, b, c, \ldots \) etc., must be determinate to insure polygons. In other words, three of the constants in the mapping function are arbitrary.

For interior of an n-sided polygon, it can map the interior of a unit circle into the interior of an n-sided polygon, it can map the interior of a unit circle into the interior of an n-sided polygon by a simple linear transformation of equation (2.85). With equation (2.84), equation (2.85) can be written

\[
\omega(\zeta) = \int_0^z \frac{dt}{\pi(t-a_k)\alpha_k} \quad , \quad \sum \alpha_k = 2, t = z \quad \text{on} \quad \Gamma \quad (2.86)
\]

Equation (2.86) is the Schwarz-Christoffel formula, which maps \( y \geq 0 \) onto the interior of a polygon. A linear transformation of the following form maps the upper half plane into a unit circle (Churchill):

\[
u = e^{i\alpha} \left[ \frac{z - z_1}{z - z_2} \right] \quad (2.87)
\]

For the case of a regular polygon, equation (2.86) and equation(2.87) form the following mapping function:

\[
\omega(\zeta) = \int_0^z \frac{dt}{\bigg(1 - t^n\bigg)^{\frac{2}{n}}} \quad (2.88)
\]

Equation (2.88) maps the interior of a regular polygon onto the interior of a unit circle.

In the case of the square, the Schwarz-Christoffel transformation, which can map the interior of the unit circle onto the interior of a square, can be promptly obtained from equation (2.88). By setting \( n = 4 \), one obtains by equation (2.88),

\[
\omega(\zeta) = A\int_0^z \frac{dt}{1 - t^4} \quad (2.89)
\]
Where 'A' is a complex constant magnification factor. The factor 'A' may be used to rotate the axes in the \( \zeta \)-plane.

The form of equation (2.89) is not suitable for computational work. In reality, it evaluates the integral by expanding the integrand into an infinite series and integrating term by term, rather than evaluate the integral in a closed form. So that it can obtain the required mapping functions by successive approximations. For example, the binomial expansion can be applied,

\[
(1 \pm x)^n = 1 \pm nx + \frac{n(n-1)}{1\times 2} x^2 \pm \frac{n(n-1)(n-2)}{1\times 2\times 3} x^3 + \ldots \tag{2.90}
\]

As a result,

\[
(1-t^4)^{\frac{1}{2}} = 1 + \frac{1}{2} t^4 + \frac{3}{8} t^8 + \frac{15}{96} t^{12} + \ldots \tag{2.91}
\]

Hence, integrating of equation (2.91) term-by-term to get

\[
\omega(\zeta) = A \left[ \zeta + \frac{\zeta^5}{10} + \frac{\zeta^9}{24} + \frac{5\zeta^{13}}{208} + \frac{35\zeta^{17}}{2176} + \frac{3\zeta^{21}}{256} + \frac{231\zeta^{25}}{25600} + \ldots \right] \tag{2.92}
\]

In order to obtain an exact mapping solution, it is required to take an infinite number of terms in the mapping function. However, this procedure cannot obtain accurate results for the normal case. For example, the concrete specimens tested were reinforced with stirrups, which had a mean radius of approximately 15.8 mm, (5/8 in.) in the corners. In fact, the condensed series has a better representation for the actual specimens and is significantly easier to deal with the problem than the exact mapping function represented by the entire infinite series.

Due to symmetry, calculations for only one-quarter section are required. The argument of \( z \) varies from 0 to \( 45^\circ \) in increments of \( 3^\circ \), the values of the mapping function can be normalized so that the point \( z = 1 \) maps onto the point in the middle of the side of the square. Also the magnification of the mapping function, a rotation of the square of \( 45^\circ \) has been capable by including \( e^{i\pi/4} \) in the complex constant \( A \). Therefore, the mapping function equation can be modified in the following expression:

\[
\omega(\zeta) = |A| \left[ \zeta - \frac{\zeta^5}{10} + \frac{\zeta^9}{24} - \frac{5\zeta^{13}}{208} + \frac{35\zeta^{17}}{2176} - \frac{3\zeta^{21}}{256} + \frac{231\zeta^{25}}{25600} - \ldots \right] \tag{2.93}
\]
In order to determine the number of terms to produce a sufficiently accurate mapping function, values for a 1,2,...,7 term mapping functions are considered (Fig 2.23 and Fig 2.24). The mapping functions with the high-order terms give better approximations to a square but these terms cause high frequency harmonics which appear as undulations along the sides of the square. The seven and eight term mapping functions appear sufficiently accurate "in the large" with respect to the sides. However, these mapping functions may be unsuitable as the undulation along the side of the square. However, this is hard to overcome by employing smoothing functions, which selectively damp the higher harmonics at the cost of more roundedness at the corners. The plots of the six and seven-term mapping functions, after smoothing, are shown in Fig 2.23 and Fig 2.24.

These are several possible smoothing functions, which can be employed to smooth mapping function, the sigma factor method of Lanczos, Fejer's arithmetic mean method and several semi-empirical weighing factors used in electronic wave analysis. The application of smoothing functions to a series not only smooth out high frequency oscillations but also is a powerful way of speeding the convergence of Fourier series.

The high frequency oscillations of a condensed series about the true function are usually small and of little consequence. In the case of a discontinuity on the true function these oscillations can reach unacceptable magnitudes on either side of the discontinuity.

By considering the required function to be the derivative of its integral, the $\sigma$ factor method for a Fourier series can be developed. Therefore, it can affect in the derivative of the Fourier series in a more accurate approximation of the true function.

If considering a condensed Fourier series,

$$ f_m(\theta) = \sum_{n=-\infty}^{n=m-1} C_n e^{i n \theta} \quad (2.94) $$

If the true function is expressed by the entire series, then the error or residual is

$$ \eta_m(\theta) = \sum_{n=m}^{\infty} \left( C_n e^{i n \theta} + C_{-n} e^{-i n \theta} \right) \quad (2.95) $$

2-51
Or,

\[ \eta_m(\theta) = e^{im\theta} \sum_{n=0}^{\infty} C_{m+n}e^{in\theta} + e^{im\theta} \sum_{n=0}^{\infty} C_{m-n}e^{in\theta} \quad (2.96) \]

Examination of the residual shows that the terms

\[ \rho_m(\theta) = \sum_{n=0}^{\infty} C_{m+n}e^{in\theta} \quad \text{and} \quad \rho_{-m}(\theta) = \sum_{n=0}^{\infty} C_{m-n}e^{-in\theta} \quad (2.97) \]

are basically a smooth function but \( e^{im\theta} \) oscillates rapidly. If one differentiates both the condensed series and function, the residual will be

\[ \eta'_m(\theta) = ime^{im\theta}\rho_m(\theta) + e^{im\theta}\rho'_m(\theta) - ime^{-im\theta}\rho_{-m}(\theta) + e^{-im\theta}\rho'_{-m}(\theta) \quad (2.98) \]

Where the terms that are linear in \( 'm' \) cause a large error. By using the following differencing equation, the \( \sigma \) factor method overcomes this problem, which is equal to the derivative in the limit as \( 'm' \) tends to infinity:

\[ D_m(f(x)) = \frac{f\left(x + \frac{\pi}{m}\right) - f\left(x - \frac{\pi}{m}\right)}{2\pi/m} \quad (2.99) \]

By substituting equation (2.91) to the residual,

\[ D_m\eta_m(\theta) = -e^{im\theta}D_m\rho_m(\theta) - e^{-im\theta}D_m\rho_{-m}(\theta) \]

Each of the terms may be necessarily smooth. If the equation (2.99) is in terms of \( e^{i\theta} \),

\[ D_m e^{i\theta} = \frac{m}{\pi} \sin \frac{n\pi}{m} e^{i\theta} \quad (2.100) \]

2-52
This method of differentiating a Fourier series is the same as multiplying the derivative of the term by

$$\sin \frac{\pi m}{n} \quad (2.101).$$

Each of the terms contains $e^{in\theta}$ in the mapping function, equation (2.93), can be multiplied by the coefficients to produce a smooth function. The results of applying these coefficients can be referred in Fig 2.23.

Fejer's arithmetic mean method can provide a smoothing process, which includes taking the arithmetic mean of the partial sums of the series. It can apply the $\sigma$ factor method, and its heavy damping causes no ripples to appear in the function but this damping lead to the corner to be noticeably more rounded than by the $\sigma$ factor method.

In electronic wave analysis, several semi-empirical smoothing functions are used. A fundamental harmonic and its associated higher harmonics can be correctly combined to produce a rectangular wave $180^\circ$ of width with a unit high.

$$\frac{4}{\pi} \left( \sin \theta + \frac{1}{3} \sin 3\theta + \ldots + \frac{1}{n} \sin n\theta \right) \quad (2.102)$$

If a condensed series is used with the coefficients of the higher harmonics that can be weighed to produce a smooth approximation. Weighting each term of the condensed series by

$$\cos^2 \left( \frac{\pi n}{2m} \right) \quad (2.103).$$

The mapping function has the effect of this factor, which can be seen in Fig 2.23. The modified mapping function illustrates that there is no considerable change over the $\sigma$ factor method in the region away from the corner and more rounding of the corner. Therefore, the seven term mapping function modified by the $\sigma$ factor smoothing function is applied in the computations.
Fig 2.23 The effect of smoothing factors on the six and seven term mapping functions.

Fig 2.24 The effect successive terms and smoothing on the fidelity of the mapping function.
2.9 Theory of Torsional Creep of Reinforced Concrete

Since there is no complete theory for analyzing the uncracked and cracked reinforced concrete under long-term torsional load, further analytical methods are required to develop for solving the rate of twist and strains of concrete and reinforcement. So that the time dependent approaches can be applied for predicting the reinforced concrete members under torsional creep condition.

Chapter 5 (Time Dependent Analysis of Reinforced Concrete under Pure Torsion) would provide a detail explanation and comparison of the results from different analytical methods.
Chapter Three

Behaviour Aspect of Structural Concrete Beams in Pure Torsion

3.1 Members with Longitudinal Steel Only

Some of the early experiments on the torsional strength of reinforcement concrete were carried out on beams with longitudinal reinforcement only. A small increase in torsional strength was recorded. This is readily explained by considering the additional torsional resistance moment due to the higher shear modulus of the steel, which replaces an equal area of concrete.

Consider a beam symmetrically reinforced with four longitudinal bars at $(\pm \frac{1}{2} x_1, \pm \frac{1}{2} y_1)$, as shown in Fig 3.1. The component shear stresses in the concrete at $(\pm \frac{1}{2} x_1, \pm \frac{1}{2} y_1)$ can be ascertained from St. Venant’s theory. Let these stresses be $\tau_{yx}$ and $\tau_{yx}$ in the two mutually perpendicular directions. Allowing for the area of concrete displaced, the additional torsional resistance moment is

$$\frac{1}{2} A_t \left( \tau_{yx} x_1 + \tau_{yx} y_1 \right) \left( \frac{G_s}{G_e} - 1 \right)$$

Where $A_t$ is the cross-sectional area of all four bars and $G_s$ and $G_e$ are the shear moduli of the concrete and the steel.

![Fig 3.1 Longitudinal torsion reinforcement.](image)
The behaviour of torsional members with longitudinal steel only is depicted by the torque-twist curve (T Vs θ) in Fig 3.2. Before cracking the torque-twist relationship is very close to that of a plain concrete member.

![Figure 3.2 Typical torque-twist curve for beams with longitudinal steel only.](image)

Fig 3.2 Typical torque-twist curve for beams with longitudinal steel only.

After cracking the beam may collapse suddenly if the beam is reinforced with light amount of steel. In the case of heavy reinforcement, the ultimate strength may exceed the cracking torque but seldom exceed it by more than 15%. This ineffectiveness of longitudinal steel in increasing the strength of the beams was thought by early experiments to be result of the location of longitudinal bars. Such longitudinal bars were always placed at the corners of a beam where the shear stress is zero according to St. Venant's stress distribution. However, later tests with longitudinal bars at the center of the faces show that longitudinal steel alone is ineffective regardless of the location.

### 3.2 Members with Longitudinal Steel and Stirrups

The torque-twist curves of a series of specimens with a square cross section (70 mm x 70 mm) and reinforced with carious amounts of torsional reinforcement (1% to 5%, including equal volumes of longitudinal steel and stirrups) are shown in Fig 3.3. Each curve can be divided into two distinct regions — before and after cracking. A horizontal plateau exists at cracking, where a member continues to twist under a constant load.
The cracking torques is plotted as a function of total steel (including longitudinal steel and stirrups) for 16 beams. It can be seen that cracking torque \( T_{cr} \) is a mild function of the total steel percentage \( \rho_t \) and can be expressed by the following equation:

\[
T_{cr} = (1 + 4\rho_t)T_{np} \quad (3.1)
\]

In equation (3.1), \( \rho_t \) is expressed in terms of the total steel ratio. For example, for a member with 3% total steel, \( T_{cr} = [1 + 4(0.03)]T_{np} = 1.12T_{np} \). In view of the small effect of \( \rho_t \) it would be simpler and conservative in practical design to neglect the favorable effect of \( \rho_t \) and to take \( T_{cr} = T_{np} \).

After cracking, the behaviour can no longer be predicted by St. Venant's theory. This is so because cracking terminates the basic premise of the theory of elasticity that the material must be continuous. Hence a new equilibrium condition is established after cracking, in which the steel picks up the tensile stresses and the concrete carries the compression. The transition from the St. Venant's equilibrium condition to the new post-cracking equilibrium condition is manifested by the horizontal plateau of torque-twist curve.

![Fig 3.3 Torque-twist curves of beams with various percentages of reinforcement.](image-url)
3.3 Post Cracking Torsional Rigidity

3.3.1 Theoretical Derivation

To derive the post cracking torsional rigidity of a reinforced concrete member, we shall start with a reinforced concrete tube of arbitrary cross-sectional shape and uniform wall thickness. Fig 3.4 shows such a reinforced concrete tube subjected to a torque, \( T \). According to the thin tube theory, the shear stress \( \tau \) should be:

\[
\tau = \frac{T}{2Ah}
\]  

(3.2)

Where

\( A \) = area within the center line of stirrup

\( h \) = wall thickness of the reinforced concrete tube

This shear stress \( \tau \) will induce stresses and strains in the reinforcement and concrete. To evaluate these stresses and strains, the reinforced concrete tube will be idealized into Rausch’s space truss, as discussed in previous section. This space truss is shown again in Fig 3.4, including the thickness of the tube, \( h \).

In Rausch’s analysis of the space truss, the forces in the longitudinal bars, in the stirrups, and in the diagonal concrete struts are designated \( X \), \( Y \) and \( D \), respectively. Each of these forces is constant throughout the tube. They also have the following:

\[
X = Y = \frac{D}{\sqrt{2}} = F = \tau h
\]  

(3.3)

Using equation (3.3), the stress \( \sigma_c \), \( \sigma_l \), and \( \sigma_h \), in the concrete struts, in the longitudinal bars, and in the stirrups, respectively, can be expressed as follows:
\[
\sigma_x = \frac{D}{\left(\frac{s}{\sqrt{2}}\right)h} = \frac{\sqrt{2} \tau h}{\left(\frac{s}{\sqrt{2}}\right)h} = 2 \tau \quad (3.4)
\]
\[
\sigma_l = \frac{X}{A_l} = \frac{\tau h}{\left(\frac{s}{\sqrt{2}}\right)h} = \frac{\tau}{r_l} \quad (3.5)
\]
\[
\sigma_h = \frac{Y}{A_h} = \frac{\tau h}{\left(\frac{s}{\sqrt{2}}\right)h} = \frac{\tau}{r_h} \quad (3.6)
\]

Fig 3.4 Space truss for a tube section.

In equation (3.5), \(A_l\) = the area of one longitudinal bar, and \(r_l = \frac{A_l}{sh}\) = the longitudinal reinforcement ratio with respect to the wall area. In equation (3.6), \(A_h\) = the area of one stirrup, and \(r_h = \frac{A_h}{sh}\) = the stirrup ratio with respect to the wall area.

The strains in the concrete struts and in the steel bars can be found from the stress-strain relationship and from Equations (3.4) through (3.6):
where $\varepsilon_c$, $\varepsilon_l$, and $\varepsilon_h$ are strains in the concrete struts, in the longitudinal bars, and in the stirrups, respectively. $E_c$ and $E_s$ are the moduli of elasticity of concrete and steel.

The strains calculated by equations (3.7) through (3.9) will cause a shear distortion of the tube. The shear distortion $\gamma$ can be obtained from the compatibility of deformations in a basic cell consists of one diagonal concrete strut and its surrounding steel bars, forming a square with a side length of $s$.

The shear distortion of cell ABCD due to a compression strain $\varepsilon_c$ in the concrete strut is shown in Fig 3.5a. The original length of the diagonal concrete strut in a cell is designated $\overline{CB}$. After shortening, the length becomes $\overline{CE}$. The shortening $\overline{BE} = \varepsilon_c \left(\sqrt{2}s\right)$. To maintain compatibility of the concrete strut with the steel bars, $\overline{CE}$ must rotate about point C, and $\overline{AB}$ must rotate about point A until points E and B meet at point F. For small deformation, $\overline{EF}$ is taken perpendicular to $\overline{CE}$ and $\overline{BF}$ perpendicular to $\overline{AB}$. From geometry, the distance $\overline{BF} = \sqrt{2} \overline{BE} = 2\varepsilon_c s$. After deformation the shape of the cell becomes a parallelogram, shown by the dotted lines. The shear distortion of the cell is then represented by the distortion angle $\gamma_c$:

$$\gamma_c = \frac{\overline{BF}}{\overline{AB}} = \frac{2\varepsilon_c s}{s} = 2\varepsilon_c$$

Similarly, the distortion $\gamma_l$ due to the lengthening of the longitudinal bars and the distortion $\gamma_h$ due to the lengthening of the stirrups are illustrated in Fig 3.5b and Fig 3.5c, respectively. Accordingly,
The total shear distortion $\gamma$ is:

$$\gamma = \gamma_c + \gamma_l + \gamma_h = 2\varepsilon_c + \varepsilon_l + \varepsilon_h \quad (3.13)$$

Fig 3.5 Compatibility of strains in a basic cell of a space truss subjected to shear distortion.

Substituting equation (3.7) through (3.9) into equation (3.13), we obtain:

$$\frac{\tau}{\gamma} = \frac{E_s}{4n + \frac{1}{r_l} + \frac{1}{r_h}}$$

Where $n = E_s/E_c$. Let us define $G_{cr} = \tau/\gamma =$ post-cracking shear modulus. The above equation becomes:

$$G_{cr} = \frac{E_s}{4n + \frac{1}{r_l} + \frac{1}{r_h}} \quad (3.14)$$

$G_{cr}$ describes the material property of a torsional member after cracking. It is a function of the moduli of elasticity of steel and concrete and the reinforcement ratios of longitudinal steel and stirrup.
It is frequently more convenient and more common to express the reinforcement ratios with respect to the solid cross-sectional area of concrete $A_c$, instead of the area of the wall. Define:

$$\rho_l = \frac{\hat{A}_l}{A_c} = \left(\frac{uh}{A_c}\right) r_l \quad (3.15)$$

$$\rho_h = \frac{A_h u}{A_c s} = \left(\frac{uh}{A_c}\right) r_h \quad (3.16)$$

Where

$\hat{A}_l$ = total cross sectional area of longitudinal steel

$A_c$ = solid cross sectional area within the outer perimeter of concrete

$u$ = perimeter of area bounded by the center line of a complete stirrup

$A_h$ = cross sectional area of one stirrup

Substituting $\rho_l$ and $\rho_h$ for $r_l$ and $r_h$ in equation:

$$G_{sr} = \frac{E_s}{\left(4n + \frac{uh}{A_c \rho_l} + \frac{uh}{A_c \rho_h}\right)} \quad (3.17)$$

The torsional geometric properties of thin tube of homogeneous materials were studied. According to Bredt's equation, the post-cracking torsional constant $J_{sr}$ of a reinforced concrete member with uniform wall thickness can be expressed as:

$$J_{sr} = \frac{4A^2_1 h}{u} \quad (3.18)$$

Where $A_1$ is defined as the area bounded by the centerline of the stirrup. It should be noted that $A_1$ was defined in previous equation as the area bounded by the centerline of the wall thickness for tubes of homogeneous materials. In the case of reinforced concrete members, however, it seems more logical to define $A_1$ by the dimension of the reinforcement, rather than that of the concrete. This is so because in the post-cracking
stage the concrete has cracked, and the torsional resistance is contributed mainly by the tension reinforcement in conjunction with the concrete in compression.

Combining equations (3.17) and (3.18) gives the post-cracking torsional rigidity, $G_{cr}J_{cr}$:

$$G_{cr}J_{cr} = \frac{4E_iA_i^2A_c}{u\left(\frac{4nA_c}{uh} + \frac{1}{\rho_l} + \frac{1}{\rho_h}\right)}$$  \hspace{1cm} (3.19)

The denominator in equation (3.19) consists of three terms. They represent, in sequence, the contributions to the post-cracking torsional rigidity of the concrete struts, the longitudinal steel, and the stirrup.

Equation (3.18) is applicable to arbitrary cross sections. It can easily be reduced to the special cases for circular and rectangular sections. For circular sections, $A_c = \pi d^2 / 4$, $A_i = \pi d_1^2 / 4$, and $u = \pi d_1$.

$$G_{cr}J_{cr} = \frac{E_i\pi d_1^2 d^2}{\left(\frac{nd^2}{d_1h} + \frac{1}{\rho_l} + \frac{1}{\rho_h}\right)}$$  \hspace{1cm} (3.20)

where $d$ and $d_1$ are, respectively, the diameters of the concrete cross section and the diameter of the circle formed by the centerline of a stirrup. For rectangular section, $A_c = xy$, $A_i = x_1y_1$, and $u = 2(x_1 + y_1)$.

$$G_{cr}J_{cr} = \frac{E_i x_1^2 y_1^2 xy}{(x_1 + y_1)^3\left(\frac{2nxy}{(x_1 + y_1)h} + \frac{1}{\rho_l} + \frac{1}{\rho_h}\right)}$$  \hspace{1cm} (3.21)

In equation (3.21), $x$ and $y$ are the shorter and longer dimensions of the rectangular concrete section. $x_1$ and $y_1$ are the shorter and longer dimensions of a rectangular stirrup.
Equations (3.19) through (3.21) appear to be quite complex. It would be more straightforward in practice to calculate $G_{cr}$ and $J_{cr}$ separately by equations (3.17) and (3.18) and then multiply them to obtain the torsional rigidity, $G_{cr}J_{cr}$.

### 3.3.2 Modifications of Theoretical Equations

**Effective Wall Thickness (Rectangular Section)**

In the foregoing theoretical derivation of post-cracking torsional rigidity we have assumed a reinforced concrete tube with a uniform wall thickness, $h$. An obvious question arises. What is the effective wall thickness, $h_e$, that is applicable to members with solid cross sections and thick hollow sections? This effective wall thickness has been derived for rectangular sections by comparing equation (3.21) with the PCA test results. Substituting the experimental values of $G_{cr}J_{cr}$ into equation (3.21), we can solve for $h$ and get the effective wall thickness, $h_e$. The non-dimensional ratio, $\rho_l + \rho_h$. It can be seen that $h_d/x$ is then plotted against the total reinforcement ratio, $\rho_l + \rho_h$, and the proportional constant is about 1.4. Hence:

$$h_e = 1.4(\rho_l + \rho_h)x \quad (3.22)$$

The term $h_e$ is usually quite small. It is an empirical quantity that fits the test results and should not be construed as the actual required wall thickness at ultimate strength.

**Vertical Intercept**

A typical torque-twist curve is shown in Fig 3.6. After cracking, the curve starts out as a straight line and then gradually curves toward horizontal when the maximum torque is approached. The slope of the straight portion represents the post-cracking torsional rigidity calculated by equation (3.21). The extrapolation of the straight portion will intersect the vertical axis, giving a vertical intercept. This vertical intercept was found to have the same parameter as $T_e = (x^2 y/3)\left(2.4\sqrt{f'}\right)$ and was defined as $\eta T_e$, where $\eta$ is a coefficient. The post-cracking torque twist relationship can therefore be expressed as:

$$T = \eta T_e + G_{cr}J_{cr}\theta \quad (3.23)$$
The coefficient $\eta$ is evaluated from series of PCA tests with identical dimensions and materials, except the wall thickness. The first series has a solid cross section, i.e. $h/x = 0.5$ where $h$ is the wall thickness and $x$ is the smaller dimension of a rectangular cross section. The section and the third series have $h/x$ values of 0.25 and 0.15, respectively. The torsional rigidity is the same, but the vertical intercept increases with the wall thickness. Consequently, $\eta$ is plotted against $h/x$. For the three series of beams the test points can be approximated by a straight line that can be expressed by:

For a solid section, $\eta = 2$.

$$\eta = 0.57 + 2.86 \frac{h}{x}$$  \hspace{1cm} (3.24)

The fact that $\eta$ is a function of the wall thickness is quite interesting. In the study of ultimate strength of a reinforced concrete torsional member, tests have shown that the concrete core has no effect on the ultimate strength. However, equation (3.24) appears to indicate that the concrete core does have an effect on the post-racking torsional behaviour via the vertical intercept.

![Diagram](image)

Fig 3.6 Typical torque-twist curve of RC beams.
3.3.3 Simplification of Torsional Rigidity

The post-cracking torsional rigidity can be greatly simplified using two assumptions:

1) Take \( \eta = 0 \); i.e., the vertical intercept is zero, and the torque-twist curve passes through the origin.

\[
G_{cr}J_{cr} = \frac{T}{\theta} \quad (3.25)
\]

2) Neglect the contribution of the concrete struts to the torsional rigidity:

\[
G_{cr} = \frac{E_s}{h_c \left( \frac{u}{A_i} + \frac{s}{A_i} \right)} \quad (3.26)
\]

When equation (3.26) is added \( J_{cr} \), it becomes:

\[
G_{cr}J_{cr} = \frac{4E_sA_i^2}{u \left( \frac{u}{A_i} + \frac{s}{A_i} \right)} \quad (3.27)
\]

Notice that \( h_c \) has been cancelled out in the multiplication.

It is interesting to note that the first assumption lowers the straight line, whereas the second assumption lowers the straight line. These two assumptions tend to cancel each other in the post-cracking region. For this reason, the simplified \( G_{cr}J_{cr} \) can often be used with acceptable accuracy in the post-cracking stage. It has been used by the post-cracking analysis of horizontally curved beams. It has also been suggested that this simplified post-cracking torsional rigidity is reasonable to represent the post-cracking elastic behaviour under repeated loading.
Chapter Four

Methods for the Time Dependent Analysis of Structures

4.1 Introduction

The time dependent analysis of concrete structures contains the determination of strains, stresses, curvatures and deflections at critical points and at critical times during the life of the structure. After the effects of creep and shrinkage has occurred, i.e. the long-term behaviour.

The creep and shrinkage characteristics of concrete are highly variable and never exactly known. Predictions of accurate numerical of time dependent behaviour are therefore not possible. But it is necessary to establish upper and lower limits to behaviour so that whether time effects are critical in any particular situation can be determined. If it is required, adjust a design to reduce undesirable long-term deformations.

If the concrete stress $\sigma$ at a point a structure remains constant as time increases, the determination of each of the strain components in equation (4.1).

$$
\varepsilon(t, \tau) = \varepsilon_s(t, \tau) + \varepsilon_c(t, \tau) + \varepsilon_{sh}(t)
$$

$$
= \frac{\sigma}{E_s(\tau)} + \frac{\sigma}{E_c(\tau)} \phi(t, \tau) + \varepsilon_{sh}(t) \quad (4.1)
$$

The predictive models can provide the numerical values of $\phi(t, \tau)$ and $\varepsilon_{sh}(t)$ and the calculation of time dependent structural behaviour is relatively straightforward.

4.2 The Effective Modulus Method (EMM)

The simplest and oldest approach for analyzing the creep in structure is Faber's Effective Modulus Method. The creep strain at time $t$ depends only on the current stress $\sigma$ and is hence independent of the previous stress history, which is defined as follows.

$$
\varepsilon(t, \tau) = \frac{\sigma}{E(s(\tau))} \left[1 + \phi(t, \tau)\right] + \varepsilon_{sh}(t) \quad (4.2)
$$

So that the effective modulus of concrete is:
Time Dependent Effects of Reinforced Concrete Subjected to Torsional Loading

\[ E_s(t, \tau) = \frac{E_c(\tau)}{1 + \phi(t, \tau)} \quad (4.3) \]

Creep is treated as delayed elastic strain and taken into account simply by reducing the elastic modulus for concrete. A time analysis using the effective modulus method is nothing more than an elastic analysis for \( E_s(t, \tau) \) is used instead of \( E_c(\tau) \).

4.2.1 Formulation

By considering the symmetrically reinforced concrete shown in Fig 4.1, that the column is subjected to a constant sustained axial force \( P \).

![Diagram of axial loading](image)

Fig 4.1 Axially loaded short column.

At time \( t \), the force equilibrium requires to satisfy the condition that the external load equal internal forces so that,

\[ P = N_c(t) + N_s(t) \quad (4.4) \]

By assuming the steel reinforcement is bonded to the concrete, satisfying the compatibility condition that the total concrete strain \( \varepsilon(t) \) and steel strain \( \varepsilon_s(t) \) equal each other at any time:

\[ \varepsilon(t) = \varepsilon_s(t) \quad (4.5) \]
By using the EMM relationships given by equation (4.2) and equation (4.3), the steel is assumed to be linear elastic with modulus of steel $E_s$.

$$\varepsilon(t, \tau) = \frac{\sigma(t)}{E_s(t)} + \varepsilon_{sh}(t) \quad (4.6)$$

$$\varepsilon_s(t) = \frac{\sigma_s(t)}{E_s} \quad (4.7)$$

Also, equation 4.4 can be rewritten as;

$$P = A_s \sigma_s(t) + A_l \sigma_l(t) \quad (4.8)$$

Rearranging the above equation, steel stresses can be eliminated.

$$\sigma_s(t) = \frac{P - \sigma(t)A_c}{A_s} \quad (4.9)$$

Substituting equation 4.6 and 4.7 to equation 4.5 gives

$$\frac{\sigma(t)}{E_s(t)} + \varepsilon_{sh}(t) = \frac{\sigma_s(t)}{E_s} \quad (4.10)$$

Putting equation 4.8 into the above equation,

$$\frac{\sigma(t)}{E_s(t)} + \varepsilon_{sh}(t) = \frac{P - \sigma(t)A_c}{A_sE_s} \quad (4.11)$$

The stress of concrete at any time can be determined in terms of the properties of the materials by rearranging the terms in equation 4.11.

$$\sigma(t) = \frac{P}{A_c(1 + n \rho)} - \frac{\varepsilon_{sh}(t)E_s \rho}{1 + n \rho} \quad (4.12)$$

Where the term $n$' and $\rho$ are defined as

$$n = \frac{E_s}{E_s(t)} \quad and \quad \rho = \frac{A_c}{A_s} \quad (4.13)$$

Thus, concrete stress $\sigma(t)$ can be obtained by substituting materials properties into equation 4.12, the steel stress $\sigma_s(t)$ may be calculated using 4.9. The steel strain $\varepsilon_s(t)$ is determined by the compatibility condition.
At any time $t$, equation 4.2 depends only on the current stress $\sigma$ and it is independent of the previous stress history. Thus, aging of concrete has been ignored. When the concrete stress is constant is old when first loaded and the effect of aging is not great. Aside the disadvantage, the EMM approach is the simplest of all methods for the time dependent analysis of concrete structures.

### 4.2.2 Numerical Example

The strain of concrete and steel is to be determined at selected times for the axially loaded cross section shown in Fig 4.2. The external load $P$ is 1000kN. Shrinkage is assumed to commence at age of first loading, $\tau_o = 10$ days. Cross sectional and materials properties and materials properties are as follows:

- $A_c = 90000 mm^2$
- $A_s = 1800 mm^2$
- $E_c(\tau_o) = 25 GPa$
- $E_s = 20 GPa$
- $\rho = 0.02$

\[
\begin{align*}
(t - \tau_o) & \quad \text{in days} = 0 & \phi(t, \tau_o) &= 0 & \varepsilon_{sh}(t - \tau_o) \times 10^{-6} &= 0 \\
(t - \tau_o) & \quad \text{in days} = 25 & \phi(t, \tau_o) &= 1.0 & \varepsilon_{sh}(t - \tau_o) \times 10^{-6} &= 200 \\
(t - \tau_o) & \quad \text{in days} = 100 & \phi(t, \tau_o) &= 2.0 & \varepsilon_{sh}(t - \tau_o) \times 10^{-6} &= 400 \\
(t - \tau_o) & \quad \text{in days} = \infty & \phi(t, \tau_o) &= 3.0 & \varepsilon_{sh}(t - \tau_o) \times 10^{-6} &= 600
\end{align*}
\]

![Fig 4.2 Cross section of axially loaded short column.](image)

At first loading $(t - \tau_o)$ in days $= 0$, $\phi(t - \tau_o) = 0$, $\varepsilon_{sh}(t - \tau_o) \times 10^6 = 0$

\[
E_c(t) = E_c(\tau_o) = 25 GPa, n' = n = \frac{E_s}{E_c(\tau_o)} = 8
\]

Stress of concrete can be determined by equation 4.12
\[
\sigma(\tau_o) = \frac{1000 \times 10^3}{90000(1 + 8 \times 0.02)} - 0 = 9.58 \text{MPa}
\]

Stress of steel can be determined by equation 4.9

\[
\sigma_s(\tau_o) = \frac{1000 \times 10^3 - 9.58 \times 90000}{1800} = 76.6 \text{MPa}
\]

As a result, strain of concrete and steel are

\[
\varepsilon_s(\tau_o) = \frac{76.6}{200000} = 383 \times 10^{-6}
\]
\[
\varepsilon_s(\tau_o) = \varepsilon_s(\tau_o) = 383 \times 10^{-6}
\]

At first loading \((t - \tau_o)\) in days = 25, \(\phi(t - \tau_o) = 1.0, \ v_{sh}(t - \tau_o) = 200 \times 10^{-6}\)

\[
E_s(t) = \frac{25}{1 + 1.0} = 12.5 \text{GPa}, n = \frac{E_s}{E_c(t)} = 16
\]

Stress of concrete can be determined by equation 4.12

\[
\sigma(\tau_o) = \frac{1000 \times 10^3}{90000(1 + 16 \times 0.02)} - \frac{200 \times 10^{-6} \times 200000 \times 0.02}{(1 + 16 \times 0.02)} = 7.81 \text{MPa}
\]

Stress of concrete can be determined by equation 4.9

\[
\sigma_s(t) = \frac{1000 \times 10^3 - 7.81 \times 90000}{1800} = 165 \text{MPa}
\]

As a result, strain of concrete and steel are

\[
\varepsilon_s(t) = \frac{165}{200000} = 825 \times 10^{-6}
\]
\[
\varepsilon_s(t) = \varepsilon_s(t) = 825 \times 10^{-6}
\]

The elastic and creep strain can be determined by Equation 4.1,

\[
\varepsilon_{el}(t) = 312 \times 10^{-6} \quad \varepsilon_{cr}(t) = 313 \times 10^{-6}
\]
At first loading \((t-t_0)\) in days = 100, \(\phi(t-t_0) = 2.0\), \(\varepsilon_{sh}(t-t_0) = 400 \times 10^{-6}\)

\[
E_s(t) = \frac{25}{1 + 2.0} = 8.33 \text{GPa}, \quad n = \frac{E_s(t)}{E_r(t)} = 24
\]

\[
\sigma(t_0) = \frac{1000 \times 10^3}{90000(1 + 24 \times 0.02)} - \frac{400 \times 10^{-6} \times 2000000 \times 0.02}{(1 + 24 \times 0.02)} = 6.43 \text{MPa}
\]

\[
\sigma_s(t_0) = \frac{6.43 \times 90000}{1800} = 234 \text{MPa}
\]

\[
\varepsilon_s(t) = \frac{234}{200000} = 1170 \times 10^{-6}
\]

\[
\varepsilon_r(t) = \varepsilon_s(t) = 1170 \times 10^{-6}
\]

\[
\varepsilon_{al}(t) = 257 \times 10^{-6} \quad \varepsilon_{cr}(t) = 514 \times 10^{-6}
\]

At first loading \((t-t_0)\) in days = \(\infty\), \(\phi(t-t_0) = 3.0\), \(\varepsilon_{sh}(t-t_0) = 600 \times 10^{-6}\)

\[
E_s(t) = \frac{25}{1 + 2.0} = 6.25 \text{GPa}, \quad n = \frac{E_s(t)}{E_r(t)} = 32
\]

\[
\sigma(t_0) = \frac{1000 \times 10^3}{90000(1 + 32 \times 0.02)} - \frac{600 \times 10^{-6} \times 2000000 \times 0.02}{(1 + 32 \times 0.02)} = 5.31 \text{MPa}
\]

\[
\sigma_s(t_0) = \frac{5.31 \times 90000}{1800} = 290 \text{MPa}
\]

\[
\varepsilon_s(t) = \frac{290}{200000} = 1450 \times 10^{-6}
\]

\[
\varepsilon_r(t) = \varepsilon_s(t) = 1450 \times 10^{-6}
\]

\[
\varepsilon_{al}(t) = 212 \times 10^{-6} \quad \varepsilon_{cr}(t) = 638 \times 10^{-6}
\]
4.3 The Age-Adjusted Effective Modulus Method (AEMM)

A simple adjustment to the effective modulus method to account for aging of concrete was introduced to analyze the structure. In general, age-adjusted effective modulus method is normally used.

4.3.1 Formulation

The column section shown in Fig 4.1 is re-analyzed by using Age Adjusted Effective Modulus Method (AEMM), the equilibrium and compatibility conditions at time $t$ are given by equation 4.4 and 4.5 respectively.

$$ P = N_r(t) + N_s(t) $$

$$ \varepsilon(t) = \varepsilon_s(t) $$

The constitutive relationship for concrete is described as before, the steel is linear-elastic:

$$ \varepsilon(t) = \frac{\sigma}{E_e(t)} + \frac{\Delta \sigma}{E_e(t)} \varepsilon_{sh}(t) \quad (4.14) $$

and

$$ \varepsilon_s(t) = \frac{\sigma_s(t)}{E_s} \quad (4.15) $$

$\sigma_o$ is the initial concrete stress at first loading, which is given by

$$ \sigma_s = \frac{P}{A_s (1 + n \rho)} \quad (4.16) $$

$\Delta \sigma(t)$ is the unknown time dependent change of stress caused by creep and shrinkage. If $\sigma_o$ is compressive, the stress increment $\Delta \sigma(t)$ is usually tensile and therefore of opposite sign. To simplify notation, the age at the first loading $t$ has been omitted from the arguments of the effective modulus $E_e(t)$ and the age-adjusted effective modulus $E_s(t)$.

At any time $t$, the concrete stress is expressed as following:

$$ \sigma(t) = \sigma_o + \Delta \sigma(t) \quad (4.17) $$
The steel stress is determined by equation 4.9

\[ \sigma_s(t) = \frac{P - \sigma(t)A_s}{A_s} \]

The analysis can be carried out by substituting equation 4.14 and 4.15 into the compatibility equation 3.5.

\[ \frac{\sigma_a}{E_s(t)} + \frac{\Delta \sigma(t)}{E_s(t)} + \varepsilon_{sh}(t) = - \frac{P - [\sigma_a + \Delta \sigma(t)]A_s}{E_s A_s} \]  \hspace{1cm} (4.18)

So that \( \Delta \sigma(t) \) can be obtained as follows:

\[ \Delta \sigma(t) = \frac{1}{1 + n' \rho} \left[ \frac{P}{A_s} - \sigma_a \left( 1 + n' \rho \right) - \varepsilon_{sh}(t)E_s \rho \right] \] \hspace{1cm} (4.19)

Where

\[ n' = \frac{E_s}{E_s(t)} \]

### 4.3.2 Numerical Example

The concrete and steel strain were determined by EMM approach in the previous sections at selected times for the axially loaded cross section in Fig 4.2. The same section is re-analyzed by using the AEMM in this section.

<table>
<thead>
<tr>
<th>( t - \tau_o ) in days</th>
<th>0</th>
<th>25</th>
<th>100</th>
<th>( \infty )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \phi(t, \tau_o) )</td>
<td>0</td>
<td>1.0</td>
<td>2.0</td>
<td>3.0</td>
</tr>
<tr>
<td>( \chi(t, \tau_o) )</td>
<td>1.0</td>
<td>0.86</td>
<td>0.8</td>
<td>0.8</td>
</tr>
<tr>
<td>( \varepsilon_{sh}(t - \tau_o) \times 10^{-6} )</td>
<td>0</td>
<td>200</td>
<td>400</td>
<td>600</td>
</tr>
</tbody>
</table>

At first loading \( t - \tau_o \) in days = 0, \( \phi(t-\tau_o) = 0, \chi(t-\tau_o) = 1.0, \varepsilon_{sh}(t-\tau_o) \times 10^{-6} = 0 \)

\[ E_s(t) = E_s(\tau_o) = 25GPa, n = \frac{E_s}{E_s(\tau_o)} = 8 \]

Stress of concrete can be determined by equation 4.12
\[ \sigma(\tau_o) = \frac{1000 \times 10^3}{90000(1 + 8 \times 0.02)} - 0 = 9.58 \text{MPa} \]

Stress of steel can be determined by equation 4.9.

\[ \sigma_s(\tau_o) = \frac{1000 \times 10^3 - 9.58 \times 90000}{1800} = 76.6 \text{MPa} \]

So that strain of concrete and steel are

\[ \varepsilon_c(t) = \frac{76.6}{200000} = 383 \times 10^{-6} \]
\[ \varepsilon_s(t) = \varepsilon_s(t) = 383 \times 10^{-6} \]

At first loading (t-\(\tau_0\)) in days = 0, \(\phi(t-\tau_0) = 0, \chi(t-\tau_0) = 1.0, \varepsilon_{sh}(t-\tau_0) \times 10^6 = 200 \times 10^6 \)

\[ E_s(t) = \frac{25}{1 + 1.0} = 12.5 \text{GPa}, n = \frac{E_s(t)}{E_c(t)} = 16 \]
\[ \overline{E_c(t)} = \frac{25}{1 + 0.86x1.0} = 13.4 \text{GPa}, \bar{n} = \frac{E_s(t)}{E_c(t)} = 14.9 \]

Change of stress of concrete can be found by equation 4.18

\[ \Delta \sigma(t) = \frac{1}{(1 + 14.9 \times 0.02)} \left[ \frac{1000 \times 10^3}{90000} - 9.58(1 + 16x0.02) - 200 \times 10^{-6} \times 200000 \times 0.02 \right] = -1.80 \text{MPa} \]

Since the concrete stress at any time can be given equation 4.17

\[ \sigma(t) = \sigma_o + \Delta \sigma(t) \]
\[ \sigma(t) = 9.58 - 1.80 = 7.78 \text{MPa} \]

Stress of steel can be determined by equation 4.9

\[ \sigma_s(\tau_o) = \frac{1000 \times 10^3 - 7.78 \times 90000}{1800} = 167 \text{MPa} \]

So, strain of concrete and steel are
\[ \varepsilon_{s}(t) = \frac{167}{200000} = 833 \times 10^{-6} \]
\[ \varepsilon_{c}(t) = \varepsilon_{s}(t) = 833 \times 10^{-6} \]

The elastic and creep strain can be found.
\[ \varepsilon_{el}(t) = 311 \times 10^{-6} \]
\[ \varepsilon_{cr}(t) = 332 \times 10^{-6} \]

At first loading \((t-t_0)\) in days = 100, \(\phi(t-t_0) = 2.0\), \(\chi(t-t_0) = 0.86\), \(\varepsilon_{sh}(t-t_0) = 400 \times 10^{-6}\)

\[ E_{e}(t) = \frac{25}{1+2.0} = 8.33 \text{ GPa} \]
\[ \frac{E_{e}}{E_{e}(t)} = 24 \]
\[ \bar{E}_{e}(t) = \frac{25}{1+0.86 \times 2.0} = 9.62 \text{ GPa} \]
\[ \frac{E_{e}}{E_{e}(t)} = 20.8 \]

\[ \Delta \sigma(t) = \frac{1}{(1+20.8 \times 0.02)} \left[ \frac{1000 \times 10^3}{90000} - 9.58(1+24 \times 0.02) - 200 \times 10^{-6} \times 200000 \times 0.02 \right] \]
\[ = -3.03 \text{ MPa} \]

\[ \sigma(t) = \sigma_{o} + \Delta \sigma(t) \]
\[ \sigma(t) = 9.58 - 3.03 = 6.28 \text{ MPa} \]

\[ \sigma_{s}(t) = \frac{1000 \times 10^3 - 6.28 \times 90000}{1800} = 241 \text{ MPa} \]

\[ \varepsilon_{s}(t) = \frac{241}{200000} = 1207 \times 10^{-6} \]
\[ \varepsilon_{c}(t) = \varepsilon_{s}(t) = 1207 \times 10^{-6} \]

\[ \varepsilon_{el}(t) = 251 \times 10^{-6} \]
\[ \varepsilon_{cr}(t) = 556 \times 10^{-6} \]

At first loading \((t-t_0)\) in days = \(\infty\), \(\phi(t-t_0) = 3.0\), \(\chi(t-t_0) = 0.8\), \(\varepsilon_{sh}(t-t_0) = 600 \times 10^{-6}\)
\[ E_c(t) = \frac{25}{1 + 2.0} = 6.2 \text{GPa}, \frac{E_s}{E_c(t)} = 32 \]

\[ \bar{E}_c(t) = \frac{25}{1 + 0.86 \times 2.0} = 7.35 \text{GPa}, \frac{E_s}{\bar{E}_c(t)} = 27.2 \]

\[ \Delta \sigma(t) = \frac{1}{(1 + 27.2 \times 0.02)} \left[ \frac{1000 \times 10^3}{90000} - 7.35(1 + 32 \times 0.02) - 200 \times 10^{-6} \times 200000 \times 0.02 \right] 
= -4.53 \text{MPa} \]

\[ \sigma(t) = \sigma_o + \Delta \sigma(t) \]
\[ \sigma(t) = 9.58 - 4.53 = 5.05 \text{MPa} \]

\[ \sigma_s(t_o) = \frac{1000 \times 10^3 - 5.05 \times 90000}{1800} = 303 \text{MPa} \]

\[ \varepsilon_s(t) = \frac{303}{200000} = 1516 \times 10^{-6} \]
\[ \varepsilon_c(t) = \varepsilon_s(t) = 1516 \times 10^{-6} \]

\[ \varepsilon_o(t) = 202 \times 10^{-6} \]
\[ \varepsilon_{cr}(t) = 714 \times 10^{-6} \]
Chapter Five

Time Dependent Analysis of Reinforced Concrete under Pure Torsion

5.1 Introduction

In most structural members, creep and shrinkage of concrete cause increases of deformation and redistribution of stresses. For some cases, the deformations caused by creep and shrinkage can lead to an increase in the internal actions in the structure and therefore, a reduction of strength. A reinforced concrete member subjected to a sustained torsional loading is an example.

There are two common cases, in which the reinforced concrete member can be subjected to an uncracked or a cracked torque and keep it for a long time.

For uncracked reinforced concrete under torsional load, there are three different analytical methods for solving the rate of twist and stresses acting on concrete and steel. They are by using composite material properties, modified analytical method and conformal mapping method. The results of each method are compared and checking their accuracy by using experimental and finite element method.

For cracked reinforced concrete subjected torsional load, Rausch's space truss model is used for solving the rate of twist and stresses acting on concrete and steel. The post-cracking torsional rigidity can be determined and can be used for further analysis. At the same time, the strains of concrete and reinforcement after cracking can be expressed from the post-cracking torsional rigidity.

The EMM and AEMM approaches are applied for solving the long-term analysis of uncracked reinforced concrete subjected to torsional loading. Only EMM approach is required for analyzing the cracked reinforced concrete under torsional loading since the arrangement for concrete and reinforcement of Rausch's space truss model limited the analysis. Further explanation is provided in the following sections.
5.2 Short Term Analysis for Uncracked RC under Torsional Load

5.2.1 Method 1 — By Composite Material Properties

There are two different cross section area of a reinforced concrete member. One contains longitudinal bars and stirrups. Another one contains longitudinal bars only. The torsional load acting on different sections has different distribution.

First, considering the part of section with stirrups, section A of the reinforced concrete member, there are three components of torques acting on the concrete cover around the stirrups, \( T_{out} \), stirrups, \( T_s \) and internal core of the reinforced concrete member which contains longitudinal bars, \( T_{in} \). The following equations show the relations between acting torque and the components of torques, also, the relations of the torques to rate of twist of section A, \( \beta_A \).

\[
T_{out} + T_{in} + T_s = T \\
T_{in} = (G_{cr})(J_{in})_{la} \beta_A \\
T_{out} = G_r(J_{out})_{la} \beta_A \\
T_s = G_s(J_{str})_{la} \beta_A
\]

\( (G_{cr})(J_{in})_{la} \) is the torsion rigidity of the internal core of the reinforced concrete member containing longitudinal bars. \( G_r(J_{out})_{la} \) is the torsion rigidity of concrete cover around the stirrups and \( G_s(J_{str})_{la} \) is the torsion rigidity of stirrup.

Thus, the rate of twist of section can be determined by equation (5.1),

\[
T = [G_r(J_{out})_{la} + (G_{cr})(J_{in})_{la} + G_s(J_{str})_{la}] \beta_A \quad (5.1)
\]

The polar moment of inertia of internal core of the reinforced concrete member that contains longitudinal bars, \( (J_{in})_{la} \) can be simply determined by considering the internal core as a solid section and with dimension of the internal core, \( x_r \) and \( y_r \), i.e. the torque acting on the internal core as following,
\[ T_{in} = \left( G_r^* \right) \left( J_{in} \right) \beta_A \]
\[ = \left( G_r^* \right) k x_r^3 y_r \beta_A \quad (5.2) \]

where \( k \) — By table(*) if knowing \( x_r / y_r \)

The shear modulus of the internal core \( \left( G_r^* \right) \), can be determined by using parallel model method when knowing shear modulus of concrete \( G_r \) and shear modulus of steel \( G_s \), as well as area of the internal core and total area of longitudinal bars.

\[ \frac{1}{\left( G_r^* \right)} = \frac{A_r}{G_r} + \frac{A_s}{G_s} \quad (5.3) \]

where \( A_1 = x_r y_r \)

\[ A_c = A_1 - A_s \]

\( A_1 \) is the area of the internal core.

\( A_c \) is the area occupied by concrete inside the internal core.

\( A_s \) is the total area of longitudinal bars.

The concrete cover around the stirrups can be treated as a thin walled hollow section. The polar moment of inertia of the concrete cover around the stirrups, \( (J_{out})_{in} \), can be determined by equation (5.3).

\[ (J_{out})_{in} = 4 \left( \text{area bounded by centerline of hollow section} \right)^2 \]
\[ = \frac{4bh}{2(b+h)} \]
\[ ds = 2(b+h) \]
\[ = \frac{2bh}{(b+h)} \quad (5.4) \]
Thus, the torque acting on the concrete cover around the stirrups is,
\[
T_{\text{out}} = G_r (J_{\text{out}})_{1a} \beta_A
= \frac{2G_r (b h)^3}{(b + h)} \beta_A
\]

Since the cross section of a stirrup along z-axis is not uniform, the thin wall thickness of the stirrup can be from zero to the diameter of the stirrup, which can be expressed by the following equation,
\[
t(z) = 2 \sqrt{\left(\frac{ds}{2}\right)^2 - z^2}
\quad (5.5)
\]

The rate of twist of a stirrup due to torque acting on the stirrup at z, \( \beta_A(z) \) can be shown by equation (5.6) as,
\[
\beta_A(z) = \frac{T_s}{4G_s \Gamma^2} \frac{ds}{t(z)} dz
\quad (5.6)
\]

where \( \Gamma = x_1 y_1 \)

\( \Gamma = \) perimeter of area bounded by the centerline of a stirrup.

Hence, the angle of twist at depth z can be expressed by equation (5.7),
\[
\theta_A(z) = \int_{-\Delta}^{\Delta} \frac{T_s}{4G_s \Gamma^2} \frac{ds}{t(z)} dz
\quad (5.7)
\]

When putting equation (5.5) into equation (5.7), the total angle of twist can be
\[
\theta_A = \frac{2T_s (x_1 + y_1)}{4G_s (x_1 y_1)^2} \int_{-\Delta}^{\Delta} \frac{1}{\left(\frac{ds}{2}\right)^2 - z^2} dz
\]
\[
= \frac{T_s (x_1 + y_1)}{4G_s (x_1 y_1)^2 \pi}
\quad (5.8)
\]
determined,
Fig 5.1 Schematic representation a stirrup.

Therefore, the torsional rigidity of the stirrup may be expressed as,

$$G_t \left( J_{stw} \right)_{ta} = \frac{4G_t}{(a_i + b_i)} \pi d_{sv} \quad (5.9)$$

And the polar moment of a stirrup, \((J_{stw})_{ta}\) may be written as

$$\left( J_{stw} \right)_{ta} = \frac{4(x_i y_i)}{(x_i + y_i)} d_{sv} \quad (5.10)$$

Second is the part of section with longitudinal bars only, section B of the reinforced concrete. This section can be considered as a whole parallel model and the rate of twist, \(\beta_g\) may be determined by the following equation,

$$T = \left( G_t \right)_2 J_g \beta_g$$

$$\Rightarrow J_g = k (2a)^3 (2b)$$

where \(k = \text{By table(2.3)}\) if knowing \(2a/2b\)

The shear modulus of the whole section \(\left( G_t \right)_2\), can be found when knowing shear modulus of concrete \(G_c\) and shear modulus of steel \(G_s\), as well as area of the section and total area of longitudinal bars.
\[ \frac{1}{G_c^*} = \left( \frac{A_c}{A_2} \right) + \left( \frac{A_s}{A_2} \right) \quad (5.11) \]

*where*  
\[ A_2 = (2a)(2b) \]  
\[ A_s = A_2 - A_i \]

\( A_2 \) is the area of the whole section.  
\( A_c \) is the area occupied by concrete inside the whole section.  
\( A_s \) is the total area of longitudinal bars.

Similarly for head and end,

\[ T = \left( G_c^* \right)_2 J_B \beta_H \quad (Head) \]
\[ T = \left( G_c^* \right)_2 J_B \beta_E \quad (End) \]

Where,

\[ \beta_B = \beta_H = \beta_E \]

But,

\[ \theta_B = \beta_B L_B \quad \text{or} \quad \beta_B s \]
\[ \theta_H = \beta_H L_H \quad \text{and} \quad \theta_E = \beta_E L_E \]

\( L_B \) is the length of section B.  
\( L_H \) is the length of the head of a member from face of head to a section A.  
\( L_E \) is the length of the head of a member from face of end to a section A.

Total angle of twist of a reinforced concrete member under pure torsion may be expressed as the equations,

\[ \theta_{\text{total}} = n \beta_A d_w + (n-1)\beta_B (s - d_w) + \beta_H L_H + \beta_E L_E \]
Where, \( n = \) no of stirrups and \( s = \) spacing between stirrups from centre to centre.

\[
\theta_{\text{total}} = (n d_{sv}) \beta_A + [(n-1)(s-d_{tv}) + L_H + L_E] \beta_B \\
= (n d_{sv}) \beta_A + [(n-1)(s-d_{sv}) + (L - (n-1)(s-d_{sv}) - nd_{sv})] \beta_B
\]  

(5.12)

5.2.2 Method 2 — Modified Analytical Method

For this method, the section is still included two different cross section areas for a reinforced concrete member. One contains longitudinal bars and stirrups. Another one contains longitudinal bars only. The torsional load also acting on different sections has different distribution. But this method can separate the torsion rigidity for concrete and steel of the internal core of the reinforced concrete member, which contains longitudinal bars.

First, considering the part of section without stirrups, section B of the reinforced concrete, there are only two components of torques. One is acting on the concrete, \( T_c \) and another one is acting on the longitudinal bars, \( T_s \). The following equations show the relations between acting torque and the components of torques, also, the relations of the torques to rate of twist of section B, \( \beta_B \).

\[
T_c + T_s = T \\
T_c = G_c (J_a)_{2b} \beta_B \\
T_s = G_s (J_s)_{2b} \beta_B
\]

\( G_c (J_a)_{2b} \) is the torsion rigidity of the concrete of the reinforced concrete member.

\( G_s (J_s)_{2b} \) is the torsion rigidity of steel of the reinforced concrete member.

Thus, the rate of twist of section can be determined by equation (5.13).

\[
T = |G_c (J_a)_{2b} + G_s (J_s)_{2b}| \beta_B
\]  

(5.13)

The problem is how to find \( G_c (J_a)_{2b} \) and \( G_s (J_s)_{2b} \).

Let’s consider the case of plain concrete under torsion. The relation between the torque, \( T \) and the rate of twist, \( \beta \) can be shown in the following equation.

\[
T = G_c J \beta
\]
The polar moment of inertia, \( J \), can be evaluated as follows:

\[
J = \frac{8a^3b}{3} \left[ \frac{1}{\pi^4} \sum_{n=0}^{\infty} \frac{1}{(2n+1)^4} - \frac{381a}{\pi^2 b} \sum_{n=0}^{\infty} \frac{\sinh k_n b}{(2n+1)^3} \right] = 16a^3b \left[ \frac{1}{3} - \frac{64a}{\pi^2 b} \sum_{n=0}^{\infty} \frac{\tanh k_n b}{(2n+1)^3} \right] = k(2a)^3(2b) \quad (5.14)
\]

Where, \( k_n = \frac{(2n+1)\pi}{2a} \) is to be determined so as to satisfy the boundary condition.

Therefore, the torque, \( T \), can be expressed as follows:

\[
T = 16G_e \beta a^3 b \left[ \frac{1}{3} - \frac{64a}{\pi^2 b} \sum_{n=0}^{\infty} \frac{\tanh k_n b}{(2n+1)^3} \right] \quad (5.15)
\]

With some calculation we find that the shearing stresses are given by the following formulas:

\[
\tau_{xz} = \frac{T}{J} \left( \frac{\partial \Psi}{\partial x} - y \right) = -\frac{16T a}{J \pi^2} \sum_{n=0}^{\infty} \frac{(-1)^n \sinh k_n y}{(2n+1)^2} \cos k_n x
\]

\[
\tau_{yz} = \frac{T}{J} \left( \frac{\partial \Psi}{\partial y} + x \right) = \frac{T}{J} \left[ 2x - \frac{16a}{\pi^2} \sum_{n=0}^{\infty} \frac{(-1)^n \cosh k_n y}{(2n+1)^2} \sin k_n x \right] \quad (5.16)
\]

With the existence of longitudinal bars, there should be additional torques due to the longitudinal bars, for \( n \) number of longitudinal bars and area of longitudinal bars are \( A_n \).

The position of the longitudinal bars with different \( (x_i, y_i) \). The additional torque should be:

\[
\sum_{n=1}^{\infty} (\tau_{yz} x_i - \tau_{xz} y_i) A_n \left( \frac{G_y}{G_e} - 1 \right)
\]
So that the relation between torque, $T$ and rate of twist, $\beta$ of the reinforced concrete member for section B change into the following expression:

$$T = 16G_c \beta_b a^3 b \left[ \frac{1}{3} \frac{64a}{\pi^3 b} \sum_{n=0}^{\infty} \frac{\tanh k_n b}{(2n+1)^3} \right] + \sum_{n=1}^{\infty} \left( \tau_{xz} x_i - \tau_{xy} y_i \right)(A_i) \left( \frac{G_t}{G_c} - 1 \right)$$

$$= G_c \left( k(2a)^3(2b) \right) \beta_b + \sum_{n=1}^{\infty} \left( \tau_{xz} x_i - \tau_{xy} y_i \right)(A_i) \left( \frac{G_t}{G_c} - 1 \right) \quad (5.17)$$

Where the shear stresses acting on the longitudinal bars can be expressed as:

$$\tau_{xz} = -\frac{16G_c \beta_b a}{\pi^3} \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^2} \frac{\sinh k_n y_i}{\cosh k_n b} \cos k_n x_i$$

$$\tau_{xy} = G_c \beta_b \left[ 2x - \frac{16a}{\pi^3} \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^2} \frac{\cosh k_n y_i}{\sinh k_n b} \sin k_n x_i \right]$$

Let $A_{bi}$ and $B_{bi}$ be the coefficients as follows:

$$A_{bi} = -\frac{16a}{\pi^3} \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^2} \frac{\sinh k_n y_i}{\cosh k_n b} \cos k_n x_i$$

$$B_{bi} = \left[ 2x - \frac{16a}{\pi^3} \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^2} \frac{\cosh k_n y_i}{\sinh k_n b} \sin k_n x_i \right]$$

Thus,

$$T = G_c \left( k(2a)^3(2b) \right) \beta_b - \left( \sum_{n=1}^{\infty} (A_{bi} x_i - B_{bi} y_i)(A_i) \right) G_c \beta_b + \left( \sum_{n=1}^{\infty} (A_{bi} x_i - B_{bi} y_i)(A_i) \right) G_c \beta_b$$

$$= \left[ \left( k(2a)^3(2b) \right) - \left( \sum_{n=1}^{\infty} (A_{bi} x_i - B_{bi} y_i)(A_i) \right) G_c + \left( \sum_{n=1}^{\infty} (A_{bi} x_i - B_{bi} y_i)(A_i) \right) G_c \right] \beta_b \quad (5.18)$$

Therefore, the torsional rigidity for concrete and for steel can be determined.
\begin{align}
G_r(j_\alpha)_{2b} &= G_r \left[ k(2a)^3(2b) - \left( \sum_{n=1}^{N} (A_{\alpha n} x_i - B_{\alpha n} y_i)(A_{\alpha i}) \right) \right] \\
G_i(j_{\gamma})_{2b} &= G_i \left( \sum_{n=1}^{N} (A_{\gamma n} x_i - B_{\gamma n} y_i)(A_{\gamma i}) \right)
\end{align} (5.19) (5.20)

For the part of section with stirrups, section A of the reinforced concrete, it can be divided into four components of torques, which are acting on the concrete cover around the stirrups, \( T_{c1} \), stirrups, \( T_{s1} \), longitudinal bars of the internal core of the reinforced concrete member \( T_{s2} \) and concrete the internal core of the reinforced concrete member, \( T_{c2} \). The relations of the torques to rate of twist of section A, \( \beta_A \) is the following.

\[
T_{s1} = G_s(j_{s1})_{2a} \beta_A \\
T_{s2} = G_s(j_{s2})_{2a} \beta_A \\
T_{c1} = G_c(j_{c1})_{2a} \beta_A \\
T_{c2} = G_c(j_{c2})_{2a} \beta_A \\
\text{So that}
\]

\[
T_{s1} + T_{s2} + T_{c1} + T_{c2} = T
\]

The polar moment of inertia of the concrete cover around the stirrups, \( (J_{c1})_{2a} \), is the same as the value for Method 1, which is,

\[
(J_{c1})_{2a} = \frac{2(bh)^3 t}{(b+h)}
\]

The polar moment of a stirrup, \( (J_{s1})_{2a} \), is also same as the one of Method 1, i.e.

\[
(J_{s1})_{2a} = \frac{4(x_i y_i)^3 \pi}{(x_i + y_i)^3} d_{iv}
\]
For the internal core, the determination of torsional rigidities is similar to the determination of section B, but the dimensions are different. The torsional rigidity for concrete and for steel are,

\[
G_c(J_a)_{2a} = G_c \left( k x_c^3 y_c \right) - \left( \sum_{n=1}^{n} (A_n x_i - B_n y_i) (A_i^i) \right) \tag{5.21}
\]

\[
G_s(J_\gamma)_{2a} = G_s \left( \sum_{n=1}^{n} (A_n x_i - B_n y_i) (A_i^i) \right) \tag{5.22}
\]

Where,

\[
A_{ai} = \frac{8x_c}{\pi^2} \sum_{n=0}^{\infty} \frac{(-1)^n \sinh 2k_n y_c \cos k_n x_i}{(2n+1)^2 \cosh k_n y_c}
\]

\[
B_{ai} = \left[ 2x - \frac{8x_c}{\pi^2} \sum_{n=0}^{\infty} \frac{(-1)^n \cosh 2k_n y_c \sin k_n x_i}{(2n+1)^2 \cosh k_n y_c} \right]
\]

Similarly for head and end,

\[
T = [G_c(J_a)_{2b} + G_s(J_\gamma)_{2b}] \beta_H \quad (Head)
\]

\[
T = [G_c(J_a)_{2b} + G_s(J_\gamma)_{2b}] \beta_E \quad (End)
\]

Where,

\[
\beta_B = \beta_H = \beta_E
\]

But,

\[
\theta_B = \beta_B L_B \quad \text{or} \quad \beta_B \theta_B
\]

\[
\theta_H = \beta_H L_H \quad \text{and} \quad \theta_E = \beta_E L_E
\]

\(L_B\) is the length of section B.

\(L_H\) is the length of the head of a member from face of head to a section A.

\(L_E\) is the length of the head of a member from face of end to a section A.
Total angle of twist of a reinforced concrete member under pure torsion is also the same as Method 1,

\[
\theta_{\text{total}} = n\beta_A d_{yw} + (n-1)\beta_B (s - d_{yw}) + \beta_H L_H + \beta_L L_E
\]

\[
\theta_{\text{total}} = (nd_{yw})\beta_A + [(n-1)(s - d_{yw}) + L_H + L_E]\beta_B
\]

\[
= (nd_{yw})\beta_A + [(n-1)(s - d_{yw}) + (L - (n-1)(s - d_{yw}) - nd_{yw})]\beta_B
\]  \hspace{1cm} (5.12)

Where, \(n\) = no of stirrups and \(s\) = spacing between stirrups from centre to centre

5.2.3 Method 3 — Idealized Reinforced Concrete Beam Method (Conformal Mapping)

With consideration of square and rectangular section, the beam with longitudinal bars and stirrups can be idealized as thin steel tube. The elastic solution can be solved by using complex variable and conformal mapping technique. Thus the boundaries of the beam cross-section become two concentric squares or rectangular with rounded corners.

The equivalent tube thickness can be obtained by calculating the total volume of steel present in a given length of beam and then determining the thickness of the equivalent tube so that the volume of the tube will equal the volume of the combined longitudinal bars and stirrups.

The solution to the torsion problem for doubly connected compound cross sections can be determined by mapping the boundaries of the cross section onto an annulus. The complex torsion functions \(f_1\) and \(f_2\) can also be expressed and the boundary conditions in terms of the annulus in the \(\zeta\)-plane.

For the case of a \(w \times w\) square and an annulus with a unit outer radius is:

\[
\omega(\zeta) = a\zeta + b\zeta^5 + c\zeta^9 + d\zeta^{13} + e\zeta^{17} + f\zeta^{21} + g\zeta^{25}
\]

\[
= 1.08 \left( \frac{w}{2} \right) \left[ \zeta - \frac{\zeta^5}{10} + \frac{\zeta^9}{24} + \frac{35\zeta^{13}}{208} + \frac{3\zeta^{17}}{2176} - \frac{3\zeta^{21}}{256} + \frac{231\zeta^{25}}{25600} \right]
\]  \hspace{1cm} (5.23)

Where,
\[ a = +0.54w \\
\[ b = -0.054w \\
\[ c = +0.0225w \\
\[ d = -0.0129808w \\
\[ e = +0.0086857w \\
\[ f = -0.0632813w \\
\[ g = +0.0048727w \\
\]

For rectangular section, the expression of \( \omega(\zeta) \), is different, which can be derived from

\[
\omega(\zeta) = \int_0^1 \frac{dt}{\pi(t - a_i) \alpha_k}, \quad \sum \alpha_i = 2, t = z \quad \text{on} \quad \Gamma, \quad \text{number of coefficients}
\]

would be increased from a to m.

Hence, the quantity \( \frac{1}{2} z \dot{z} \) can be expressed as following:

\[
\frac{1}{2} z \dot{z} = A + \frac{B}{2} (\zeta^4 + \zeta^{-4}) + \frac{C}{2} (\zeta^8 + \zeta^{-8}) + \frac{D}{2} (\zeta^{12} + \zeta^{-12}) + \frac{E}{2} (\zeta^{16} + \zeta^{-16})
\]

\[
+ \frac{F}{2} (\zeta^{20} + \zeta^{-20}) + \frac{G}{2} (\zeta^{24} + \zeta^{-24}) \quad (5.24)
\]

\[
= A + \frac{B}{2} \rho^4 (\sigma^4 + \sigma^{-4}) + \frac{C}{2} \rho^8 (\sigma^8 + \sigma^{-8}) + \frac{D}{2} \rho^{12} (\sigma^{12} + \sigma^{-12}) + \frac{E}{2} \rho^{16} (\sigma^{16} + \sigma^{-16})
\]

\[
+ \frac{F}{2} \rho^{20} (\sigma^{20} + \sigma^{-20}) + \frac{G}{2} \rho^{24} (\sigma^{24} + \sigma^{-24}) \quad (5.25)
\]

However, \( \sigma^n + \sigma^{-n} = 2 \cos(n\theta) \).

Thus,

\[
\frac{1}{2} z \dot{z} = A + B \rho^4 \cos(4\theta) + C \rho^8 \cos(8\theta) + D \rho^{12} \cos(12\theta) + E \rho^{16} \cos(16\theta)
\]

\[
+ F \rho^{20} \cos(20\theta) + H \rho^{24} \cos(24\theta) \quad (5.26)
\]
Where,

\[
A = \rho^2 \left( a^2 + b^2 \rho^8 + c^2 \rho^{16} + d^2 \rho^{24} + e^2 \rho^{32} + f^2 \rho^{40} + g^2 \rho^{48} \right)
\]

\[
B = \rho^6 \left( ab + bcp^8 + cdp^4 \rho^{16} + de \rho^{24} + ef \rho^{32} + fg \rho^{40} \right)
\]

\[
C = \rho^{10} \left( ac + bdp^8 + cep^4 \rho^{16} + dcp^4 \rho^{24} + ecp^4 \rho^{32} \right)
\]

\[
D = \rho^{14} \left( ad + bep^8 + cfp^3 \rho^{16} + dgp^3 \rho^{24} \right)
\]

\[
E = \rho^{18} \left( ae + bf \rho^8 + cg \rho^{16} \right)
\]

\[
F = \rho^{22} \left( af + bg \rho^8 \right)
\]

\[
G = \rho^{26} \left( ag \right)
\]

The coefficients for the mapping function and the product \( \frac{1}{2} z \overline{z} \) on the inner boundary can be calculated by first determining the inner radius \( \rho_1 \). This radius is a function of the side of the inner square, Fig 2.22. Newton's method can provide a convergent solution for calculating these numerical inner radii.

\[
x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \quad (5.27)
\]

Where,

\( x_n \) = approximation to the root

\( f(x) \) = functional relationship

\( f'(x) \) = derivative of functional relationship

So that,

\[
\omega(\zeta) = \frac{w}{2} - t = a\rho_1 + b\rho_1^5 + c\rho_1^9 + d\rho_1^{13} + e\rho_1^{17} + f\rho_1^{21} + g\rho_1^{25}
\]

where \( t = \text{thickness of equivalent tube} \)

\( \rho_2 = 1 \) (unit circle)

If the outer boundary, \( \rho_2 \) is considered, then for the problem under consideration
\[ \varphi_2 = b_0^* + \sum_{k=1}^{\infty} \left[ \rho_2^k a_k^* - \rho_2^{-k} a_{-k}^* \right] \sin(k\theta) + \left[ \rho_2^k b_k^* + \rho_2^{-k} b_{-k}^* \right] \cos(k\theta) \]

\[ = A_2 + B_2 \cos(4\theta) + C_2 \cos(8\theta) + D_2 \cos(12\theta) + E_2 \cos(16\theta) + F_2 \cos(20\theta) + G_2 \cos(24\theta) + k_2 \]  
(5.28)

Since there are no \( \sin(k\theta) \) terms present in \( \frac{1}{2} z \bar{z} \) one can conclude that

\[ a_k^* = a_{-k}^* = 0 \]  
(5.29)

Therefore,

\[ \varphi_2 = b_0^* + \sum_{k=1}^{\infty} \left[ \rho_2^k b_k^* + \rho_2^{-k} b_{-k}^* \right] \cos(k\theta) \]  
(5.30)

Hence,

\[ b_0^* = (A_2 + k_2) \]
\[ b_k^* + b_{-k}^* = B_2 \]
\[ b_k^* + b_{-k}^* = C_2 \]

\[ b_{12}^* + b_{-12}^* = D_2 \]
\[ b_{16}^* + b_{-16}^* = E_2 \]
\[ b_{20}^* + b_{-20}^* = F_2 \]
\[ b_{24}^* + b_{-24}^* = G_2 \]  
(5.31)

In a similar way if one considers the inner boundary, \( \rho_1 \), then

\[ \varphi_1 = b_0^* + \sum_{k=1}^{\infty} \left[ b_k^* \rho_1^k \cos(k\theta) \right] \]  
(5.32)

Since there are no \( \sin(k\theta) \) terms in either \( \frac{1}{2} z \bar{z} \) or \( \varphi_2 \). Hence,

\[ \frac{\partial \varphi_1}{\partial \rho} = \sum_{k=1}^{\infty} b_k^* \cos(k\theta) \left[ k\rho_1^{k-1} \right] \]  
(5.33)

\[ \frac{\partial \varphi_2}{\partial \rho} = \sum_{k=1}^{\infty} \left[ k\rho_1^{k-1} b_k^* - k\rho_1^{-(k+1)} b_{-k}^* \right] \cos(k\theta) \]  
(5.34)
It follows that

\[ kb_i^* \rho_i^{k-1} = k \rho_i^{k-1} b_i^* - k \rho_i^{(k+1)} b_{-i}^* \quad (5.35) \]

Finally, if one substitutes the expressions for \( \varphi_1, \varphi_2 \) and \( \frac{1}{2} z z^* \), then,

\[ G_2 \left( \rho_1^4 b_i^* + \rho_1^4 b_{-i}^* \right) - G_i b_i^* \rho_i^k = (G_2 - G_1) A_k \quad (5.36) \]

It can be rearranged as follows, where, \( A_k = A, B, C, \ldots \)

\[ v \rho_i^{24} b_i^* + b_{-i}^* = uA_k \rho_i^k \quad (5.37) \]

and,

\[ v = \frac{G_2 - G_1}{G_2 + G_1} \]

Equation (5.37) can be expressed in the following form

\[ v \rho_i^8 b_i^* + b_{-4}^* = uB_i \rho_i^8 \]
\[ v \rho_i^{16} b_8^* + b_{-8}^* = uC_i \rho_i^8 \]
\[ v \rho_i^{24} b_{12}^* + b_{-12}^* = uD_i \rho_i^{12} \]
\[ v \rho_i^{48} b_{16}^* + b_{-16}^* = uE_i \rho_i^{16} \]
\[ v \rho_i^{40} b_{20}^* + b_{-20}^* = uF_i \rho_i^{20} \]
\[ v \rho_i^{48} b_{24}^* + b_{-24}^* = uG_i \rho_i^{24} \quad (5.37a) \]

Substituting equations (5.31) into equation (5.37a) for like values of \( K \) and solving for \( b_i^* \), one obtains
\[ b_4^{*} = \left[ \frac{\nu B_1 \rho_1^3 - B_2}{\nu p_1^4} \right] \]

\[ b_8^{*} = \left[ \frac{\nu C_1 \rho_1^6 - C_2}{\nu \rho_1^{16}} \right] \]

\[ b_{12}^{*} = \left[ \frac{\nu D_1 \rho_1^{12} - D_2}{\nu \rho_1^{24}} \right] \]

\[ b_{16}^{*} = \left[ \frac{\nu E_1 \rho_1^{16} - E_2}{\nu \rho_1^{32}} \right] \]

\[ b_{20}^{*} = \left[ \frac{\nu F_1 \rho_1^{20} - F_2}{\nu \rho_1^{40}} \right] \]

\[ b_{24}^{*} = \left[ \frac{\nu G_1 \rho_1^{24} - G_2}{\nu \rho_1^{48}} \right]. \quad (5.38) \]

The complex torsion functions \( f_1 \) and \( f_2 \) can be determined, with consideration of the above coefficients. For the provided mapping, \( I \) and \( D_0 \) can be evaluated for both regions using the residues theorem of complex variable theory. The residues and values of the integral for \( I \) and \( D_0 \) for both regions can be developed, starting with the outer region.

With the expressions for \( f_1, f_2 \) and \( \omega(\zeta) \), the residues of the above integrals are

\[ (D_0)_2 = (D_0)_{2\Gamma_2} + (D_0)_{2\Gamma_1} = -\frac{1}{4} \int_{\Gamma_1} \left[ f_2(\zeta_1) + \overline{f_2(\zeta_1)} \right] d[\omega(\zeta_1)\overline{\omega(\zeta_1)}] \]

\[ -\frac{1}{4} \int_{\Gamma_2} \left[ f_2(\zeta_2) + \overline{f_2(\zeta_2)} \right] d[\omega(\zeta_2)\overline{\omega(\zeta_2)}] \quad (5.39) \]

\[ \sum \text{Residues on } \Gamma_2 = -2\pi \left[ 4B_2 (b_{4}^{*} - b_{4}^{*}) + 8C_2 (b_{8}^{*} - b_{8}^{*}) + 12D_2 (b_{12}^{*} - b_{12}^{*}) + 16E_2 (b_{16}^{*} - b_{16}^{*}) + 20F_1 (b_{20}^{*} - b_{20}^{*}) + 24G_2 (b_{24}^{*} - b_{24}^{*}) \right] \quad (5.40) \]
\[ \sum \text{Residues on } \Gamma_1 = -2 \left[ 4B_i \left( \rho_i^4 b_i^4 - \rho_i^{12} b_i^{12} \right) + 8C_i \left( \rho_i^8 b_i^8 - \rho_i^{10} b_i^{10} \right) \right] \]

\[ + 12D_i \left( \rho_i^2 b_i^2 - \rho_i^{12} b_i^{12} \right) + 16E_i \left( \rho_i^{10} b_i^{10} - \rho_i^{10} b_i^{10} \right) \]

\[ + 20F_i \left( \rho_i^{20} b_i^{20} - \rho_i^{20} b_i^{20} \right) + 24G_i \left( \rho_i^{24} b_i^{24} - \rho_i^{24} b_i^{24} \right) \]  

(5.41)

Hence,

\[ (D_0)_2 = 2\pi \left( -\frac{1}{4} \right) \left[ \text{Residues on } \Gamma_2 + \text{Residues on } \Gamma_1 \right] \]  

(5.42)

In a similar manner \((D_0)_i\) for inner region is

\[ (D_0)_i = -\frac{1}{4} \int_{\Gamma_i} \left[ f_i(\zeta_1) + \overline{f_i(\zeta_1)} \right] \left[ \omega(\zeta_1) + \overline{\omega(\zeta_1)} \right] \]  

(5.43)

Thus, the residues are

\[ \sum \text{Residues on } \Gamma_1 = -i \left[ 8B_i \rho_i^4 b_i^4 + 32C_i \rho_i^8 b_i^8 + 32D_i \rho_i^{12} b_i^{12} + 16E_i \rho_i^{10} b_i^{10} + 40F_i \rho_i^{20} b_i^{20} + 48G_i \rho_i^{24} b_i^{24} \right] \]  

(5.44)

By the residues theorem

\[ (D_0)_i = -\pi \left[ 4B_i \rho_i^4 b_i^4 + 8C_i \rho_i^8 b_i^8 + 12D_i \rho_i^{12} b_i^{12} + 16E_i \rho_i^{10} b_i^{10} + 20F_i \rho_i^{20} b_i^{20} + 24G_i \rho_i^{24} b_i^{24} \right] \]  

(5.45)

It can provide a check on the numerical result

\[ |(D_0)_i| = |(D_0)_{2\Gamma_i}| \]  

(5.46)

The polar moment of inertia can be determined for both the inner and outer regions. Since the polar moment of inertia for the outer region is the following.

If the expression \( z = \omega(\zeta) \), then the residues are

\[ I_z = I_{r_2} - I_{r_1} \]  

(5.47)

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\[ \sum \text{Re} \text{ sidues} = a^4 \rho^3 + 12a^2 b^2 \rho^{11} + 5(2ac + b^2) \rho^{19} + 28(ad + bc)^2 \rho^{27} + 9(2ae + 2bd + c^2) \rho^{35} + 44(af + be + cd) \rho^{43} + 13(2ag + 2bf + 2ce + d^2) \rho^{51} + 60(bg + cf + de) \rho^{59} + 17(2cg + 2df + e^2) \rho^{67} + 76(dg + cf) \rho^{75} + 21(2eg + f^2) \rho^{83} + 92(fg)^2 \rho^{91} + 25g^4 \rho^{99} \quad (5.48) \]

The polar moment of inertia inside the required boundary for the region is

\[ I = 2\pi i \sum \text{Re} \text{ sidues} (5.48) \]

\[ T = \left[ G_1 \left[ I_{11} + (D_0) \right] + G_2 \left[ (I_{22} - I_{11}) + (D_0) \right] \right] \beta \quad (5.49) \]

Therefore, the relationship between torque, \( T \) and rate of twist, \( \beta \) has the following expression,

The maximum shearing stress will occur at the middle of the side of the square, and hence,

\[ \tau_{\text{max}} = (\tau_{\rho})_{\theta=90^\circ} = (\tau_{\theta})_{\theta=0} \quad (5.50) \]

Only expressions for \( (\tau_{\theta})_{\theta=0} \) on the inner and outer boundaries require to be developed as the maximum shear stresses occur on all the sides. The point of maximum stress for the concrete and steel lies on the positive real axis.

From equation (5.23) the derivative and conjugate of the mapping function can be determined as follows:

\[ \omega'(\zeta) = a + 5b\zeta^4 + 9c\zeta^8 + 13d\zeta^{12} + 17e\zeta^{16} + 21f\zeta^{20} + 25g\zeta^{24} \quad (5.51) \]

\[ \overline{\omega(\zeta)} = a\overline{\zeta} + b\overline{\zeta}^5 + c\overline{\zeta}^9 + d\overline{\zeta}^{13} + e\overline{\zeta}^{17} + f\overline{\zeta}^{21} + g\overline{\zeta}^{25} \quad (5.52) \]

Also, \( |\omega'(\zeta)| = \omega'(\zeta) \) on the positive real axis

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The derivatives of the complex torsion function \( f_1 \) and \( f_2 \) are expressed:

\[
f_1' (\zeta) = i \left[ 4b_4 \zeta^3 + 8b_6 \zeta^7 + 12b_4 \zeta^{11} + 16b_4 \zeta^{15} + 20b_4 \zeta^{19} + 24b_4 \zeta^{23} \right] \quad (5.53)
\]

\[
f_2' (\zeta) = i \left[ 24b_8 \zeta^{21} - 4b_8 (\zeta)^{-5} - 8b_8 (\zeta)^{-9} - 12b_8 (\zeta)^{-13} \right] \quad (5.54)
\]

\[-16b_8 (\zeta)^{-17} - 20b_8 (\zeta)^{-21} - 24b_8 (\zeta)^{-25}\]

The resulting torque of the specimen can be expressed as a function of the maximum shearing stress of the concrete and the torsional rigidity of the specimen. This relationship can be easily obtained by combining the expressions for \( D \) or \( GJ \) and \( \tau \) in order to cancel the rate of twist.

\[
\tau_o = \frac{GJ\beta}{k_3} \quad (5.55)
\]

\[
k_3 = k_2 (xy^2) \quad (5.56)
\]

\[
T = torque = \frac{D\tau_{\text{max}}}{k_2 Gc} \quad (5.57)
\]

The torsional rigidity and the maximum shear stress can be compared with the case of a homogenous prism by setting the shearing modules equal. This provides a check on the accuracy of the mapping function and the subsequent calculations. According to table (5.1), the difference between the values for both the torsional rigidity and maximum shear stress computed by the above method and those tabulated by Timoshenko are less than about 1.5 percent.

Table (5.1) shows the values of \( D \) or \( GJ \), the measured torque and the predicted maximum shearing stresses for the various specimens.
Table 5.1

Data for compound Torsion Problem

<table>
<thead>
<tr>
<th>$\omega(\zeta)$</th>
<th>$1 - \frac{zz}{z} \cdot \rho_2$ on $\zeta_2$</th>
<th>$1 - \frac{zz}{z} \cdot \rho_1$ on $\zeta_2$</th>
<th>K</th>
<th>Specimens with 2 percent balanced reinforcement</th>
</tr>
</thead>
<tbody>
<tr>
<td>$a \to g$</td>
<td>$A_2 \to G_2$</td>
<td>$A_2 \to G_2$</td>
<td></td>
<td>$\nu = 0.64$</td>
</tr>
<tr>
<td>+3.2314476</td>
<td>+5.2788810</td>
<td>+5.0174524</td>
<td>0</td>
<td>-</td>
</tr>
<tr>
<td>-0.3124034</td>
<td>-1.0547722</td>
<td>-0.9035604</td>
<td>4</td>
<td>-1.1206568</td>
</tr>
<tr>
<td>+0.1172773</td>
<td>+0.4007379</td>
<td>+0.4007379</td>
<td>8</td>
<td>+0.4186525</td>
</tr>
<tr>
<td>-0.0562475</td>
<td>-0.1924016</td>
<td>-0.1924016</td>
<td>12</td>
<td>-0.1986300</td>
</tr>
<tr>
<td>+0.0282273</td>
<td>+0.0959106</td>
<td>+0.0959106</td>
<td>16</td>
<td>+0.0981701</td>
</tr>
<tr>
<td>-0.0131937</td>
<td>-0.0441632</td>
<td>-0.0441632</td>
<td>20</td>
<td>-0.0449153</td>
</tr>
<tr>
<td>+0.0048924</td>
<td>+0.0158095</td>
<td>+0.0158095</td>
<td>24</td>
<td>+0.0159964</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$\omega(\zeta)$</th>
<th>$1 - \frac{zz}{z} \cdot \rho_2$ on $\zeta_2$</th>
<th>$1 - \frac{zz}{z} \cdot \rho_1$ on $\zeta_1$</th>
<th>K</th>
<th>Specimens with 1 percent balanced reinforcement</th>
</tr>
</thead>
<tbody>
<tr>
<td>$a \to g$</td>
<td>$A_2 \to G_2$</td>
<td>$A_1 \to G_1$</td>
<td></td>
<td>$\nu = 0.74$</td>
</tr>
<tr>
<td>+3.2314476</td>
<td>+5.2788810</td>
<td>+5.0174524</td>
<td>0</td>
<td>-</td>
</tr>
<tr>
<td>-0.3124034</td>
<td>-1.0547722</td>
<td>-0.9035604</td>
<td>4</td>
<td>-1.1259888</td>
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<tr>
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<td>-0.1354140</td>
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<tr>
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<td>+0.0959106</td>
<td>+0.0613674</td>
<td>16</td>
<td>+0.0987883</td>
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<tr>
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<td>-0.0441632</td>
<td>-0.0257305</td>
<td>20</td>
<td>-0.0451703</td>
</tr>
<tr>
<td>+0.0048924</td>
<td>+0.0158095</td>
<td>+0.0084100</td>
<td>24</td>
<td>+0.0160723</td>
</tr>
</tbody>
</table>

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Table 5.1

Data for compound Torsion Problem (continued)

<table>
<thead>
<tr>
<th>GJ or D value for 2 percent balanced reinforcement</th>
</tr>
</thead>
<tbody>
<tr>
<td>((D_0)_2\Gamma_2 = -22.2402463), ((D_0)_2\Gamma_1 = +16.3023987) (ckeck), ((D_0)_1 = -16.3024015),</td>
</tr>
<tr>
<td>(I_{\Gamma_1} = -199.3794795), (I_{\Gamma_1} = -179.0919959),</td>
</tr>
<tr>
<td>(GJ = D = 14.3496360G_\Gamma + 162.7895944G_\Gamma),</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>GJ or D value for 1 percent balanced reinforcement</th>
</tr>
</thead>
<tbody>
<tr>
<td>((D_0)_2\Gamma_2 = -22.4611667), ((D_0)_2\Gamma_1 = +18.4836266) (ckeck), ((D_0)_1 = -18.4836263),</td>
</tr>
<tr>
<td>(I_{\Gamma_1} = -199.3794795), (I_{\Gamma_1} = -186.2957186),</td>
</tr>
<tr>
<td>(GJ = D = 9.1062208G_\Gamma + 167.8120923G_\Gamma),</td>
</tr>
</tbody>
</table>

5.2.4 Comparison Between Three Methods of Uncracked Analysis

In order to compare to difference of the results for the three analytical methods of reinforced concrete section under uncracked torque, a series of numerical test is provided for showing the results obtained by each method of analysis. The external pure torque, \(T\) acting on the member is 400 Nm. Cross sectional, material properties and dimensions are as follows:

\[
2a = 120mm \quad 2b = 120mm
\]
\[
x_1 = 80mm \quad y_1 = 80mm
\]
\[
d_{x_1} = 6mm \quad d_{y_1} = 6mm \quad (4 \text{ longitudinal bars})
\]
\[
E_r(\tau_r) = 25GPa \quad \nu_r = 0.15 \quad (\text{Poisson ratio for concrete})
\]
\[
E_s = 200GPa \quad \nu_s = 0.3 \quad (\text{Poisson ratio for steel})
\]
\[
L = 1000mm, \quad (\text{Length of member})
\]
\[
n = 13, (\text{No of stirrups}) \quad s = 80mm \quad (\text{Spacing of stirrups})
\]

First, there is the comparison for plain concrete and the three analytical methods of angles of twist, \(\theta\) by changing the ratio of the dimensions of section and stirrups, where the ratios of the section and stirrup are set to the same value. Also, the amount and
diameter of longitudinal bars is constant. The results are presented in Table 5.2 and Fig 5.2.

<table>
<thead>
<tr>
<th>2a</th>
<th>x1</th>
<th>y1/x1</th>
<th>B/a</th>
<th>2b</th>
<th>y1</th>
<th>Steel Ratio (%)</th>
<th>θ (Method1)</th>
<th>θ (Method2)</th>
<th>θ (Method3)</th>
<th>θ (Plain concrete)</th>
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</table>

Table 5.2 Comparison of θ Vs different b/a and diameter of steel bars is constant.

![Comparison of θ Vs diff b/a and dia of steel bars = constant](image)

Fig 5.2

From the calculations, we found that the angles of twist obtained by Method 2 are quite closed to the values of plain concrete and they are slightly smaller than the values of plain concrete. The values calculated from Method 1 are smaller than the one given by
Method 2. When the ratio of $b/a$ increases i.e. the rectangle section with greater length-width ratio, the difference becomes larger. This means the value calculated by Method 1 becomes much smaller as the $b/a$ ratio increase. The results computed by Method 3 are much lesser than Method 1 and Method 2; in especial for large $b/a$ ratio, it gives extremely small values of angle of twist. It tends to zero for a section, which is a very flat one.

This may be the fact that the parallel model (Method 1) cannot model the effect of different location of longitudinal bars. For example, if the one longitudinal bar is originally placed at the corner and touching the stirrup, changed its position to the centre of the section, this method can only give the same value of angle of twist, i.e. cannot model the effect caused by different location of longitudinal bars. For analytical Method 3, the condensed mapping series would have a great fluctuation if the shape of the change from a square to rectangle. It gives extremely small results if the section is a flat one.

The following is the next comparison for plain concrete and the three analytical methods of angles of twist, $\theta$ by setting the ratio of the dimensions of section and stirrups to be constant equals one, where the value if dimensions of the section and stirrup are increasing but with the same ratio. The amount and diameter of longitudinal bars and stirrups are constant. The results are presented in Table 5.3 and Fig 5.3.
<table>
<thead>
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<th>2a</th>
<th>xl</th>
<th>y1/x1</th>
<th>b/a</th>
<th>2b</th>
<th>y1</th>
<th>Steel Ratio (%)</th>
<th>θ (Method1)</th>
<th>θ (Method2)</th>
<th>θ (Method3)</th>
<th>θ (Plain concrete)</th>
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</tbody>
</table>

Table 5.3 Comparison of θ Vs different size of cross section and stirrups, diameter of steel bars and stirrups is constant.

![Comparison of θ Vs diff size and dia of steel bars = constant b/a = 1](image)

Fig 5.3

The results given by three methods are quite close, less than 30 % difference. Method 1 provides largest values while Method 3 gives smaller values but the differences are not too significant. This illustrates three methods can provide similar results even steel ratio is changing.

The three comparison is the plain concrete and the three analytical methods of angles of twist, θ by setting the ratio of the dimensions of section and stirrups to be
constant equals one, where the value if dimensions of the section and stirrup are increasing but with the same ratio. The amount and diameter of longitudinal bars is constant, but increasing for stirrups. The results are shown in Table 5.4 and Fig 5.4.

<table>
<thead>
<tr>
<th>2a</th>
<th>x1</th>
<th>y1/x1</th>
<th>b/a</th>
<th>2b</th>
<th>y1</th>
<th>Steel Ratio (%)</th>
<th>dsv (mm)</th>
<th>θ (Method1)</th>
<th>θ (Method2)</th>
<th>θ (Method3)</th>
<th>θ (Plain concrete)</th>
</tr>
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<td>1.173E-03</td>
<td>1.573E-03</td>
</tr>
<tr>
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<td>80</td>
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<td>1</td>
<td>120</td>
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</table>

Table 5.4 Comparison of θ Vs same size of cross section and stirrups, different diameter of stirrups and the diameter of longitudinal bars is constant.

![Comparison of θ Vs diff dia of stirrups dsv and same cross section area](image)

Fig 5.4

The results given by three methods are also quite close, less than 30 % difference, apart from the diameter of 25 mm stirrup. Method 1 still provides largest values while Method 3 gives smaller values but the differences are not too significant for small diameter stirrups. This shows three methods can provide similar results for square section
and shape of the section can be a sensitive factor affecting the results of Method 1 and Method 3. The comparison can provide more evidence for supporting this point.

Similarly, instead of different diameters of stirrups, different diameters of longitudinal bars are tested for the three analytical methods. The results are shown in Table 5.5 and Fig 5.5.

<table>
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<th>y1/x1</th>
<th>b/a</th>
<th>2b</th>
<th>y1</th>
<th>Steel Ratio (%)</th>
<th>ds (mm)</th>
<th>θ (Method1)</th>
<th>θ (Method2)</th>
<th>θ (Method3)</th>
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</table>

Table 5.5 Comparison of θ Vs same size of cross section and stirrups, different diameters of longitudinal bars and the diameter of stirrups is constant.

![Comparison of θ Vs diff dia of longitudinal bars ds and same cross section area](image)

Fig 5.5
Time Dependent Effects of Reinforced Concrete Subjected to Torsional Loading

The results are similar to the above condition. The three methods give close results. Method 1 still gives largest values while Method 3 gives smaller values but the differences are not too significant for small diameter longitudinal bars.

The following are further verification for showing that the shape of the rectangular section is the significant factor affecting the results given by Method 1 and Method 3. In order to show the effect of the shape of rectangular section, the b/a ratio used for the numerical test is start from one to ten. The following tables and figures show the change of the angle of twist of different methods.

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<tr>
<th>2a</th>
<th>x1</th>
<th>y1/x1</th>
<th>b/a</th>
<th>2b</th>
<th>y1</th>
<th>Steel Ratio (%)</th>
<th>ds (mm)</th>
<th>θ (Method1)</th>
<th>θ (Method2)</th>
<th>θ (Method3)</th>
<th>θ (Plain concrete)</th>
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Table 5.6 Comparison of θ Vs different size of stirrups and the b/a ratio = 1 and x1/y1 ratio = 1.

<table>
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5-28
Table 5.7 Comparison of $\theta$ Vs different size of stirrups and the $b/a$ ratio = 1.2 and $x_i/y_i$ ratio = 1.2.

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Table 5.8 Comparison of $\theta$ Vs different size of stirrups and the $b/a$ ratio = 1.5 and $x_i/y_i$ ratio = 1.5.

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Table 5.9 Comparison of $\theta$ Vs different size of stirrups and the $b/a$ ratio = 2 and $x_i/y_i$ ratio = 2.

5-29
Time Dependent Effects of Reinforced Concrete Subjected to Torsional Loading

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Table 5.10 Comparison of θ Vs different size of stirrups and the b/a ratio = 3 and x1/y1 ratio = 3.

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Table 5.11 Comparison of θ Vs different size of stirrups and the b/a ratio = 5 and x1/y1 ratio = 5.

5-30
Table 5.12 Comparison of $\theta$ Vs different size of stirrups and the $b/a$ ratio = 10 and $x_1/y_1$ ratio = 10.

<table>
<thead>
<tr>
<th>2a</th>
<th>x1</th>
<th>y1/x1</th>
<th>b/a</th>
<th>2b</th>
<th>y1</th>
<th>Steel Ratio (%)</th>
<th>ds (mm)</th>
<th>$\theta$ (Method1)</th>
<th>$\theta$ (Method2)</th>
<th>$\theta$ (Plain concrete)</th>
<th>Shape Factor (Method2)</th>
<th>Shape Factor (Method3)</th>
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<td>1100</td>
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</tr>
</tbody>
</table>

Fig 5.6
Comparison of $\theta$ Vs diff size of stirrups x1 and same cross section area, b/a = 1.2, x1/y1 = 1.2

Fig 5.7

Comparison of $\theta$ Vs diff size of stirrups x1 and same cross section area, b/a = 1.5, x1/y1 = 1.5

Fig 5.8
Comparison of $\theta$ Vs diff size of stirrups $x_1$ and same cross section area, $b/a = 2$, $x_1/y_1 = 2$

Fig 5.9

Comparison of $\theta$ Vs diff size of stirrups $x_1$ and same cross section area, $b/a = 3$, $x_1/y_1 = 3$

Fig 5.10
When the $b/a$ ratio equals one, i.e. square section, the results obtained from the three methods are quite close too, which is independent of the sizes of stirrup.
However, when the b/a ratio increases to 1.2, the results given by Method 3 are rather sensitive and have moderate change. A shape factor may be introduced for Method 3 so that it can provide reasonable values for angle of twist. While Method 1 can still give similar results to Method 2.

The recommended shape factors are shown in the above tables for Method 1 and Method 3. When the b/a ratio increases to 1.5, Method 1 requires a shape factor of 2. The b/a ratio increases to 10, the required shape factor raise to 75. The difference of the value given by Method 1 and Method 2 is nearly constant as the size of the stirrup increase.

However, it has a sharp increase of shape factor for Method 3 of 460, as the b/a ratio changes to 2. When the b/a ratio greater than 2, Methods 3 gives unrealistic small value for angle of twist. The shape factor never valid to the analysis for Method 3 because it can only give results, which tend to zero.

Therefore, the three analytical methods perform good results for square sections, which is independent of size of stirrups and longitudinal bars. The shape factors are required to apply for the analysis of Method 1 when the b/a ratio increases from 1.5. More reliable results can be provided for Method 1 when the shape factors are utilized. Method 2 can give reliable results for computations, which is independent of the shape of the section, i.e. it can provide dependable results even with a flat section of reinforced concrete member.

Nevertheless, Method 3 can operate for the b/a ratio less than 2. If the b/a ratio gets bigger, it cannot perform properly for analysis.
5.3 Uncracked Analysis by EMM and AEMM Approach for Various Analytical Methods

5.3.1 Time Dependent Analysis – EMM and AEMM Approach for Analytical Method 1

First, focus on EMM approach (Effective Modulus Method) for analytical Method 1. By assuming the equations effective elastic modulus can be applied to effective shear modulus. Further verification will present in section 6.4.2 (Time Effect on Plain Concrete). If the instantaneous and creep components of strain are combined, a reduced or effective shear modulus for concrete $G_e(t,\tau)$ can be defined as follows:

$$
\gamma(t,\tau) = \frac{\tau}{G_e(t,\tau)[1+\phi(t,\tau)]} = \frac{\tau}{G_e(t,\tau)} \tag{5.58}
$$

where

$$
G_e(t,\tau) = \frac{G_e(\tau)}{[1+\phi(t,\tau)]} \tag{5.59}
$$

and

$$
[(G_e)_{1}(t,\tau)] = \frac{1}{\left(\frac{A_1}{A_1} + \frac{A_1}{G_e(t,\tau)} + \frac{A_1}{G_s(t,\tau)}\right)} \tag{5.60}
$$

$$
[(G_e)_{2}(t,\tau)] = \frac{1}{\left(\frac{A_2}{A_2} + \frac{A_2}{G_e(t,\tau)} + \frac{A_2}{G_s(t,\tau)}\right)} \tag{5.61}
$$

Creep is treated as a delayed elastic shear strain and is taken into account simply by reducing the elastic modulus for concrete. A time analysis using the effective modulus method is nothing more than an elastic analysis in which $G_e(t,\tau)$ is used instead of $G_e(\tau)$.

Consider the reinforced concrete member, which is subjected to a constant sustained torque, T. For section A, the external torsional load T is resisted by the internal
torques in the concrete cover around the stirrups, $T_{out}(t)$, stirrups, $T_s(t)$ and internal core of the reinforced concrete member which contains longitudinal bars, $T_{in}(t)$. The redistribution of internal torques due to the gradual development of creep strains is to be examined and the time-dependent stresses and strains in both the concrete and the steel are to be calculated.

The torsion load $T$, acting on different parts of the section has the following distribution:

$$T_{out}(t) + T_{in}(t) + T_s(t) = T(t) = T \quad (5.62)$$

For angle of twist $\theta_{total}(t)$ after time $t$,

$$\theta_{total}(t) = (nd_{sv})\beta_A(t) + [(n - 1)(s - d_{sv}) + (L - (n - 1)(s - d_{sv}) - nd_{sv})] \beta_B(t) \quad (5.63)$$

The rate of twist of section $A$ after time $t$, has the relation with different torsion component as the following:

$$\beta_A(t) = \frac{T_s(t)}{G_e(J_{sv})_{ha}} \quad (5.64)$$

$$\beta_A(t) = \frac{T_{in}(t)}{[(G_e J_{sv})_a(t)]_{ha}} \quad (5.65)$$

$$\beta_A(t) = \frac{T_{out}(t)}{G_e(J_{out})_{ha}} \quad (5.66)$$

Therefore, rate of twist of section $\beta_A(t)$ is

$$\beta_A(t) = \frac{T}{G_e(J_{sv})_{ha} + [(G_e J_{sv})_a(t)]_{ha} + G_e(J_{out})_{ha}} \quad (5.67)$$

Therefore, the shear stress acting on the concrete cover around the stirrups, $\tau_{out}(t)$, stirrups, $\tau_s(t)$ and internal core of the reinforced concrete member, which contains longitudinal bars, $\tau_{in}(t)$ are the following:
\( \tau_s(t) = \frac{G_s(J_{sr})_{la} \beta_A(t)}{2 \Gamma_s t(z)} \)  
\( \tau_{in}(t) = \frac{[(G_c)_c(t) J_{in})_{la} \beta_A(t)}{k_2 x_c y_c^2} \)  
\( \tau_{out}(t) = \frac{G_e(t) J_{out})_{la} \beta_A(t)}{2 \Gamma_{out} t} \)  

The rate of twist of section B after time \( t \), the external torque has the following relation to \( \beta_B(t) \),

\[
T = [(G_c)_c^e(t)] J_B \beta_B(t) \\
\beta_B(t) = \frac{T}{[(G_c)_c^e(t)] J_B} 
\]

Hence, the total angle of twist can be expressed as,

\[
\theta_{total}(t) = T \left\{ \frac{(nd_{w})}{G_s(J_{sr})_{la} + [(G_c)_c^e(t)] J_{in})_{la} + G_e(t) J_{out})_{la} + [(n-1)(s-d_{w}) + (L - (n-1)(s-d_{w}) - nd_{w})] [(G_c)_c^e(t)] J_B} \right\} 
\]

Then, consider the AEMM approach (Aged-adjusted Effective Modulus Method) for analytical Method 1. The age-adjusted effective shear modulus for concrete \( \bar{G}_e(t, \tau) \) can be given by,

\[
\bar{G}_e(t, \tau) = \frac{G_s(t)}{1 + \chi(t, \tau) \phi(t, \tau)} 
\]

and

\[
[(G_c)_c^e(t, \tau)] = \frac{A_1}{\frac{A_1}{G_s(t, \tau)} + \frac{A_1}{G_s}} 
\]
The total shear strain at time $t$ may be expressed as the sum of the shear strains produced by $\tau_o$ (instantaneous and creep), the strains produced by the gradually applied stress increment $\Delta \tau(t)$ (instantaneous and creep). It is assume that shrinkage does not affect the angle of twist of concrete. Hence, shrinkage strain is ignored in this analysis.

$$
\gamma(t) = \frac{\tau_o}{G_e(t, \tau)} [1 + \phi(t, \tau)] + \frac{\Delta \tau(t)}{G_e(t, \tau)} [1 + \chi(t, \tau) \phi(t, \tau)] \\
= \frac{\tau_o}{G_e(t, \tau)} + \frac{\Delta \tau(t)}{G_e(t, \tau)} \quad (5.75)
$$

For section A, the external shear stress $\tau$ distribute into the different internal shear stresses in the concrete cover around the stirrups, $\tau_{out}(t)$, stirrups, $\tau_s(t)$ and internal core of the reinforced concrete member which contains longitudinal bars, $\tau_{in}(t)$. The redistribution of internal torques due to the gradual development of creep strains is to be examined and the time-dependent stresses and strains in both the concrete and the steel are to be calculated.

The rate of twist of section A after time $t$, has the relation with different shear stress component as the following:

$$
\beta_A(t) = \frac{2\Gamma t(z) \tau(_s(t))}{G_e(J_{str})_{ta}} \quad (5.76)
$$

$$
\beta_A(t) = \frac{(\tau_{in})(k_{x_e} y_e)^2}{[G_e(J_{in})_{ta}]} + \frac{(\Delta \tau_{in}(t))(k_{x_e} y_e)^2}{[G_e(J_{in})_{ta}]} \quad (5.77)
$$

$$
\beta_A(t) = \frac{2\Gamma_{out}t(\tau_{out})}{G_e(J_{out})_{ta}} + \frac{2\Gamma_{out} t(\Delta \tau_{out}(t))}{G_e(J_{out})_{ta}} \quad (5.78)
$$

There are two unknowns, $\Delta \tau_{in}(t)$ and $\Delta \tau_{out}(t)$ in the above equations, where $(_{in})_e$ and $(_{out})_e$ are the instantaneous stresses.

The internal torque acting on the stirrup can also find the rate of twist of section A,
\[
T_s(t) = T - T_{\text{in}}(t) - T_{\text{out}}(t) \\
= T - \tau_{\text{in}}(t)\left(k_2x_r, y_r^2\right) - \tau_{\text{out}}(t)2\Gamma_{\text{out}}t \\
= T - \left[\left(\tau_{\text{in}}(t)e + \Delta\tau_{\text{in}}(t)e\right)\left(k_2x_r, y_r^2\right) - \left[\left(\tau_{\text{out}}(t)e + \Delta\tau_{\text{out}}(t)e\right)\right]2\Gamma_{\text{out}}t\right) \quad (5.79)
\]

\[
\beta_s(t) = \frac{T_s(t)}{G_s(J_{\text{str}})_{la}} \\
= \frac{T - T_{\text{in}}(t) - T_{\text{out}}(t)}{G_s(J_{\text{str}})_{la}}
\]

From equations (5.77) and (5.78),

\[
\beta_s(t) = \frac{\tau_{\text{in}}(t)e\left(k_2x_r, y_r^2\right) + \Delta\tau_{\text{in}}(t)e\left(k_2x_r, y_r^2\right)}{\left[\left(G_e\right)_{J_{\text{in}}}(t)\right]_{la}} + \frac{\left[\left(G_e\right)_{J_{\text{in}}}(t)\right]_{la}}{\left[\left(G_e\right)_{J_{\text{in}}}(t)\right]_{la}} = \frac{T - T_{\text{in}}(t) - T_{\text{out}}(t)}{G_s(J_{\text{str}})_{la}}
\]

\[
\beta_s(t) = \frac{2\Gamma_{\text{out}}t\tau_{\text{out}}(t)e}{G_e(J_{\text{out}})_{la}} + \frac{2\Gamma_{\text{out}}t\Delta\tau_{\text{out}}(t)e}{G_e(J_{\text{out}})_{la}} = \frac{T - T_{\text{in}}(t) - T_{\text{out}}(t)}{G_s(J_{\text{str}})_{la}}
\]

Substitute equation (5.79) into the above equations, two equations for determining the two unknowns can be formed as follows:

\[
\frac{\tau_{\text{in}}(t)e\left(k_2x_r, y_r^2\right) + \Delta\tau_{\text{in}}(t)e\left(k_2x_r, y_r^2\right)}{\left[\left(G_e\right)_{J_{\text{in}}}(t)\right]_{la}} + \frac{\left[\left(G_e\right)_{J_{\text{in}}}(t)\right]_{la}}{\left[\left(G_e\right)_{J_{\text{in}}}(t)\right]_{la}} = \frac{T - \left[\left(\tau_{\text{in}}(t)e + \Delta\tau_{\text{in}}(t)e\right)\left(k_2x_r, y_r^2\right) - \left[\left(\tau_{\text{out}}(t)e + \Delta\tau_{\text{out}}(t)e\right)\right]2\Gamma_{\text{out}}t\right) \quad (5.80)
\]

\[
\frac{2\Gamma_{\text{out}}t\tau_{\text{out}}(t)e}{G_e(J_{\text{out}})_{la}} + \frac{2\Gamma_{\text{out}}t\Delta\tau_{\text{out}}(t)e}{G_e(J_{\text{out}})_{la}} = \frac{T - \left[\left(\tau_{\text{in}}(t)e + \Delta\tau_{\text{in}}(t)e\right)\left(k_2x_r, y_r^2\right) - \left[\left(\tau_{\text{out}}(t)e + \Delta\tau_{\text{out}}(t)e\right)\right]2\Gamma_{\text{out}}t\right) \quad (5.81)
\]

Let unknowns be, \(X = \Delta\tau_{\text{in}}(t)\) and \(Y = \Delta\tau_{\text{out}}(t)\).

Let the coefficients be, \(a = (\tau_{\text{in}})_{e}\), \(b = (\tau_{\text{out}})_{e}\), \(x = \left(k_2x_r, y_r^2\right)\), \(y = 2\Gamma_{\text{out}}t\),

\[
A = \left[\left[\left(G_e\right)_{J_{\text{in}}}(t)\right]_{la}\right], \quad B = \left[\left.G_e\right(J_{\text{out}})_{la}\right], \quad C = \left[\left.G_s\right(J_{\text{str}})_{la}\right], \quad D = \left[\left[\left.G_e\right)_{J_{\text{in}}}(t)\right]_{la}\right], \quad E = \\
G_e(J_{\text{out}})_{la}, \quad K = T - \left(\tau_{\text{in}}(t)e\left(k_2x_r, y_r^2\right) - \left(\tau_{\text{out}}(t)e\right)2\Gamma_{\text{out}}t\right) = T - ax - by
\]
Then the equations (5.80) and (5.81) becomes,

\[
\frac{ax + x}{D} + \frac{X}{A} = \frac{K - xX - yY}{C} \tag{5.82}
\]

\[
\frac{by + y}{E} + \frac{Y}{B} = \frac{K - xX - yY}{C} \tag{5.83}
\]

By solving equations (5.82) and (5.83), the unknowns X and Y can be determined as follows:

\[
X = \frac{ADEK - aAE(B + C)x - bABDy}{xDE(A + B + C)}
= \frac{A}{(A + B + C)} \left[ xT \frac{b(E - B)y}{Ex} - \frac{a(B + C + D)}{D} \right]
\]

\[
Y = \frac{BDEK - aABEx - bBD(A + C)y}{yDE(A + B + C)}
= \frac{B}{(A + B + C)} \left[ yT \frac{a(D - A)x}{Dy} - \frac{b(A + C + E)}{E} \right]
\]

Therefore,

\[
\beta_A(t) = \frac{ax + x}{D} + \frac{X}{A}
= \frac{ax}{D} + \frac{1}{(A + B + C)} \left[ T - \frac{ax(B + C + D)}{D} - \frac{by(E - B)}{E} \right]
= \frac{1}{(A + B + C)} \left[ T - \frac{ax(D - A)}{D} - \frac{by(E - B)}{E} \right]
\]
Put $X = \Delta \tau_{in}(t)$ into equation (5.83) so that,

$$
\Delta \tau_{in}(t) = \frac{\left\{ \left[ \frac{\left[ \begin{array}{c} G_{r} \\ \dot{G}_{r} \end{array} \right] \right]_{e}(t) \right\} \left[ J_{in} \right]_{la}}{\left[ \left[ \begin{array}{c} G_{r} \\ \dot{G}_{r} \end{array} \right] \right]_{e}(t) \left[ J_{in} \right]_{la} + G_{e}(t)\left[ J_{out} \right]_{la} + G_{s}(J_{str})_{la}} \left[ \begin{array}{c} \frac{T}{k_{2}x_{c}y_{c}^{2}} \left( \tau_{out} \right)_{e} \left[ \begin{array}{c} G_{r} \\ \dot{G}_{r} \end{array} \right]_{e}(t) - \bar{G}_{e}(t) \\
- \tau_{in} \left[ \begin{array}{c} G_{r} \\ \dot{G}_{r} \end{array} \right]_{e}(t) \left[ J_{in} \right]_{la} + G_{e}(J_{str})_{la} + \left[ \begin{array}{c} G_{r} \\ \dot{G}_{r} \end{array} \right]_{e}(t) \left[ J_{in} \right]_{la} \right] \right]}
$$

(5.84)

$$
\Delta \tau_{out}(t) = \frac{\left[ \begin{array}{c} \frac{\left[ \begin{array}{c} G_{r} \\ \dot{G}_{r} \end{array} \right] \right]_{e}(t) \left[ J_{in} \right]_{la} + G_{e}(t)\left[ J_{out} \right]_{la} + G_{s}(J_{str})_{la}}{\left[ \left[ \begin{array}{c} G_{r} \\ \dot{G}_{r} \end{array} \right] \right]_{e}(t) \left[ J_{in} \right]_{la} + G_{e}(J_{str})_{la} + G_{e}(t)\left[ J_{out} \right]_{la}} \left[ \begin{array}{c} \frac{T}{k_{2}x_{c}y_{c}^{2}} \left( \tau_{in} \right)_{e} \left[ \begin{array}{c} G_{r} \\ \dot{G}_{r} \end{array} \right]_{e}(t) - \left[ \begin{array}{c} G_{r} \\ \dot{G}_{r} \end{array} \right]_{e}(t) \\
\left( \tau_{out} \right)_{e} \left[ \begin{array}{c} G_{r} \\ \dot{G}_{r} \end{array} \right]_{e}(t) \left[ J_{in} \right]_{la} + G_{e}(J_{str})_{la} + G_{e}(t)\left[ J_{out} \right]_{la} \right] \right]}
$$

(5.85)

Therefore the rate of twist of section A after time $t$, can be expressed as,

$$
\beta_{A}(t) = \frac{1}{(A + B + C)} \left[ T - \frac{ax(D - A)}{D} - \frac{by(E - B)}{E} \right]
$$

$$
= \frac{1}{\left[ \left[ \begin{array}{c} G_{r} \\ \dot{G}_{r} \end{array} \right] \right]_{e}(t) \left[ J_{in} \right]_{la} + G_{e}(t)\left[ J_{out} \right]_{la} + G_{s}(J_{str})_{la}} \left[ \begin{array}{c} \frac{T}{k_{2}x_{c}y_{c}^{2}} \left( \tau_{in} \right)_{e} \left[ \begin{array}{c} G_{r} \\ \dot{G}_{r} \end{array} \right]_{e}(t) - \left[ \begin{array}{c} G_{r} \\ \dot{G}_{r} \end{array} \right]_{e}(t) \\
\left( \tau_{out} \right)_{e} \left[ \begin{array}{c} G_{r} \\ \dot{G}_{r} \end{array} \right]_{e}(t) \left[ J_{in} \right]_{la} + G_{e}(J_{str})_{la} + G_{e}(t)\left[ J_{out} \right]_{la} \right] \right]}
$$

(5.86)

The rate of twist of section B after time $t$, for AEMM approach is the same as the calculation as EMM approach. Since, the whole section is treated as one creeping material and there is no distribution of stress steel in consideration. Thus,

$$
\beta_{B}(t) = \frac{T}{\left[ \left[ \begin{array}{c} G_{r} \\ \dot{G}_{r} \end{array} \right] \right]_{e}(t)J_{B}}
$$

(5.87)
Hence, the total angle of twist can be determined by equation (5.63),

$$\theta_{\text{total}}(t) = (nd_{sv})\beta_a(t) + [(n-1)(s-d_{sv}) + (L-(n-1)(s-d_{sv})-nd_{sv})]\beta_b(t)$$  \hspace{1cm} (5.63)

5.3.1.1 Numerical Example

In order to illustrate EMM and AEMM approach, a numerical example is presented. The concrete and steel shear strains as well as the angle of twist are to be determined at selected times for the pure torsion. The external pure torque $T$ is 800 Nm. Cross sectional, material properties and dimensions are as follows:

\begin{align*}
2a &= 120\text{mm} \quad 2b = 120\text{mm} \\
x_i &= 80\text{mm} \quad y_i = 80\text{mm} \\
d_{sv} &= 6\text{mm} \quad d_s = 6\text{mm} \quad (4 \text{ longitudinal bars}) \\
E_c(\tau_o) &= 25\text{GPa} \quad \nu_c = 0.15 \quad (\text{Poisson ratio for concrete}) \\
E_s &= 200\text{GPa} \quad \nu_s = 0.3 \quad (\text{Poisson ratio for steel}) \\
L &= 1000\text{mm}, \quad (\text{Length of member}) \\
n &= 13, (\text{No of stirrups}) \quad s = 80\text{mm} \quad (\text{Spacing of stirrups})
\end{align*}

At first loading ($t - \tau_o$) in days = 0, $\phi(t - \tau_o) = 0$,

$$E_s(t, \tau) = E_s(\tau_o) = 25\text{GPa}$$

$$G_s(\tau_o) = \frac{E_s(\tau_o)}{2(1+\nu_s)} = 10.87\text{GPa}$$

$$G_s = \frac{E_s}{2(1+\nu_s)} = 76.9\text{GPa}$$
Time Dependent Effects of Reinforced Concrete Subjected to Torsional Loading

\[
\frac{1}{(G_c)_h} = \frac{\left( \frac{A_x}{A} \right)}{G_r} + \frac{\left( \frac{A_x}{A} \right)}{G_s} = \frac{(5476 - 452.4)}{5476} + \frac{(452.4)}{76.9} \Rightarrow (G_c)_h = 11.7 \text{GPa}
\]

where \( A = x, y \),
\( = (80 - 6)(80 - 6) \)
\( = 5476 \text{mm}^2 \)
\( A_s = 4\pi (6)^2 \)
\( = 452.4 \text{mm}^2 \)

\[
\frac{1}{(G_c)_2} = \frac{\left( \frac{A_x}{A} \right)}{G_r} + \frac{\left( \frac{A_x}{A} \right)}{G_s} = \frac{(14400 - 452.4)}{14400} + \frac{(452.4)}{76.9} \Rightarrow (G_c)_2 = 11.17 \text{GPa}
\]

where \( A = (2a)(2b) \),
\( = (120)(120) \)
\( = 14400 \text{mm}^2 \)
\( A_s = 4\pi (6)^2 \)
\( = 452.4 \text{mm}^2 \)

The polar moment of inertia of the concrete cover around the stirrups, \( (J_{ou})_{1a} \),

\[
(J_{ou})_{1a} = \frac{2(\bar{h})^3 t}{(b + h)} = \frac{2((103)(103))^3}{(103 + 103)} = 18576359 \text{mm}^4
\]
The polar moment of a stirrup, \((J_{sr})_{la}\),

\[
(J_{sr})_{la} = \frac{4(x_1 y_1)^2 \pi}{(x_1 + y_1)} d_w
\]

\[
= \frac{4(80 \times 80)^2 \pi}{(80 + 80)} x_6
\]

\[
= 19301945 \text{mm}^4
\]

The polar moment of a internal core, \((J_{in})_{la}\),

\[
(J_{in})_{la} = k x_7^3 y_7
\]

\[
= 0.141x(74^3)x(74)
\]

\[
= 4228107 \text{mm}^4
\]

The polar moment of section B, \(J_B\),

\[
J_B = k (2a)^3 (2b)
\]

\[
= 0.141x(120)^3 x(120)
\]

\[
= 29237760 \text{mm}^4
\]

The instantaneous rate of twist of section A and section B can be determined as follows:

\[
\beta_A = \frac{T}{G_c (J_{out})_{la} + (G_c)(J_{in})_{la} + G_c (J_{sr})_{la}}
\]

\[
= \frac{10.87(18576359) + (11.7)(4228107) + (76.9)(19301945)}{800}
\]

\[
= 4.6 \times 10^{-7} \text{Rad/mm}
\]

\[
\beta_B = \frac{T}{(G_c)(J_B)}
\]

\[
= \frac{800}{(11.17)x(29237760)}
\]

\[
= 2.45 \times 10^{-6} \text{Rad/mm}
\]
Thus, the total angle of twist of the reinforced concrete member is,

\[ \theta_{\text{Total}} = (nd_{n'}) \beta_A + \left[ (n-1)(s-d_{n'}) + (L-(n-1)(s-d_{n'})-nd_{n'}) \right] \beta_B \]

\[ = (13x6)x(4.6x10^{-7}) + \left[ (13-1)(80-6) + (1000-(13-1)(80-6)-(13x6)) \right] x(2.45x10^{-6}) \]

\[ = 2.30x10^{-3} \text{ Rad} \]

The shear stresses acting on different component of the reinforced concrete member of section A at initial are follows,

\[ \tau_s = \frac{G_s (J_{n'}) \beta_{A}}{2 \Gamma_{f}(z)} = \frac{G_s (J_{n'}) \beta_{A}}{2 \Gamma_{f}d_{n'}} \]

\[ = \frac{(76.9)(19301945)(4.6x10^{-7})}{2x(80x80)x6} x1000 \]

\[ = 8.9 \text{MPa} \]

For initial maximum shear stress acting on the internal core,

\[ \left( \tau_{in} \right)_c = \tau_{in} (t) = \frac{[(G_c')(r)](t)(J_{in}) \beta_A}{(k_1 x_r y_r^2)} \]

\[ = \frac{(11.7)(4228107)(4.6x10^{-7})}{(0.208x74x74^2)} x1000 \]

\[ = 0.27 \text{MPa} \]

For initial maximum shear stress acting on the concrete cover around the stirrups,

\[ \left( \tau_{out} \right)_c = \tau_{out} (t) = \frac{G_c (t)(J_{out}) \beta_{A}}{2 \Gamma_{out}t} \]

\[ = \frac{(10.87)(18576359)(4.6x10^{-7})}{2x(103x103)x17} x1000 \]

\[ = 0.26 \text{MPa} \]

First, the section is analyzed by EMM approach.
At loading $(t - \tau_0)$ in days $= 25$, $\phi(t - \tau_0) = 1.0$,

$$G_e(\tau) = \frac{G_e(\tau)}{1 + \phi(\tau)} \quad (5.59)$$

$$= \frac{10.87}{1 + 1} = 5.435 \text{ GPa}$$

$$\left[ \left( \frac{G_i}{G_e} \right)_{i,\tau} \right] = \frac{A_i}{\left( A_i \frac{G_e}{G_i} + A_i \right)} \quad (5.60)$$

$$= \frac{5476}{\left( \frac{5476 - 452.5}{5.435} + \frac{452.5}{76.9} \right)} = 5.887 \text{ GPa}$$

$$\left[ \left( \frac{G_i}{G_e} \right)_{i,\tau} \right] = \frac{A_2}{\left( A_2 \frac{G_e}{G_2} + A_2 \right)} \quad (5.61)$$

$$= \frac{14400}{\left( \frac{14400 - 452.5}{5.435} + \frac{452.5}{76.9} \right)} = 5.598 \text{ GPa}$$
The rate twist of section A can be determined from equation (5.67).

$$\beta_A(t) = \frac{T}{G_s(J_{sv})_{IA} + \left\{(G_r)_{IA}(t)\right\} J_{va} + G_r(t)J_{oa}} \frac{800}{800}$$

$$= \frac{[76.9(19301945) + (5.887)(4228107) + (5.435)(18576359)]}{800}$$

$$= 4.97 \times 10^{-7} \text{ Rad/mm}$$

The rate twist of section B can be determined from equation (5.87),

$$\beta_B(t) = \frac{T}{\frac{1}{((G_r)_{IB}(t)) J_B}} \frac{800}{800}$$

$$= \frac{5.598(29237760)}{800}$$

$$= 4.90 \times 10^{-6} \text{ Rad/mm}$$

Hence, the total angle of twist is,

$$\theta_{total}(t) = (nd_{sv})\beta_A(t) + [(n-1)(s-d_{sv}) + (L - (n-1)(s-d_{sv}) - nd_{sv})] \beta_B(t)$$

$$= (13 \times 6)(4.97 \times 10^{-7}) + [(13-1)(80-6) + (1000 - (13-1)(80-6) - 13 \times 6)] \times 4.9 \times 10^{-6}$$

$$= 4.56 \times 10^{-3} \text{ Rad}$$

The shear stresses acting on different components of the reinforced concrete member of section A at initial are follows,

$$\tau_s(t) = \frac{G_s(J_{sv})_{IA} \beta_A(t)}{2\Gamma_s t(e)} = \frac{G_s(J_{sv})_{IA} \beta_A(t)}{2\Gamma_s d_{sv}}$$

$$= \frac{76.9(19301945)(4.97 \times 10^{-7})}{2 \times (80 \times 80) x 6} \times 1000$$

$$= 9.61 \text{ MPa}$$

For maximum shear stress acting on the internal core,
\[ \tau_{in}(t) = \frac{\left( \left( G_e \right)_1(t) \right) \left( J_{in} \right)_a \beta_A(t)}{k_{2c} y_c^2} \]
\[ = \frac{(5.85)(4228107)(4.97 \times 10^{-7})}{(0.208 \times 74 \times 74^2)} \times 1000 \]
\[ = 0.145 \text{MPa} \]

For maximum shear stress acting on the concrete cover around the stirrups,

\[ \tau_{out}(t) = \frac{G_e(t)J_{out}A_B(t)}{2 \Gamma_{out}} \]
\[ = \frac{(5.585)(18576359)(4.97 \times 10^{-7})}{2 \times (103 \times 103) \times 17} \times 1000 \]
\[ = 0.143 \text{MPa} \]

At loading \((t - \tau_o)\) in days = 100, \(\phi(t - \tau_o) = 2.0\),

\[ G_e(t, \tau) = 3.623 \text{GPa}, \quad \left[ \left( G_e \right)_1(t, \tau) \right] = 3.933 \text{GPa}, \quad \left[ \left( G_e \right)_2(t, \tau) \right] = 3.735 \text{GPa}, \]
\[ \beta_A(t) = 5.1 \times 10^{-7} \text{Rad/mm}, \quad \beta_B(t) = 7.348 \times 10^{-6} \text{Rad/mm}, \quad \theta_{total}(t) = 6.8 \times 10^{-3} \text{Rad} \]
\[ \tau_s(t) = 9.86 \text{MPa}, \quad \tau_{in}(t) = 0.10 \text{MPa}, \quad \tau_{out}(t) = 0.095 \text{MPa} \]

At loading \((t - \tau_o)\) in days = 10000, \(\phi(t - \tau_o) = 3.0\),

\[ G_e(t, \tau) = 2.718 \text{GPa}, \quad \left[ \left( G_e \right)_1(t, \tau) \right] = 2.953 \text{GPa}, \quad \left[ \left( G_e \right)_2(t, \tau) \right] = 2.802 \text{GPa}, \]
\[ \beta_A(t) = 5.17 \times 10^{-7} \text{Rad/mm}, \quad \beta_B(t) = 9.797 \times 10^{-6} \text{Rad/mm}, \quad \theta_{total}(t) = 9.07 \times 10^{-3} \text{Rad} \]
\[ \tau_s(t) = 9.99 \text{MPa}, \quad \tau_{in}(t) = 0.076 \text{MPa}, \quad \tau_{out}(t) = 0.072 \text{MPa} \]

The concrete and steel shear stresses and angle of twist are now re-analyzed by AEMM approach.

<table>
<thead>
<tr>
<th>(t - \tau_o) in days</th>
<th>(\phi(t - \tau_o))</th>
<th>(\chi(t - \tau_o))</th>
</tr>
</thead>
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<td>0</td>
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<td></td>
</tr>
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<td>0.86</td>
<td></td>
</tr>
<tr>
<td>100</td>
<td>0.8</td>
<td></td>
</tr>
<tr>
<td>10000</td>
<td>0.8</td>
<td></td>
</tr>
</tbody>
</table>
At first loading \((t - \tau_o)\) in days = 0, \(\phi(t - \tau_o) = 0\), The results are the same as EMM approach.

\[
G_e(t, \tau) = 10.87 \text{GPa}, \quad \left[ G_e^* \right]_{e} (t, \tau) = 11.7 \text{GPa}, \quad \left[ G_e^* \right]_{\tau} (t, \tau) = 11.17 \text{GPa}, \\
\beta_A(t) = 4.6 \times 10^{-7} \text{Rad/mm}, \quad \beta_p(t) = 2.45 \times 10^{-6} \text{Rad/mm}, \quad \phi_{\text{total}}(t) = 2.30 \times 10^{-3} \text{Rad} \\
\tau_s(t) = 8.9 \text{MPa}, \quad \tau_{in}(t) = 0.27 \text{MPa}, \quad \tau_{out}(t) = 0.26 \text{MPa}
\]

At first loading \((t - \tau_o)\) in days = 25, \(\phi(t - \tau_o) = 1.0\), \(\chi(t - \tau_o) = 0.86\),

\[
\bar{G}_e(t, \tau) = \frac{G_e(t)}{1 + \chi(t, \tau)\phi(t, \tau)} = \frac{10.87}{1 + (0.86)(1.0)} = 5.844 \text{GPa}
\]

\[
\left[ \left( \bar{G}_e \right)_e (t, \tau) \right] = \frac{A_i}{\left( \frac{A_i}{G_e(t, \tau)} + A_i \right)} = \frac{5476}{\left( \frac{5476 - 452.4}{5.844} + \frac{452.5}{76.9} \right)} = 6.327 \text{GPa}
\]

By equations (5.84) and (5.85) \(\Delta \tau_{in}(t)\) and \(\Delta \tau_{out}(t)\) can be found,

\[
\Delta \tau_{in}(t) = \frac{\left[ \left( \bar{G}_e \right)_e (t) J_{in} \right]_{ta}}{\left( \bar{G}_e \right)_e (t) J_{in} + \bar{G}_e J_{out} + G_s (J_{str})_{ta}} \\
\times \left[ T \left( \frac{\tau_{in}}{\tau_{out}} \right) \left( \frac{G_e(t) J_{in}}{\left[ \left( \bar{G}_e \right)_e (t) J_{in} \right]_{ta}} \right) + G_s (J_{str})_{ta} + \left[ \left( \bar{G}_e \right)_e (t) J_{in} \right]_{ta} \right] \\
\times \frac{(6.327)(4228107)}{((6.327)(4228107) + (5.844)(18576359) + (76.9)(19301945))} \\
\times \frac{0.208x74x74^2}{(0.26/1000)(2x103x103x17)(5.435) - (5.844)} \\
= -0.133 \text{MPa}
\]

5-37
\[
\Delta \tau_{out}(t) = \frac{G_e(t)(J_{out})_{la}}{\left(\left(\frac{(G_e)^r_{la}}{J_{in}}(t) + G_e(t)(J_{out})_{la} + G_e(J_{str})_{la}\right)\left(2\Gamma_{out,t}\right)\right) x (5.435)(18576359)}
\]

\[
\frac{(\tau_{in})_{c}(k_{2x}x_{c}y_{c}^2)\left(\left((G_e)^r_{la}(t)\right)(J_{in})_{la} - \left((G_e)^r_{la}(t)\right)(J_{in})_{la}\right)}{(\left(2\Gamma_{out,t}\right)(\left((G_e)^r_{la}(t)\right)(J_{in})_{la})\right)}
\]

\[
= \frac{(6.29)(4228107) + (5.844)(18576359) + (76.9)(19301945)}{(0.26/1000)(5.844)(18576359) + (76.9)(19301945) + (6.327)(4228107)}
\]

\[
\frac{800}{(2x80x80x6)} + \frac{(0.27/1000)(0.208x74x74^2)(5.887) - (6.327)}{(2x80x80x6)(5.887)}
\]

\[
= -0.129 \text{MPa}
\]

Therefore, the rate of twist for section A by using AEMM approach is,

\[
\beta_A(t) = \frac{1}{\left(\left((G_e)^r_{la}(t)\right)(J_{in})_{la} + G_e(t)(J_{out})_{la} + G_e(J_{str})_{la}\right)\left(2\Gamma_{out,t}\right)\right) x (5.435)(18576359)}
\]

\[
\frac{(\tau_{in})_{c}(k_{2x}x_{c}y_{c}^2)\left(\left((G_e)^r_{la}(t)\right)(J_{in})_{la} - \left((G_e)^r_{la}(t)\right)(J_{in})_{la}\right)}{(\left(2\Gamma_{out,t}\right)(\left((G_e)^r_{la}(t)\right)(J_{in})_{la})\right)}
\]

\[
= \frac{1}{((6.327)(4228107) + (5.85)(18576359) + (76.9)(19301945))}
\]

\[
\frac{800 - (0.27/1000)(0.208x74x74^2)(5.887) - (6.327)}{5.887}
\]

\[
= (0.26/1000)(2x103x103x17)(5.435 - 5.84)
\]

\[
= 4.99 \times 10^{-7} \text{Rad/mm}
\]
Since the rate of twist of section B equal the value of EMM approach, the total angle of twist is,

\[
\theta_{\text{total}}(t) = (nd_{v})\beta_{A}(t) + [(n-1)(s-d_{v}) + (L-(n-1)(s-d_{v})-nd_{v})]\beta_{B}(t)
\]

\[
= (13\times6)(4.99\times10^{-7}) + [(13-1)(80-6) + (1000-(13-1)(80-6)-13\times6)](4.90\times10^{-6})
\]

\[
= 4.57\times10^{-3} \text{Rad}
\]

The shear stresses acting on different component of the reinforced concrete member of section A at initial are follows,

\[
\tau_{\text{max, shear stress on stirrup}} = \frac{G_{s}(J_{mr})_{in} \beta_{A}(t)}{2\Gamma_{s}(z)} = \frac{G_{s}(J_{mr})_{in} \beta_{A}(t)}{2\Gamma_{s}d_{sv}}
\]

\[
= \frac{(76.9)(19301945)(4.99\times10^{-7})}{2\times(80\times80)\times6} \times 1000
\]

\[
= 9.65\text{MPa}
\]

For maximum shear stress acting on the internal core,

\[
\tau_{\text{in}}(t) = (\tau_{\text{in}})_{c} + \Delta\tau_{\text{in}}(t)
\]

\[
= 0.27 - 0.133
\]

\[
= 0.137\text{MPa}
\]

For maximum shear stress acting on the concrete cover around the stirrups,

\[
\tau_{\text{out}}(t) = (\tau_{\text{out}})_{c} + \Delta\tau_{\text{out}}(t)
\]

\[
= 0.26 - 0.129
\]

\[
= 0.131\text{MPa}
\]

At loading (t - \tau_{o}) in days = 100, \phi(t - \tau_{o}) = 2.0, \chi(t - \tau_{o}) = 0.8,

\[
G_{s}(t, \tau) = 3.623\text{GPa}, \quad \ln(G_{s}^{*})_{2}(t, \tau) = 3.933\text{GPa}, \quad \ln(G_{s}^{*})_{2}(t, \tau) = 3.735\text{GPa},
\]

\[
\overline{G}_{s}(t, \tau) = 4.18\text{GPa}, \quad \ln(\overline{G}_{s})_{2}(t, \tau) = 4.535\text{GPa}
\]

\[
\beta_{A}(t) = 5.17\times10^{-7} \text{Rad/mm}, \quad \beta_{B}(t) = 7.350\times10^{-6} \text{Rad/mm}, \quad \theta_{\text{total}}(t) = 6.82\times10^{-3} \text{Rad}
\]

\[
\tau_{s}(t) = 9.998\text{MPa}, \quad \tau_{\text{in}}(t) = 0.075\text{MPa}, \quad \tau_{\text{out}}(t) = 0.072\text{MPa}
\]
At loading \((t - \tau_o)\) in days = 10000, \(\phi(t - \tau_o) = 3.0, \chi(t - \tau_o) = 0.8,\)

### 5.3.2 Time Dependent Analysis – EMM and AEMM Approach for Analytical Method 2

First, consider the EMM approach (Effective Modulus Method) for analytical Method 2. If the instantaneous and creep components of strain are combined, a reduced or effective shear modulus for concrete \(G_e(t, \tau)\) is the same as Method 1, which can be defined as follows:

\[
\gamma(t, \tau) = \frac{\tau}{G_e(t, \tau)}[1 + \phi(t, \tau)]
\]

\[
= \frac{\tau}{G_e(t, \tau)} \quad (5.58)
\]

where

\[
G_e(t, \tau) = \frac{G_e(t)}{[1 + \phi(t, \tau)]} \quad (5.59)
\]

A constant sustained torque, \(T\) is acting on the reinforced concrete member. For section B, the external torsional load, \(T\) is mainly resisted by internal torque of concrete, \(T_r(t)\) and internal torque of the longitudinal bars, \(T_s(t)\). The redistribution of internal torques due to the gradual development of creep strains is to be examined and the time-dependent stresses and strains in both the concrete and the steel are to be calculated. The torsion load \(T\), acting on different parts of the section has the following distribution:

\[
T_r(t) + T_s(t) = T(t) = T \quad (5.88)
\]

For angle of twist \(\theta_{total}(t)\) after time \(t\),

\[
\theta_{total}(t) = (nd_{sv})\beta_A(t) + [(n - 1)(s - d_{sv}) + (L - (n - 1)(s - d_{sv}) - nd_{sv})]\beta_B(t) \quad (5.63)
\]
The rate of twist of section \( B \) after time \( t \), has the relation with different torsion component as the following:

\[
\beta_b(t) = \frac{T_1(t)}{G_1(J_\gamma)_{2b}} \quad (5.89)
\]

\[
\beta_a(t) = \frac{(T_2(t))}{G_s(J_\alpha)_{2b}} \quad (5.90)
\]

Therefore, rate of twist of section \( \beta_b(t) \) is

\[
\beta_b(t) = \frac{T}{G_1(J_\gamma)_{2b} + G_s(J_\alpha)_{2b}} \quad (5.91)
\]

Therefore, the maximum shear stress acting on the edge of the concrete, \( \tau_c(t) \) and each longitudinal bar of two different directions, \( \tau_{c1}(t) \) and \( \tau_{c2}(t) \) are the following:

\[
\tau_c(t) = \frac{G_s(J_\alpha)_{2b} \beta_b(t)}{(k_2(2a)_{2b})^2} \quad (5.92)
\]

\[
\tau_{c1}(t) = G_s A_c \beta_b(t) \quad (5.93)
\]

\[
\tau_{c2}(t) = G_s B_c \beta_b(t) \quad (5.94)
\]

For section \( A \), there are four components of torques acting on the concrete cover around the stirrups, \( T_{c1}, \) stirrups, \( T_{c1}, \) longitudinal bars of the internal core of the reinforced concrete member \( T_{c2} \) and concrete the internal core of the reinforced concrete member, \( T_{c3}, \) in order to resist a constant external torque, \( T \) acting on the reinforced concrete member.

The torsion load \( T \), acting on different parts of the section has the following distribution:

\[
T_{c1}(t) + T_{c1}(t) + T_{c2}(t) + T_{c3}(t) = T(t) = T \quad (5.95)
\]

And,

\[
T_{c1}(t) = G_s(J_{c1})_{2a} \beta_A(t) \quad (5.96)
\]

\[
T_{c2}(t) = G_s(J_{c2})_{2a} \beta_A(t) \quad (5.97)
\]

\[
T_{c1}(t) = G_s(J_{c1})_{2a} \beta_A(t) \quad (5.98)
\]

\[
T_{c2}(t) = G_s(J_{c2})_{2a} \beta_A(t) \quad (5.99)
\]
Therefore, the rate of twist of section A after time t, the external torque has the following relation to \( \beta_A(t) \),

\[
\beta_A(t) = \frac{T}{G_s \left[ (J_y)_{2a} + (J_{sl})_{2a} \right] + G_s \left( \frac{(J_a)_{2a}}{(J_{cl})_{2a}} \right)}
\]

(5.100)

Therefore, the maximum shear stress acting on the edge of the concrete, \( \tau_{sl}(t) \) and each longitudinal bar of two different directions, \( \tau_{xz}(t) \) and \( \tau_{yz}(t) \), as well as the stirrups, \( \tau_{st}(t) \) are the following:

\[
\tau_{xz}(t) = \frac{G_s \left( \frac{(J_a)_{2a}}{(J_{cl})_{2a}} \right) \beta_A(t)}{k_2 (x_c, y_c)^2}
\]

(5.101)

\[
\tau_{yz}(t) = \frac{G_s \left( \frac{(J_a)_{2a}}{(J_{cl})_{2a}} \right) \beta_A(t)}{(2\Gamma_{out})}
\]

(5.102)

\[
\tau_{st}(t) = G_s \cdot A_m \beta_A(t)
\]

(5.103)

\[
\tau_{st}(t) = G_s \cdot B_m \beta_A(t)
\]

(5.104)

\[
\tau_{sl}(t) = \frac{G_s \left( \frac{(J_{sl})_{2a}}{(J_{cl})_{2a}} \right) \beta_A(t)}{2\Gamma (z)}
\]

(5.105)

Hence, the total angle of twist can be expressed as the following.

\[
\theta_{Total}(t) = T \left[ \frac{(nd_{iv})}{G_s \left[ (J_y)_{2a} + (J_{sl})_{2a} \right] + G_s \left( \frac{(J_a)_{2a}}{(J_{cl})_{2a}} \right) + \left[ n \left( s - d_{iv} \right) + \left( l - n \right) \left( s - d_{iv} \right) nd_{iv} \right]} \right]
\]

(5.106)

Then, consider the AEMM approach (Aged-adjusted Effective Modulus Method) for analytical Method 2. The age-adjusted effective shear modulus for concrete \( \bar{G}_s(t, \tau) \) can be given by,

\[
\bar{G}_s(t, \tau) = \frac{G_s(\tau)}{[1 + \chi(t, \tau)\phi(t, \tau)]}
\]

(5.73)
The total shear strain at time $t$ may be expressed as the sum of the shear strains produced by $\tau_o$ (instantaneous and creep), the strains produced by the gradually applied stress increment $\Delta \tau(t)$ (instantaneous and creep).

\[
\gamma(t) = \frac{\tau_o}{G_o(t)} \left[1 + \phi(t, \tau)\right] + \frac{\Delta \tau(t)}{G_o(t)} \left[1 + \chi(t, \tau)\phi(t, \tau)\right] + \frac{\Delta \tau(t)}{G_e(t, \tau)}
\]

\[\tag{5.75}\]

For section B, the external shear stress $\tau$ distribute into the different internal shear stresses on the concrete, $\tau_c(t)$ and on the longitudinal bars, $\tau_s(t)$. The redistribution of internal torques due to the gradual development of creep strains is to be examined and the time-dependent stresses and strains in both the concrete and the steel are to be calculated.

The rate of twist of section B after time $t$, has the relation with shear stress component of concrete as the following:

\[
\beta(t) = \left(\frac{\tau_c}{G_e(t)}\right)\left(k_x(2a)(2b)^2\right) + \left(\frac{\Delta \tau_c(t)}{G_e(t)}\right)\left(k_x(2a)(2b)^2\right)
\]

\[\tag{5.107}\]

There is one unknown, $\Delta \tau_c(t)$ in the above equation, where $\tau_c(t)$ is the instantaneous stress.

The internal torque acting on the stirrups can also find the rate of twist of section B,

\[
T_s(t) = T - T_c(t)
\]
\[
= T - \tau_s(t)(k_x(2a)(2b)^2)
\]
\[
= T - [(\tau_c(t) + \Delta \tau_c(t))(k_x(2a)(2b)^2)]
\]

\[\tag{5.108}\]

And so,
\[
\beta_b(t) = \frac{T_s(t)}{G_s(J_y)_{2b}} = \frac{T - T_s(t)}{G_s(J_y)_{2b}} = \frac{T - \left(\tau_c + \Delta \tau_c(t)\right)\left(k_2(2a)(2b)^2\right)}{G_s(J_y)_{2b}}
\]

From equations (*) and (**),

\[
\beta_b(t) = \left(\tau_c\right)_{e}\left(k_2(2a)(2b)^2\right) + \frac{\left(\Delta \tau_c(t)\right)\left(k_2(2a)(2b)^2\right)}{G_s(J_y)_{2b}} = \frac{T - \left(\tau_c + \Delta \tau_c(t)\right)\left(k_2(2a)(2b)^2\right)}{G_s(J_y)_{2b}}
\]

Therefore,

\[
\Delta \tau_c(t) = \frac{G_s(t)TG_s(t)(J_a)_{2b} - \left[G_s(t)(J_a)_{2b} + G_s(J_y)_{2b}\right]\left(\tau_c\right)_{e}\left(k_2(2a)(2b)^2\right)}{G_s(t)G_s(t)(J_a)_{2b} + G_s(J_y)_{2b}}
\]

(5.110)

Substitute equation (*) into the equation (*) so that,

\[
\beta_b(t) = \frac{G_s(t)\beta_s(t)}{G_s(t)G_s(t)(J_a)_{2b} + G_s(J_y)_{2b}}
\]

(5.111)

Therefore, the maximum shear stress acting on the edge of the concrete, \(\tau_c(t)\) and each longitudinal bar of two different directions, \(\tau_{sz}(t)\) and \(\tau_{sy}(t)\) are the following:

\[
\tau_c(t) = \tau_c(t) + \Delta \tau_c(t) = \frac{G_s(t)(J_a)_{2b}\beta_s(t)}{\left(k_2(2a)(2b)^2\right)}
\]

(5.112)

\[
\tau_{sz}(t) = G_sA_{sb}\beta_s(t)
\]

(5.114)

\[
\tau_{sy}(t) = G_sB_{sb}\beta_s(t)
\]

(5.115)

For section A, the external shear stress \(\tau\) distribute into the different internal shear stresses in the concrete cover around the stirrups, \(\tau_{c1}(t)\), stirrups, \(\tau_{s1}(t)\), longitudinal bars
of the internal core of the reinforced concrete member $\tau_{x_2}$ and the internal core of the reinforced concrete member which contains longitudinal bars, $\tau_{x_2}(t)$.

The rate of twist of section A after time $t$, has the relation with shear stress component of concrete and the torques for steel as the following equations:

$$\beta_{A}(t) = \frac{(\tau_{c_1})(2\Gamma_{out}t)}{[G_{e}(t)J_{c_1}]} + \frac{(\Delta\tau_{c_1}(t))(2\Gamma_{out}t)}{[G_{e}(t)J_{c_1}]}$$  \hspace{1cm} (5.116)

$$\beta_{A}(t) = \frac{(\tau_{c_2})(k_{x_2}y_{c_2})}{[G_{e}(t)J_{a}]} + \frac{(\Delta\tau_{c_2}(t))(k_{x_2}y_{c_2})}{[G_{e}(t)J_{a}]}$$  \hspace{1cm} (5.117)

$$\beta_{A}(t) = \frac{T_{s_1}(t)}{G_{s}(J_{s_1})}$$  \hspace{1cm} (5.118)

$$\beta_{A}(t) = \frac{T_{s_2}(t)}{G_{s}(J_{s_2})}$$  \hspace{1cm} (5.119)

There are two unknowns, $\Delta\tau_{c_1}(t)$ and $\Delta\tau_{c_2}(t)$ in the above equations, where $(\tau_{c_1})_e$ and $(\tau_{c_2})_e$ are the instantaneous stresses.

The internal torque acting on the stirrup can also find the rate of twist of section A,

$$T_{s_1}(t) = T_{s_1}(t) + T_{s_2}(t) = T - T_{c_1}(t) - T_{c_2}(t)$$

$$T_{s_1}(t) = T - (\tau_{c_1}(t)(2\Gamma_{out}t) - (\tau_{c_2}(t)(k_{x_2}y_{c_2})$$

$$[G_{s}(J_{s_1}) + G_{s}(J_{s_2})] \beta_{A}(t) = T - [(\tau_{c_1}(t) + \Delta\tau_{c_1}(t))(2\Gamma_{out}t) - [(\tau_{c_2}(t) + \Delta\tau_{c_2}(t))(k_{x_2}y_{c_2})$$  \hspace{1cm} (5.120)

Where $T_{s_1}(t)$ and $T_{s_2}(t)$ are the torques with respect to time acting on the stirrups and longitudinal bars of the internal core of the reinforced concrete member respectively.

Let unknowns be, $X = \Delta\tau_{c_1}(t)$ and $Y = \Delta\tau_{c_2}(t)$ so that,

$$\frac{(\tau_{c_1})(2\Gamma_{out}t)}{[G_{e}(t)J_{c_1}]} + \frac{(\Delta\tau_{c_1}(t))(2\Gamma_{out}t)}{[G_{e}(t)J_{c_1}]} = T - [(\tau_{c_1}(t) + \Delta\tau_{c_1}(t))(2\Gamma_{out}t) - [(\tau_{c_2}(t) + \Delta\tau_{c_2}(t))(k_{x_2}y_{c_2})$$

$$\frac{(\tau_{c_2})(k_{x_2}y_{c_2})}{[G_{e}(t)J_{a}]} + \frac{(\Delta\tau_{c_2}(t))(k_{x_2}y_{c_2})}{[G_{e}(t)J_{a}]} = T - [(\tau_{c_1}(t) + \Delta\tau_{c_1}(t))(2\Gamma_{out}t) - [(\tau_{c_2}(t) + \Delta\tau_{c_2}(t))(k_{x_2}y_{c_2})$$  \hspace{1cm} (5.121)

$$\frac{(\tau_{c_1})(2\Gamma_{out}t)}{[G_{e}(t)J_{c_1}]} + \frac{(\Delta\tau_{c_1}(t))(2\Gamma_{out}t)}{[G_{e}(t)J_{c_1}]} = T - [(\tau_{c_1}(t) + \Delta\tau_{c_1}(t))(2\Gamma_{out}t) - [(\tau_{c_2}(t) + \Delta\tau_{c_2}(t))(k_{x_2}y_{c_2})$$

$$\frac{(\tau_{c_2})(k_{x_2}y_{c_2})}{[G_{e}(t)J_{a}]} + \frac{(\Delta\tau_{c_2}(t))(k_{x_2}y_{c_2})}{[G_{e}(t)J_{a}]} = T - [(\tau_{c_1}(t) + \Delta\tau_{c_1}(t))(2\Gamma_{out}t) - [(\tau_{c_2}(t) + \Delta\tau_{c_2}(t))(k_{x_2}y_{c_2})$$  \hspace{1cm} (5.122)
Let the coefficients be, \( a = (\tau_{c1}), \ b = (\tau_{c2}), \ x = (2\Gamma_{out}), \ y = (k_x, y_c^2) \),

\[
\begin{align*}
A &= \bar{G}_x(t)(J_{x1})_{1n}, \ B = \bar{G}_x(t)(J_{x2})_{2n}, \ C = [G_x(J_{x1})_{2n} + G_y(J_{y1})_{2n}], \ D = [G_x(J_{x1})_{1n}], \ E = [G_x(t)(J_{x2})_{2n}], \ K = T - (\tau_{c1})(2\Gamma_{out}) - (\tau_{c2})(k_x, y_c^2) = T - ax - by.
\end{align*}
\]

Then the equations (*) and (**) becomes,

\[
\begin{align*}
\frac{ax}{D} + \frac{xX}{A} &= \frac{K - xX - yY}{C} \\
\frac{by}{E} + \frac{yY}{B} &= \frac{K - xX - yY}{C}
\end{align*}
\]

By solving equations (*) and (**), the unknowns \( X \) and \( Y \) can be determined as follows:

\[
X = \frac{ADEK - aAE(B + C)x - bABDy}{xDE(A + B + C)} = \frac{A}{(A + B + C)} \left[ T - \frac{b(E - B)y}{Ex} - \frac{a(B + C + D)}{D} \right]
\]

\[
Y = \frac{BDEK - aABEx - bBD(A + C)y}{yDE(A + B + C)} = \frac{B}{(A + B + C)} \left[ T - \frac{a(D - A)x}{Dy} - \frac{b(A + C + E)}{E} \right]
\]

Therefore,

\[
\beta_A(t) = \frac{ax}{D} + \frac{xX}{A} = \frac{ax}{D} + \frac{1}{(A + B + C)} \left[ T - \frac{ax(B + C + D)}{D} - \frac{by(E - B)}{E} \right]
\]

\[
= \frac{1}{(A + B + C)} \left[ T - \frac{ax(D - A)}{D} - \frac{by(E - B)}{E} \right]
\]
Put \( \Delta \tau_{c_1} (t) \) into equation (*) and \( \Delta \tau_{c_2} (t) \) into equation (***) so that,

\[
\Delta \tau_{c_1} (t) = \frac{\left[ G_\epsilon (t) \right]_{J_{c_1}}_{2a} - \left[ G_\epsilon (t) \right]_{J_{a}}_{2a} + G_s (J_{s_1})_{2a} + G_s (J_{s_2})_{2a} \left[ G_\epsilon (t) \right]_{J_{c_1}}_{2a} \right]}{\left( 2 \Gamma_{out} t \right)} \frac{\left( \tau_{c_2} \right)_{e} \left( k_2 x_c y_c \right)}{\left[ G_\epsilon (t) \right]_{J_{a}}_{2a} + G_s (J_{s_1})_{2a} + G_s (J_{s_2})_{2a} + \left[ G_\epsilon (t) \right]_{J_{c_1}}_{2a} \right]}
\]

\[
\Delta \tau_{c_2} (t) = \frac{\left[ G_\epsilon (t) \right]_{J_{c_1}}_{2a} - \left[ G_\epsilon (t) \right]_{J_{a}}_{2a} + G_s (J_{s_1})_{2a} + G_s (J_{s_2})_{2a} \left[ G_\epsilon (t) \right]_{J_{c_1}}_{2a} \right]}{\left( 2 \Gamma_{out} t \right)} \frac{\left( \tau_{c_1} \right)_{e} \left( k_2 x_c y_c \right)}{\left[ G_\epsilon (t) \right]_{J_{a}}_{2a} + G_s (J_{s_1})_{2a} + G_s (J_{s_2})_{2a} + \left[ G_\epsilon (t) \right]_{J_{c_1}}_{2a} \right]}
\]

(5.123)

(5.124)

Therefore the rate of twist of section A after time t, can be expressed as,

\[
\beta_A (t) = \frac{1}{(A + B + C)} \left[ T - \frac{ax(D - A) - by(E - B)}{D} \right]
\]

\[
= \frac{1}{\left( G_\epsilon (t) \right)_{J_{c_1}}_{2a} + \left[ G_\epsilon (t) \right]_{J_{a}}_{2a} + G_s (J_{s_1})_{2a} + G_s (J_{s_2})_{2a} \left[ G_\epsilon (t) \right]_{J_{c_1}}_{2a} \right]}
\]

\[
\left[ T - \left[ \left( \tau_{c_1} \right)_{e} (2 \Gamma_{out} t) + \left( \tau_{c_2} \right)_{e} \left( k_2 x_c y_c \right) \right] \left( \frac{G_\epsilon (t) - G_s (t)}{G_\epsilon (t)} \right) \right]
\]

(5.125)

Therefore, the maximum shear stress acting on the edge of the concrete, \( \tau_c (t) \) and each longitudinal bar of two different directions, \( \tau_{x_1} (t) \) and \( \tau_{x_2} (t) \), as well as the stirrups are the following:
\[ \tau_{c1}(t) = (\tau_{c1})_e + \Delta \tau_{c1}(t) \]  
(5.126)
\[ \tau_{c2}(t) = (\tau_{c2})_e + \Delta \tau_{c2}(t) \]  
(5.127)
\[ \tau_{sl}(t) = G_s A_{sl} B_A(t) \]  
(5.128)
\[ \tau_{st}(t) = G_s B_{sl} B_A(t) \]  
(5.129)
\[ \tau_{sl}(t) = \frac{G_s (J_{sl})_{2a} B_A(t)}{2\Gamma_i t(z)} \]  
(5.130)

Hence, the total angle of twist can be determined by equation (*), which is the same as Method 1,

\[ \theta_{total}(t) = (nd_{sr})_B A(t) + [(\tau - n - 1)(s - d_{sr}) + (L - (n - 1)(s - d_{sr}) - nd_{sr})] B_B(t) \]  
(5.63)

\[ G_s (t, \tau) = 3.623 GPa, \quad \left[\left[G_s^{(1)}\right] (t, \tau)\right] = 2.953 GPa, \quad \left[\left[G_s^{(2)}\right] (t, \tau)\right] = 2.802 GPa, \]

\[ G_s (t, \tau) = 3.20 GPa, \quad \left[\left[G_s^{(1)}\right] (t, \tau)\right] = 3.472 GPa \]

\[ \beta_A(t) = 5.26 \times 10^{-7} \text{ Rad/mm}, \quad \beta_B(t) = 9.798 \times 10^{-6} \text{ Rad/mm}, \quad \theta_{total}(t) = 9.074 \times 10^{-3} \text{ Rad} \]

\[ \tau_{sl}(t) = 10.177 \text{ MPa}, \quad \tau_{st}(t) = 0.043 \text{ MPa}, \quad \tau_{out}(t) = 0.041 \text{ MPa} \]

**5.3.2.1 Numerical Example**

The concrete and steel shear strains as well as the angle of twist are to be determined at selected times for the pure torsion. The external pure torque T is 800 Nm. Cross sectional, material properties and dimensions are the same as previous example and the four longitudinal bars are located at (34, 34), (34, -34), (-34, 34) and (-34, -34) from the origin of the section.

At first loading \((t - \tau_0)\) in days = 0, \(\phi(t - \tau_0) = 0\),

\[ E_s(t, \tau) = E_s(\tau_0) = 25 GPa \]
\[ G_s(\tau_0) = \frac{E_s(\tau_0)}{2(1 + \nu_s)} = 10.87 GPa \]
\[ G_s = \frac{E_s}{2(1 + \nu_s)} = 76.9 GPa \]
The polar moment of inertia of the concrete cover around the stirrups, \( (J_{c1})_{2a} = (J_{cw1})_{2a} \),

\[
(J_{c1})_{2a} = \frac{2(bh)^{3}t}{(b + h)} = \frac{2[(103)(103)]^{2}(17)}{(103 + 103)} = 18576359 \text{mm}^4
\]

The polar moment of a stirrup, \( (J_{s1})_{2a} = (J_{str1})_{2a} \),

\[
(J_{s1})_{2a} = \frac{4(x_{s}y_{s})^{2}\pi}{(x_{s} + y_{s})}d_{sv} = \frac{4(80 \times 80)^{2}\pi}{(80 + 80)} \times 6 = 19301945 \text{mm}^4
\]

The polar moment of a internal core for concrete, \( (J_{a})_{2a} \),

\[
(J_{a})_{2a} = \left(kx_{c}^{3}y_{c}ight) - \sum_{n=1}^{5} \left(A_{ai}x_{i} - B_{ai}y_{i}\right)(A_{i})_{i}
\]

\[
= 0.141 \times (74)^{3} \times (74) - \sum_{n=1}^{4} \left(A_{ai}x_{i} - B_{ai}y_{i}\right)6
\]

\[
= 4227470 \text{mm}^4
\]
Where,

\[ A_{m} = -\frac{8x_c}{\pi^2} \sum_{n=6}^{\infty} \frac{(-1)^n \sinh 2k_n y_i \cos k_n x_i}{(2n+1)^2 \cosh k_n y_c} \]

\[ = -\frac{8x}{\pi^2} \sum_{n=6}^{\infty} \frac{(-1)^n \sinh 2k_n (\pm 34) \cos k_n (\pm 34)}{(2n+1)^2 \cosh k_n (74)} \]

\[ = \pm 9.35 \]

\[ B_{m} = \frac{2x - \frac{8x_c}{\pi^2} \sum_{n=0}^{\infty} (-1)^n \cosh 2k_n y_i \sin k_n x_i}{2(\pm 34) - \frac{8x}{\pi^2} \sum_{n=0}^{\infty} \frac{(-1)^n \cosh 2k_n (\pm 34) \sin k_n (\pm 34)}{(2n+1)^2 \cosh k_n (74)}} \]

\[ = \frac{2(\pm 34)}{\pm 9.35} \]

The polar moment of longitudinal bars, \((J_y)_a\),

\[ (J_y)_a = \left( \sum_{n=1}^{\infty} (A_{ni} x_i - B_{ni} y_i) (A_i)_i \right) \]

\[ = \sum_{n=1}^{\infty} ((\pm 9.35)(\pm 34) - (\pm 9.35)(\pm 34))(6) \]

\[ = 636 mm^4 \]

The polar moment for concrete of section B, \((J_a)_b\),

\[ (J_a)_b = k(2a)^3 (2b) - \left( \sum_{n=1}^{\infty} (A_{bi} x_i - B_{bi} y_i) (A_i)_i \right) \]

\[ = 0.141x(120)^3x(120) - \sum_{n=1}^{\infty} (A_{bi} x_i - B_{bi} y_i)(6) \]

\[ = 29235999 mm^4 \]

Where,
Time Dependent Effects of Reinforced Concrete Subjected to Torsional Loading

\[ A_{bi} = -\frac{16a}{\pi^2} \sum_{n=6}^{\infty} \frac{(-1)^n \sinh k_n y_i \cos k_n x_i}{(2n+1)^2 \cosh k_n b} \]
\[ = -\frac{16 \times 60}{\pi^2} \sum_{n=0}^{\infty} \frac{(-1)^n \sinh k_n (\pm 34) \cos k_n (\pm 34)}{(2n+1)^2 \cosh k_n (74)} \]
\[ = \pm 25.9 \]

\[ B_{bi} = \left[ 2x - \frac{16a}{\pi^2} \sum_{n=0}^{\infty} \frac{(-1)^n \cosh k_n y_i \sin k_n x_i}{(2n+1)^2 \cosh k_n b} \right] \]
\[ = \left[ 2(\pm 34) - \frac{8 \times 60}{\pi^2} \sum_{n=0}^{\infty} \frac{(-1)^n \cosh k_n (\pm 34) \sin k_n (\pm 34)}{(2n+1)^2 \cosh k_n (74)} \right] \]
\[ = \pm 25.9 \]

The polar moment for longitudinal bars of section B, \( (J_a)_{2b} \),

\[ (J_a)_{2b} = \left( \sum_{n=1}^{\infty} \left( A_{bi} x_i - B_{bi} y_i x(A_i) x \right) \right) \]
\[ = \sum_{n=1}^{4} \left( (\pm 25.9)(\pm 34) - (\pm 25.9)(\pm 34) \right) \]
\[ = 1760.6 mm^4 \]

The instantaneous rate of twist of section A and section B can be determined as follows:
\[ \beta_A = \frac{T}{\left[ G_s \left( J_{s1} \right)_{2a} + G_s \left( J_{r1} \right)_{2a} + G_c \left( J_{c1} \right)_{2a} + G_c \left( J_{c} \right)_{2a} \right]} \]

\[ = \frac{800}{(76.9)(19301945) + (76.9)(636) + (10.87)(18576359) + (10.87)(29235999)} \]

\[ = 4.62 \times 10^{-7} \text{ Rad/mm} \]

\[ \beta_B = \frac{T}{\left[ G_r \left( J_{r} \right)_{2b} + G_A \left( J_{r} \right)_{2b} \right]} \]

\[ = \frac{800}{(10.87)(29235999) + (76.9)(1760.6)} \]

\[ = 2.52 \times 10^{-6} \text{ Rad/mm} \]

Thus, the total angle of twist of the reinforced concrete member is,

\[ \theta_{\text{total}} = (nd_{s1}) \beta_A + [(n-1)(s-d_{s1}) + (L-(n-1)(s-d_{s1})-nd_{s1})] \beta_B \]

\[ = (13 \times 6)(4.62 \times 10^{-7}) + [(13-1)(80-6) + (1000 - (13-1)(80-6)-(13 \times 6))](2.52 \times 10^{-6}) \]

\[ = 2.36 \times 10^{-3} \text{ Rad} \]

The shear stresses acting on different component of the reinforced concrete member of section A at initial are follows,

\[ \tau_s = \frac{G_s \left( J_{s1} \right)_{2a} \beta_A}{2 \Gamma_s I(z)} = \frac{G_c \left( J_{c1} \right)_{2a} \beta_A}{2 \Gamma_c d_{sv}} \]

\[ = \frac{(76.9)(19301945)(4.62 \times 10^{-7})}{2 \times (80 \times 80) \times 6} \times 1000 \]

\[ = 8.93 \text{ MPa} \]

For initial maximum shear stress acting on the internal core,
\[
(\tau_{c2})_e = \tau_{c2}(t) = \frac{[G_x(t)][J_\alpha]_{2a} \beta_\lambda}{(k_2 x_t y_e^2)} = \frac{(10.87)(4227470)(4.62 \times 10^{-7})}{(0.208 \times 74 \times 74^2)} \times 1000 = 0.26 \text{MPa}
\]

For initial maximum shear stress acting on the concrete cover around the stirrups,

\[
(\tau_{c1})_e = \tau_{c1}(t) = \frac{G_x(t)[J_\alpha]_{2a} \beta_\lambda}{2 \Gamma_{\text{out}} t} = \frac{(10.87)(18576359)(4.62 \times 10^{-7})}{2 \times (103 \times 103 \times 17)} \times 1000 = 0.26 \text{MPa}
\]

For initial maximum shear stress acting on each longitudinal bar of z and y direction,

\[
\tau_{z}(t) = G_x A_{z} \beta_\lambda = (76.9)(9.35)(4.62 \times 10^{-7}) = \pm 0.33 \text{MPa}
\]

\[
\tau_{y}(t) = G_x B_{y} \beta_\lambda = (76.9)(\pm 9.35)(4.62 \times 10^{-7}) = \pm 0.33 \text{MPa}
\]

The shear stresses acting on different component of the reinforced concrete member of section B at initial are follows,

For initial maximum shear stress acting on the concrete,

\[
(\tau_{c2})_e = \tau_{c2}(t) = \frac{[G_x(t)][J_\alpha]_{2b} \beta_\beta}{(k_2(2a)(2b)^2)} = \frac{(10.87)(29235999)(2.52 \times 10^{-6})}{(0.208 \times 120 \times 120^2)} \times 1000 = 2.22 \text{MPa}
\]

For initial maximum shear stress acting on each longitudinal bar of z and y direction,
\[
\tau_{a}(t) = G_A \beta_a \\
= (76.9) \pm 25.9 \times 10^{-6} \\
= \pm 5.0 \text{MPa}
\]

\[
\tau_{b}(t) = G_B \beta_b \\
= (76.9) \pm 25.9 \times 10^{-6} \\
= \pm 5.0 \text{MPa}
\]

First, the section is analyzed by EMM approach.

At loading \((t - \tau_a)\) in days = 25, \(\phi(t - \tau_a) = 1.0\),

\[
G_e(t, \tau) = \frac{G_c(\tau)}{1 + \phi(t, \tau)} \\
= \frac{10.87}{1.58} = 5.435 \text{GPa}
\]

The rate twist of section A can be determined from equation (5.100),

\[
\beta_a(t) = \frac{T}{G_e((J_y)_{2a} + (J_{x1})_{2a}) + G_e(t)((J_{x1})_{2a} + (J_{x1})_{2a})} \\
= \frac{800}{76.9 \times (636 + 19301945) + \phi \times 5.435 \times (4227470 + 18576359)} \\
= 4.97 \times 10^{-7} \text{ Rad }/\text{ mm}
\]

The rate twist of section B can be determined from equation (5.91),

\[
\beta_b(t) = \frac{T}{G_e((J_y)_{2b} + G_e(t)(J_{x1})_{2b})} \\
= \frac{800}{(76.9)(1760.6) + \phi \times 5.435 \times (29235999)} \\
= 5.03 \times 10^{-6} \text{ Rad }/\text{ mm}
\]

Hence, the total angle of twist is,
\[ \theta_{\text{total}}(t) = (nd_{s,1}) \beta_A(t) + \left[ (n-1)(s-d_{c,v}) + (L-(n-1)(s-d_{c,v})-nd_{c,v}) \right] \beta_B(t) \quad (5.63) \]
\[
= (13 \times 6) \times (4.97 \times 10^{-7}) + \left[ (13-1)(80-6) + (1000 - (13-1)(80-6) - 13 	imes 6) \right] \times 5.03 \times 10^{-6} \\
= 4.67 \times 10^{-3} \text{ Rad} 
\]

The maximum shear stress acting on the edge of the concrete, \( \tau_e(t) \) and each longitudinal bar of two different directions, \( \tau_{x\alpha}(t) \) and \( \tau_{y\alpha}(t) \) of section B are the following.

\[
\tau_e(t) = \frac{G_e(t)(J_{\alpha})_{2b} \beta_B(t)}{(k_2(2a)(2b)^2)} \\
= \frac{5.435 \times 29235999 \times 5.03 \times 10^{-6}}{0.208 \times 120 \times 120^2} \times 1000 \\
= 2.22 \text{MPa} 
\]

\[
\tau_{x\alpha}(t) = G_s A_{x\alpha} \beta_B(t) \\
= (76.9)(\pm 25.9)(5.03 \times 10^{-6}) \times 1000 \\
= \pm 10.0 \text{MPa} 
\]

\[
\tau_{y\alpha}(t) = G_s B_{y\alpha} \beta_B(t) \\
= (76.9)(\pm 25.9)(5.03 \times 10^{-6}) \times 1000 \\
= \pm 10.0 \text{MPa} 
\]

The maximum shear stress acting on the edge of the concrete, \( \tau_e(t) \) and each longitudinal bar of two different directions, \( \tau_{x\alpha}(t) \) and \( \tau_{y\alpha}(t) \), as well as the stirrups, \( \tau_{s1}(t) \) are the following:

\[
\tau_{x\alpha}(t) = \frac{G_e(t)(J_{\alpha})_{2a} \beta_A(t)}{(k_2(x_v(y_v)^3)} \\
= \frac{5.435 \times 4227470 \times 4.97 \times 10^{-7}}{0.208 \times 74 \times 74^2} \times 1000 \\
= 0.136 \text{MPa} 
\]

\[
\tau_{s1}(t) = \frac{G_e(t)(J_{\alpha})_{2a} \beta_A(t)}{2(T_{ou\alpha})} \\
= \frac{5.435 \times 18576359 \times 4.97 \times 10^{-7}}{2 \times 103 \times 103 \times 17} \times 1000 \\
= 0.139 \text{MPa} 
\]
\[ \tau_{x1}(t) = G_A A_2 \beta_A \beta_A(t) \]
\[ = (76.9) \pm 9.35 \left(4.97 \times 10^{-1}\right) \times 1000 \]
\[ = \pm 0.357 \text{MPa} \]
\[ \tau_{y2}(t) = G_B A_2 \beta_A \beta_A(t) \]
\[ = (76.9) \pm 9.35 \left(4.97 \times 10^{-1}\right) \times 1000 \]
\[ = \pm 0.357 \text{MPa} \]
\[ \tau_{y1}(t) = \frac{G_A (J_{11})_{2n} \beta_A \beta_A(t)}{2 \Gamma_x(\tau)} \]
\[ = \frac{(76.9)(19301945)\left(4.97 \times 10^{-1}\right)}{2 \times 80 \times 80 \times 6} \]
\[ = 9.61 \text{MPa} \]

At loading \((t - \tau_0)\) in days = 100, \(\phi(t - \tau_0) = 2.0,\)

\[ G_A(t, \tau) = 3.623 \text{GPa}, \quad \beta_A(t) = 5.10 \times 10^{-7} \text{ Rad / mm}, \quad \beta_B(t) = 7.543 \times 10^{-6} \text{ Rad / mm}, \]
\[ \theta_{\text{Total}}(t) = 6.99 \times 10^{-3} \text{ Rad}, \quad \tau_r(t) = 2.22 \text{MPa}, \]
\[ \tau_{x1}(t) = \pm 15.0 \text{MPa}, \quad \tau_{y2}(t) = \pm 15.0 \text{MPa}, \quad (\text{For section B}) \]
\[ \tau_{x2}(t) = 0.092 \text{MPa}, \quad \tau_{y1}(t) = 0.095 \text{MPa}, \]
\[ \tau_{x1}(t) = \pm 0.36 \text{MPa}, \quad \tau_{y1}(t) = \pm 0.36 \text{MPa}, \quad \tau_{y1} = 9.87 \text{MPa} \quad (\text{For section A}) \]

At loading \((t - \tau_0)\) in days = 10000, \(\phi(t - \tau_0) = 3.0,\)

\[ G_A(t, \tau) = 2.717 \text{GPa}, \quad \beta_A(t) = 5.17 \times 10^{-7} \text{ Rad / mm}, \quad \beta_B(t) = 10.0 \times 10^{-6} \text{ Rad / mm}, \]
\[ \theta_{\text{Total}}(t) = 9.30 \times 10^{-3} \text{ Rad}, \quad \tau_r(t) = 2.22 \text{MPa}, \]
\[ \tau_{x1}(t) = \pm 20.0 \text{MPa}, \quad \tau_{y2}(t) = \pm 20.0 \text{MPa}, \quad (\text{For section B}) \]
\[ \tau_{x2}(t) = 0.070 \text{MPa}, \quad \tau_{y1}(t) = 0.072 \text{MPa}, \]
\[ \tau_{x1}(t) = \pm 0.37 \text{MPa}, \quad \tau_{y1}(t) = \pm 0.37 \text{MPa}, \quad \tau_{y1} = 10.0 \text{MPa} \quad (\text{For section A}) \]

The concrete and steel shear stresses and angle of twist are now re-analyzed by AEMM approach.

At first loading \((t - \tau_0)\) in days = 0, \(\phi(t - \tau_0) = 0,\) the results are the same as EMM approach.
Time Dependent Effects of Reinforced Concrete Subjected to Torsional Loading

\[
\begin{align*}
(t - \tau_a) & \quad \text{in days} = 0 \quad \phi(t - \tau_a) = 0 \quad \chi(t - \tau_a) = 1.0 \\
(t - \tau_a) & \quad \text{in days} = 25 \quad \phi(t - \tau_a) = 1 \quad \chi(t - \tau_a) = 0.86 \\
(t - \tau_a) & \quad \text{in days} = 100 \quad \phi(t - \tau_a) = 2 \quad \chi(t - \tau_a) = 0.8 \\
(t - \tau_a) & \quad \text{in days} = 10000 \quad \phi(t - \tau_a) = 3 \quad \chi(t - \tau_a) = 0.8
\end{align*}
\]

At first loading \((t - \tau_a)\) in days = 25, \(\phi(t - \tau_a) = 1.0, \chi(t - \tau_a) = 0.86\),

\[
\bar{G}_{c}(t, \tau) = \frac{G_{e}(\tau)}{1 + \chi(t, \tau)\phi(t, \tau)}
\]

\[
= \frac{10.87}{1 + (0.86)(1.0)} = 5.844 \text{GPa}
\]

For section B, the one unknown of \(\Delta \tau_c(t)\), is required to be determined,

\[
\Delta \tau_c(t) = \frac{\bar{G}_{e}(t)TG_{e}(t)(J_a)_{2b} - \left[G_{e}(t)(J_a)_{2b} + G_{s}(J_{r})_{2b}\right]k_z(2a)(2b)^2}{\left[G_{e}(t)(J_a)_{2b} + G_{s}(J_{r})_{2b}\right]}
= \frac{5.844(800)(5.435)(29235999) - \left[(5.435)(29235999) + (76.9)(1760.6)(0.00222)(359424)\right]}{5.435(5.844(29235999) + (76.9)(1760.6))} \times 1000
= -0.000947 \text{MPa}
\]

\[
\tau_c(t) = (\tau_c) + \Delta \tau_c(t)
= 2.22 - 0.000947 = 2.219 \text{MPa}
\]

Therefore, the rate if twist of section B is,

\[
\beta_B(t) = \frac{G_{e}(t)T + \left[\bar{G}_{e}(t) - G_{e}(t)\right]k_z(2a)(2b)^2}{G_{e}(t)\left[G_{e}(t)(J_a)_{2b} + G_{s}(J_{r})_{2b}\right]}
= \frac{(5.435)(800) + (5.844 - 5.435)(0.00222)(359424)}{(5.435)(5.844(29235999) + (76.9)(1760.6))}
= 5.03 \times 10^{-6} \text{Rad/mm}
\]

The maximum shear stress acting on each longitudinal bar of two different directions, \(\tau_{x1}(t)\) and \(\tau_{x2}(t)\), are the following.
\[ \tau_{xz}(t) = G_s A_{wz} \beta_y(t) \]
\[ = (76.9) \pm 25.9 \times 10^{-6} \times 1000 \]
\[ = \pm 10.0 \text{MPa} \]
\[ \tau_{yx}(t) = G_s A_{wy} \beta_y(t) \]
\[ = (76.9) \pm 25.9 \times 10^{-6} \times 1000 \]
\[ = \pm 10.0 \text{MPa} \]

For section A, there are mainly two unknowns for solving the rate of twist, which are \[ \Delta \tau_{e1}(t) \] and \[ \Delta \tau_{e2}(t) \] can be found,

\[ \Delta \tau_{e1}(t) = \frac{\text{\( G_e(t) J_{a1} \) \( J_{a2} \)} \text{\( J_{a1} \)}}{\left[ \frac{T}{(2 \Gamma_{out} r)^2} \left( 2 \Gamma_{out} r \right) G_e(t) \right] \left[ (5.844 \times 18576359) + (5.844 \times 4227470) + (76.9 \times 19301945) + (76.9 \times 636) \right]}
\]
\[ \times \left[ (\tau_{e1}) \left( G_e(t) J_{a1} \right) + G_s (J_{a1})^2 + G_s (J_{a2})^2 + G_s (J_{a1} J_{a2}) + G_s (J_{a1}) (J_{a2}) \right] \]
\[ = \frac{800}{2 \times 103 \times 103 \times 17} \left[ (0.258 / 1000) (5.844 \times 4227470) + (76.9 \times 19301945) + (76.9 \times 636) + (5.435 \times 18576359) \right]
\[ \times (5.435 \times 18576359) \]
\[ = -0.127 \text{MPa} \]
\[
\Delta \tau_{e2}(t) = \frac{\overline{G_e}(t)(J_{el})_{2a}}{\left[\left(\overline{G_e}(t)(J_{el})_{2a} + G_e(J_{el})_{2a} + G_s(J_{sl})_{2a} + G_s(J_{fr})_{2a}\right) \right.} \\
\left. \left[ -\frac{T}{k_{x1}} \frac{(\tau_{el})_e(2\Gamma_{out})}{G_e(t)} + \frac{(\tau_{el})_e(k_{x1}y_r^2)}{G_e(t)} \right] \right] \\
\left. + \frac{\overline{G_e}(t)(J_{el})_{2a} + G_e(J_{el})_{2a} + G_s(J_{sl})_{2a} + G_s(J_{fr})_{2a}}{G_e(t)(J_{el})_{2a}}\right) \\
(5.844)(4227470) \\
= \frac{(5.844)(18576359) + (5.844)(4227470) + (76.9)(19301945) + (76.9)(636)}{800 - (0.252/1000)(2x103x103x17)(5.435 - 5.844)} \\
\frac{84286.6}{(84286.6)(5.435)} \\
- \frac{(0.252/1000)(5.844)(18576359) + (76.9)(19301945) + (76.9)(636) + (5.435)(4227470)}{(5.435)(4227470)} \\
= -0.124\text{MPa}
\]

Therefore, the rate of twist for section A by using AEMM approach is,

\[
\beta_A(t) = \frac{1}{\left[\left(\overline{G_e}(t)(J_{el})_{2a} + G_e(J_{el})_{2a} + G_s(J_{sl})_{2a} + G_s(J_{fr})_{2a}\right) \right.} \\
\left. \left[ -\frac{T}{k_{x1}} \frac{(\tau_{el})_e(2\Gamma_{out})}{G_e(t)} + \frac{(\tau_{el})_e(k_{x1}y_r^2)}{G_e(t)} \right] \right] \\
\left. + \frac{1}{(5.844)(18576359) + (5.844)(4227470) + (76.9)(19301945) + (76.9)(636)} \\
\frac{800 - (0.252/1000)(2x103x103x17) + (0.252/1000)(84286.6)(5.85 - 6.29)}{84286.6} \right] \\
= 4.99\times10^{-7}\text{ Rad} / \text{mm}
\]

Hence, the total angle of twist is,

\[
\theta_{\text{total}}(t) = (nd_w)\beta_A(t) + [(n-1)(s-d_w) + (L-n)(s-d_w) - nd_w]\beta_B(t) \\
= (13\times6)(4.99\times10^{-7}) + [(13-1)(80-6) + (1000-(13-1)(80-6)-13x6)](5.03\times10^{-6}) \\
= 4.67\times10^{-3}\text{ Rad}
\]
The shear stresses acting on different component of the reinforced concrete member of section A at initial are as follows,

\[ \tau_{c1}(t) = (\tau_{c1})_0 + \Delta \tau_{c1}(t) = 0.258 - 0.127 = 0.131 MPa \]

\[ \tau_{c2}(t) = (\tau_{c2})_0 + \Delta \tau_{c2}(t) = 0.252 - 0.124 = 0.128 MPa \]

\[ \tau_{x1}(t) = G_i A_n \beta_A(t) = (76.9) \pm 9.35(4.99 \times 10^{-7}) \times 1000 = \pm 0.359 MPa \]

\[ \tau_{x2}(t) = G_i B_n \beta_A(t) = (76.9) \pm 9.35(4.99 \times 10^{-7}) \times 1000 = \pm 0.359 MPa \]

\[ \tau_{11}(t) = \frac{G_i (J_{11} \beta_2 \beta_A(t))}{2\Gamma_{11}(z)} = \frac{(76.9)(19301945)(4.99 \times 10^{-7})}{2 \times 80 \times 80 \times 6} \times 1000 = 9.66 MPa \]

At loading \((t - \tau_0)\) in days = 100, \(\phi(t - \tau_0) = 2.0, \chi(t - \tau_0) = 0.8\),

\[ G_i(t, \tau) = 3.623 GPa, \quad \bar{G}_e(t, \tau) = 4.181 GPa, \]

\[ \beta_A(t) = 5.10 \times 10^{-7} \text{ Rad / mm}, \quad \beta_B(t) = 7.544 \times 10^{-6} \text{ Rad / mm}, \]

\[ \theta_{total}(t) = 7.00 \times 10^{-2} \text{ Rad}, \quad \tau_{11}(t) = 1.71 MPa, \]

\[ \tau_{22}(t) = \pm 15.0 MPa, \quad \tau_{22}(t) = \pm 15.0 MPa, \quad (\text{For section B}) \]

\[ \tau_{c1}(t) = 0.069 MPa, \quad \tau_{c1}(t) = 0.072 MPa, \]

\[ \tau_{c2}(t) = 0.372 MPa, \quad \tau_{x1}(t) = 0.372 MPa, \quad \tau_{x1} = 10.0 MPa \quad (\text{For section A}) \]

At loading \((t - \tau_0)\) in days = 10000, \(\phi(t - \tau_0) = 3.0, \chi(t - \tau_0) = 0.8\).
\[ G_e(t,\tau) = 2.717 \text{GPa}, \quad \beta_a(t) = 5.27 \times 10^{-7} \text{Rad/mm}, \quad \beta_b(t) = 10.0 \times 10^{-6} \text{Rad/mm}, \]

\[ \theta_{\text{total}}(t) = 9.31 \times 10^{-3} \text{Rad}, \quad \tau_c(t) = 1.96 \text{MPa}, \]

\[ \tau_{s_1}(t) = \pm 20.0 \text{MPa}, \quad \tau_{s_2}(t) = \pm 20.0 \text{MPa}, \quad \text{(For section B)} \]

\[ \tau_{c_1}(t) = 0.040 \text{MPa}, \quad \tau_{c_2}(t) = 0.041 \text{MPa}, \]

\[ \tau_{s_1}(t) = \pm 0.378 \text{MPa}, \quad \tau_{s_2}(t) = \pm 0.378 \text{MPa}, \quad \tau_{s_3} = 10.2 \text{MPa} \quad \text{(For section A)} \]

5.3.3 Time Dependent Analysis – EMM and AEMM Approach for Analytical Method 3

First, consider the EMM approach (Effective Modulus Method) for analytical Method 3. If the instantaneous and creep components of strain are combined, a reduced or effective shear modulus for concrete \( G_e(t,\tau) \) is the same as Method 1 and Method 2, which can be defined as follows:

\[
\gamma(t,\tau) = \frac{\tau}{G_e(t,\tau)} \left[ 1 + \phi(t,\tau) \right]
\]

\[
= \frac{\tau}{G_e(t,\tau)} \quad (5.58)
\]

where

\[
G_e(t,\tau) = \frac{G_e(\tau)}{1 + \phi(t,\tau)} \quad (5.59)
\]

A constant sustained torque, \( T \) is acting on the reinforced concrete member. The external torsional load, \( T \) is mainly resisted by internal torque of concrete, \( T_c(t) \) and equivalent steel, \( T_s(t) \). The redistribution of internal torques due to the gradual development of creep strains is to be examined. The time-dependent stresses and strains in both the concrete and the steel are to be calculated which are similar as Method 1 and Method 2.

The torsion load \( T \), acting on different parts of the section has the following distribution:

\[ T_c(t) + T_s(t) = T(t) = T \quad (5.88) \]

For angle of twist \( \theta_{\text{total}}(t) \) after time \( t \),

\[ \theta_{\text{total}}(t) = \beta(t) x L \]
The rate of twist of section B after time \( t \), has the relation with different torsion component as the following:

\[
\beta(t) = \frac{T_0(t)}{G_i \left( I_{\Gamma_z} - I_{\Gamma_i} \right) + \left( D_0 \right)_{\Gamma_z} - \left( D_0 \right)_{\Gamma_i}} \tag{5.131}
\]

\[
\beta(t) = \frac{T_0(t)}{G_i \left( I_{\Gamma_i} + \left( D_0 \right)_i \right)} \tag{5.132}
\]

Therefore, rate of twist of the whole member \( \beta(t) \) is

\[
\beta(t) = \frac{T}{G_i \left( I_{\Gamma_z} - I_{\Gamma_i} \right) + \left( D_0 \right)_{\Gamma_z} - \left( D_0 \right)_{\Gamma_i}} + G_i \left( I_{\Gamma_i} + \left( D_0 \right)_i \right) \tag{5.133}
\]

Therefore, the maximum shear stress acting on the edge of the concrete, \( \tau_o(t) \) and the inner boundary are the following, where the substituting of \( \rho_2 = \zeta \) for outer boundary and \( \rho_1 = \zeta \) for inner boundary:

\[
\tau_o = \frac{GJ \beta(t)}{k_3} \tag{5.134}
\]

Therefore, \( k_3 = k_3 \left( \lambda \right) \tag{5.135} \)

Then, consider the AEMM approach (Aged-adjusted Effective Modulus Method) for analytical Method 3, which is the same as Method 2. The age-adjusted effective shear modulus for concrete \( \overline{G}_\varepsilon(t, \tau) \) can be given by,

\[
\overline{G}_\varepsilon(t, \tau) = \frac{G_i(\tau)}{\left[ 1 + \chi(t, \tau) \phi(t, \tau) \right]} \tag{5.73}
\]

The total shear strain at time \( t \) may be expressed as the sum of the shear strains produced by \( \tau_o \) (instantaneous and creep), the strains produced by the gradually applied stress increment \( \Delta \tau(t) \) (instantaneous and creep), which is also the same as Method 1 and Method 2.

\[
\gamma(t) = \frac{\tau_o}{\overline{G}_\varepsilon(t, \tau)} \left[ 1 + \phi(t, \tau) \right] + \frac{\Delta \tau(t)}{\overline{G}_\varepsilon(t, \tau)} \left[ 1 + \chi(t, \tau) \phi(t, \tau) \right]
\]

\[
= \frac{\tau_o}{\overline{G}_\varepsilon(t, \tau)} + \frac{\Delta \tau(t)}{\overline{G}_\varepsilon(t, \tau)} \tag{5.75}
\]
The external shear stress \( \tau \) distribute into the different internal shear stresses in concrete, \( \tau_c(t) \) and the equivalent steel tube, \( \tau_s(t) \). The redistribution of internal torques due to the gradual development of creep strains is to be examined and the time-dependent stresses and strains in both the concrete and the steel are to be calculated, which is similar to the calculation of Method 1 and Method 2.

The rate of twist, \( \beta \), after time \( t \), has the relation with different shear stress components as the following:

\[
\beta(t) = \frac{(k_3) \tau_s(t)}{G_s \left[ \frac{1}{I_{r1}} - \frac{1}{I_{r2}} + \frac{(D_0)_{2r1}}{(D_0)_{2r2}} \right]} \quad (5.136)
\]

\[
\beta(t) = \frac{(k_3) \tau_c(t)}{G_c \left[ \frac{1}{I_{r1}} + \frac{(D_0)_{1r2}}{(D_0)_{1r1}} \right]} + \frac{(k_3) \Delta \tau_s(t)}{G_s \left[ \frac{1}{I_{r1}} + \frac{(D_0)_{2r1}}{(D_0)_{2r2}} \right]} \quad (5.137)
\]

There is one unknown, \( \Delta \tau_s(t) \) in the above equation, where \( \tau_c(t) \) is the instantaneous stress.

The internal torque acting on the steel tube can also find the rate of twist, \( \beta \),

\[
T_s(t) = T - T_c(t) = T - \tau_c(t)(k_3)_c = T - \left[ (\tau_c)_c + \Delta \tau(t) \right](k_3)_c \quad (5.138)
\]

And so,

\[
\beta(t) = \frac{T_s(t)}{T - T_c(t)} = \frac{T_s(t)}{G_s \left[ \frac{1}{I_{r1}} - \frac{1}{I_{r2}} + \frac{(D_0)_{2r1}}{(D_0)_{2r2}} \right]} + \frac{(k_3) \Delta \tau_s(t)}{G_s \left[ \frac{1}{I_{r1}} + \frac{(D_0)_{2r1}}{(D_0)_{2r2}} \right]} \]

From equations (5.136) and (5.137),
\[ \beta(t) = \frac{(\tau_c)(k_3)}{[\bar{G}_e(t)]^2 + (D_0)h} + \frac{((\Delta \tau_e)(t))(k_3)}{[\bar{G}_e(t)]^2 + (D_0)h} = \frac{T - [(\tau_c) + (\Delta \tau_e)(t))(k_3)]}{G_s[I_{I_1} - I_{I_1} + (D_0)2I_1 - (D_0)2I_1]} \] (5.139)

Therefore,

\[ \Delta \tau_e(t) \]
\[ = \frac{\bar{G}_e(t)T\bar{G}_e(t)[I_{I_1} + (D_0)h] - [\bar{G}_e(t)[I_{I_1} + (D_0)h] + G_s[I_{I_1} - I_{I_1} + (D_0)2I_1 - (D_0)2I_1]^{I_{I_1}}(k_3)]}{[\bar{G}_e(t)[I_{I_1} + (D_0)h] + G_s[I_{I_1} - I_{I_1} + (D_0)2I_1 - (D_0)2I_1]} \] (5.140)

So that,

\[ \beta(t) = \frac{G_e(t)^2 + [\bar{G}_e(t) - G_e(t)](k_3)}{[\bar{G}_e(t)[I_{I_1} + (D_0)h] + G_s[I_{I_1} - I_{I_1} + (D_0)2I_1 - (D_0)2I_1]} \] (5.141)

Therefore, the maximum shear stress acting on the edge of the concrete, \( \tau_c(t) \) and the equivalent steel tube, \( \tau_s(t) \) is the following:

\[ \tau_c(t) = (\tau_c) + \Delta \tau_e(t) = \frac{G_e(t)[I_{I_1} + (D_0)h] \beta(t)}{(k_3)} \] (5.142)

\[ \tau_s(t) = \frac{G_s[I_{I_1} - I_{I_1} + (D_0)2I_1 - (D_0)2I_1] \beta(t)}{(k_3)} \] (5.143)

5.3.3.1 Numerical Example

The complex variable and conformal mapping technique is used in the following calculations to an idealized reinforced concrete beam, which has the same cross sectional, material properties and dimensions as previous examples. Instead of a discrete system of longitudinal bars and stirrups, those beams with balanced reinforcement can be idealized as thin steel tube. Thus the boundaries of the beam cross-section can be treated as two
concentric squares with rounded corners. By using the mapping function, it is possible to map approximately the above cross section onto annulus.

The equivalent tube thickness for the beam is 0.575 mm, i.e. the inner square has width of 59.42 mm. These value is obtained by calculating the total volume of steel present in a given length of beam and then determining the thickness of the equivalent tube so that the volume of the tube will equal the volume of the longitudinal bars and stirrups.

For the case of a 120 by 120 square and annulus with a unit outer radius is:

\[ \omega(\zeta) = a\zeta + b\zeta^5 + c\zeta^9 + d\zeta^{13} + e\zeta^{17} + f\zeta^{21} + g\zeta^{25} \]

Where,

\[ a = +64.8 \]
\[ b = -6.265 \]
\[ c = +0.910 \]
\[ d = -1.128 \]
\[ e = +0.566 \]
\[ f = -0.256 \]
\[ g = +0.098 \]

The coefficients for the mapping function and the product \( \frac{1}{2} z \zeta \) on the inner boundary depend upon the inner radius \( \rho_1 \) and setting the outer radius \( \rho_2 \) as 1.

Newton's method gives a convergent solution for computing the value of inner radii when substituting \( \rho_1 \) into the above equation.

In several iterative cycles this method gives the following values, accurate to three decimal places:

\[ \rho_1 = 0.984 \]
\( \rho_2 = 1 \) (unit circle)

The coefficients of the quantity \( \frac{1}{2} zz \) can be determined as:

\[
\begin{align*}
A_i &= \rho_i^{2} \left( a^2 + b^2 \rho_i^8 + c^2 \rho_i^{16} + d^2 \rho_i^{24} + e^2 \rho_i^{32} + f^2 \rho_i^{40} + g^2 \rho_i^{48} \right) \\
B_i &= \rho_i^{6} \left( ab + bcp_i^8 + cd\rho_i^{16} + dep_i^{24} + ef\rho_i^{32} + fgp\rho_i^{40} \right) \\
C_i &= \rho_i^{10} \left( ac + bdp_i^8 + cep_i^{16} + dfp_i^{24} + egp_i^{32} \right) \\
D_i &= \rho_i^{14} \left( ad + bep_i^8 + cfp_i^{16} + dgp_i^{24} + egp_i^{32} \right) \\
E_i &= \rho_i^{18} \left( ae + bf\rho_i^8 + cgp_i^{16} \right) \\
F_i &= \rho_i^{22} \left( af + bg\rho_i^8 \right) \\
G_i &= \rho_i^{26} \left( ag \right)
\end{align*}
\]

for \( i = 1, 2 \)

The values of \( b_i^* \) are first calculated and then \( b_i^* \) can be evaluated using equations.

These coefficients for both types of specimens are presented in Table 5.13.

\[
\begin{align*}
b_4^* &= \left[ \frac{uB_1\rho_1^4 - B_2}{u\rho_1^8 - 1} \right] \\
b_8^* &= \left[ \frac{uC_1\rho_1^8 - C_2}{u\rho_1^{16} - 1} \right] \\
b_{12}^* &= \left[ \frac{uD_1\rho_1^{12} - D_2}{u\rho_1^{24} - 1} \right] \\
b_{16}^* &= \left[ \frac{uE_1\rho_1^{16} - E_2}{u\rho_1^{24} - 1} \right] \\
b_{20}^* &= \left[ \frac{uF_1\rho_1^{20} - F_2}{u\rho_1^{40} - 1} \right] \\
b_{24}^* &= \left[ \frac{uG_1\rho_1^{24} - G_2}{u\rho_1^{48} - 1} \right]
\end{align*}
\]

Hence, the value of \( b_{44}^* \) can be evaluated as follows,
\[ b_0' = (A_2 + k_2) \]
\[ b_4' + b_{-4}' = B_2 \]
\[ b_8' + b_{-8}' = C_2 \]
\[ b_{12}' + b_{-12}' = D_2 \]
\[ b_{16}' + b_{-16}' = E_2 \]
\[ b_{20}' + b_{-20}' = F_2 \]
\[ b_{24}' + b_{-24}' = G_2 \]

For the given mapping, the values of I and D₀ can be given using the residues theorem.

\[
\sum \text{Residues on } \Gamma_2 = -2i \left[ 4B_2 \left( b_4' - b_{-4}' \right) + 8C_2 \left( b_8' - b_{-8}' \right) + 12D_2 \left( b_{12}' - b_{-12}' \right) \right] \\
\sum \text{Residues on } \Gamma_1 = -2i \left[ 4B_1 \left( \rho_1^{12} b_{12}' - \rho_1^{12} b_{-12}' \right) + 8C_1 \left( \rho_1^{16} b_{16}' - \rho_1^{16} b_{-16}' \right) \right] \\
\sum \text{Residues on } \Gamma_1 = -2i \left[ 16E_1 \left( \rho_1^{20} b_{20}' - \rho_1^{20} b_{-20}' \right) + 24G_1 \left( \rho_1^{24} b_{24}' - \rho_1^{24} b_{-24}' \right) \right]
\]

Hence,

\[
(D_0)_2 = 2\pi \left(-\frac{1}{4}\right) \left[ \text{Residues on } \Gamma_2 + \text{Residues on } \Gamma_1 \right]
\]

Where,

\[
\sum \text{Residues on } \Gamma_1 = -i \left[ 8B_1 \rho_1^{12} b_{12}' + 16C_1 \rho_1^{12} b_{-12}' + 24D_1 \rho_1^{12} b_{12}' \right] \\
\sum \text{Residues on } \Gamma_1 = -i \left[ 32E_1 \rho_1^{16} b_{16}' + 40F_1 \rho_1^{20} b_{20}' + 48G_1 \rho_1^{24} b_{24}' \right]
\]

By the residues theorem

\[
(D_0)_1 = -\pi \left[ 4B_1 \rho_1^{12} b_{12}' + 8C_1 \rho_1^{16} b_{16}' + 12D_1 \rho_1^{12} b_{12}' + 16E_1 \rho_1^{16} b_{16}' + 20F_1 \rho_1^{20} b_{20}' + 24G_1 \rho_1^{24} b_{24}' \right]
\]

The polar moment of inertia for the outer region is
The residues are

\[ \sum \text{Residues} = a^4 \rho^3 + 12a^2b^2 \rho^{11} + 5(2ac + b^2)^2 \rho^{19} + 28(ad + bc)^2 \rho^{27} + 9(2ae + 2bd + c^2)^2 \rho^{35} + 44(af + be + cd)^2 \rho^{43} + 13(2ag + 2bf + 2ce + d^2)^2 \rho^{51} + 60(bg + cf + de)^2 \rho^{59} + 17(2cg + 2df + e^2)^2 \rho^{67} + 76(dg + cf)^2 \rho^{75} + 21(2eg + f^2)^2 \rho^{83} + 92(fg)^2 \rho^{91} + 25g^4 \rho^{99} \]

The polar moment of inertia for the region inside the required boundary is

\[ I = 2\pi \sum \text{Residues} \]

The values for I, D_0 and D appear in table 5.13 for both types of specimens.

<table>
<thead>
<tr>
<th>( b_i^- )</th>
<th>( b_{-k}^- )</th>
<th>( b_k^- )</th>
<th>Specimens with 2 percent balanced reinforcement</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \omega(\zeta) )</td>
<td>( \frac{1}{2} \zeta ) on ( \rho_2 )</td>
<td>( \frac{1}{2} \zeta ) on ( \rho_1 )</td>
<td>( K )</td>
</tr>
<tr>
<td>( +3.2314476 )</td>
<td>+5.2788810</td>
<td>+5.0174524</td>
<td>0</td>
</tr>
<tr>
<td>( -0.3124034 )</td>
<td>-1.0547742</td>
<td>-0.9035604</td>
<td>4</td>
</tr>
<tr>
<td>( +0.1172773 )</td>
<td>+0.4007379</td>
<td>+0.4007379</td>
<td>8</td>
</tr>
<tr>
<td>( -0.0562475 )</td>
<td>-0.1924016</td>
<td>-0.1924016</td>
<td>12</td>
</tr>
<tr>
<td>( +0.0282273 )</td>
<td>+0.0959106</td>
<td>+0.0959106</td>
<td>16</td>
</tr>
<tr>
<td>( -0.0131937 )</td>
<td>-0.0441632</td>
<td>-0.0441632</td>
<td>20</td>
</tr>
<tr>
<td>( +0.0048924 )</td>
<td>+0.0158095</td>
<td>+0.0158095</td>
<td>24</td>
</tr>
</tbody>
</table>

Table 5.13
Therefore,

\[
\begin{align*}
ix & \sum \text{Residues} \quad \text{on} \quad \Gamma_2 = 1852450.123 \\
ix & \sum \text{Residues} \quad \text{on} \quad \Gamma_1 = 1527954.373 \\
(D_0)_2 &= -2909821.849 \\
(D_0)_1 &= -2400105.116 \\
I_{r_2} &= 31419648.055 \\
I_{r_1} &= 29390853.896
\end{align*}
\]

The relationship between torque, T and rate of twist, \( \beta \) can be expressed as follows:

\[
T = \left[ G_r \left( I_{r_2} + (D_0)_1 \right) + G_s \left( (D_0)_2 r_2 - (D_0)_2 r_1 \right) \right] \beta
\]

\[
T = \left[ (10.87)(26990748.779) + (200)(1519077.427) \right] \beta
\]

\[
T = 4.10 \times 10^{11} \beta
\]

\[
D = GJ = 4.10 \times 10^{11} Nmm^2
\]

At first loading \((t - \tau_0)\) in days = 0, \( \phi(t - \tau_0) = 0 \).

\[
E_r(t, \tau) = E_r(\tau_0) = 25 GPa
\]

\[
G_s(\tau_0) = \frac{E_s}{2(1 + \nu_1)} = 10.87 GPa
\]

\[
G_s = \frac{E_s}{2(1 + \nu_1)} = 76.9 GPa
\]

If the external torque, \( T = 800 Nmm \), then the rate of twist, \( \beta = 1.95 \times 10^{-6} \text{ Rad / mm} \) and hence, angle of twist, \( \theta = 1.95 \times 10^{-3} \text{ Rad} \).

The shear stresses acting on the inner and outer boundaries are:

\[
\tau_0 = \frac{GJ \beta}{k_3} \quad (5.134)
\]

\[\Rightarrow (\tau_r)_c = \tau_c(t) = \frac{10.87 \times 26990748.779 \times 1.95 \times 10^{-6}}{359424} \times 1000 = 1.59 \text{MPa}\]

\[\Rightarrow (\tau_s)_c = \tau_s(t) = \frac{76.9 \times 1519077.427 \times 1.95 \times 10^{-6}}{365268.29} \times 1000 = 0.62 \text{MPa}\]
where
\[ k_3 = k_2 (xy^2) = \frac{0.208x(120x120^2)}{0.984} = 365268.29 \text{ (for inner boundary)} \]
\[ = 0.208x(120x120^2) = 359424 \text{ (for outer boundary)} \]

First, the section is analyzed by EMM approach.

At loading \((t - \tau_0)\) in days = 25, \(\phi(t - \tau_0) = 1.0\),

\[
G_x(t, \tau) = \frac{G_x(\tau)}{1 + \phi(t, \tau)}
\]
\[= \frac{10.87}{1+1} = 5.435 \text{ GPa} \]

The rate of twist of the whole member \(\beta(t)\) is

\[
\beta(t) = \frac{T}{G_x \left( I_{t_2} - I_{r_1} \right) + (D_0)_{2r_1} - (D_0)_{2r_2} + G_x(t) \left( I_{r_1} + (D_0) \right)}
\]
\[= \frac{800}{(76.9)(26990748.779) + (5.435)(15190777.427)} = 3.04 \times 10^{-6} \text{ Rad/mm} \]

\[
\theta_{total}(t) = 3.04 \times 10^{-6} \times 1000 = 3.04 \times 10^{-3} \text{ Rad} \]

The maximum shear stress acting on the edge of the concrete, \(\tau_\phi(t)\) and the inner boundary are the following:
\[ \tau_\theta = \frac{GJ\beta}{k_3} \quad (5.134) \]
\[ \Rightarrow \tau_c(t) = \frac{5.435 \times 26990748.779 \times 3.04 \times 10^{-6}}{359424} \times 1000 \]
\[ = 1.24 \text{MPa} \]
\[ \Rightarrow \tau_s(t) = \frac{76.9 \times 1519077.427 \times 3.04 \times 10^{-6}}{365268.29} \times 1000 \]
\[ = 0.97 \text{MPa} \]

At loading \((t - \tau_0)\) in days = 100, \(\phi(t - \tau_0) = 2.0\),

\[ G_c(t, \tau) = 3.623 \text{GPa}, \quad \beta(t) = 3.73 \times 10^{-6} \text{Rad/mm}, \quad \theta_{\text{Total}}(t) = 3.73 \times 10^{-3} \text{Rad}, \]
\[ \tau_c(t) = 1.01 \text{MPa}, \quad \tau_s(t) = 1.19 \text{MPa} \]

At loading \((t - \tau_0)\) in days = 10000, \(\phi(t - \tau_0) = 3.0\),

\[ G_c(t, \tau) = 2.717 \text{GPa}, \quad \beta(t) = 4.21 \times 10^{-6} \text{Rad/mm}, \quad \theta_{\text{Total}}(t) = 4.21 \times 10^{-3} \text{Rad}, \]
\[ \tau_c(t) = 0.86 \text{MPa}, \quad \tau_s(t) = 1.35 \text{MPa} \]

The concrete and steel shear stresses and angle of twist are now re-analyzed by AEMM approach.

At first loading \((t - \tau_0)\) in days = 0, \(\phi(t - \tau_0) = 0\), the results are the same as EMM approach.

\[(t - \tau_0) \quad \text{in days} = 0 \quad \phi(t - \tau_0) = 0 \quad \chi(t - \tau_0) = 1.0\]
\[(t - \tau_0) \quad \text{in days} = 25 \quad \phi(t - \tau_0) = 1 \quad \chi(t - \tau_0) = 0.86\]
\[(t - \tau_0) \quad \text{in days} = 100 \quad \phi(t - \tau_0) = 2 \quad \chi(t - \tau_0) = 0.8\]
\[(t - \tau_0) \quad \text{in days} = 10000 \quad \phi(t - \tau_0) = 3 \quad \chi(t - \tau_0) = 0.8\]

At first loading \((t - \tau_0)\) in days = 25, \(\phi(t - \tau_0) = 1.0\), \(\chi(t - \tau_0) = 0.86\),

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\[
\bar{G}_r(t, \tau) = \frac{G_r(\tau)}{[1 + \chi(t, \tau)\phi(t, \tau)]} = \frac{10.87}{[1 + (0.86)(1.0)]} = 5.844 \text{ GPa}
\]

The difference of shear stress acting on concrete:

\[
\Delta \tau_r(t) = \frac{\bar{G}_r(t)TG_r(t)I_{\Gamma_1} + (D_0)_{\Gamma_1} - [\bar{G}_r(t)I_{\Gamma_1} + (D_0)_{\Gamma_1}] + G_r[I_{\Gamma_2} - I_{\Gamma_1}] + (D_0)_{2\Gamma_2} - (D_0)_{2\Gamma_1}]\{\tau_r\}_3 (k_3) \gamma_r}{G_r(t)I_{\Gamma_1} + (D_0)_{\Gamma_1} + G_r[I_{\Gamma_2} - I_{\Gamma_1}] + (D_0)_{2\Gamma_2} - (D_0)_{2\Gamma_1}} \tag{5.140}
\]

\[
= \frac{5.844 \times 800 \times 5.435 \times 26990748.779 - [5.435 \times 26990748.779 + 76.9 \times 1519077.427] \times 0.00159 \times 359424}{5.435 \times 5.844 \times 26990748.779 + 76.9 \times 1519077.427}
\]

\[
= -0.36 \text{ MPa}
\]

Therefore,

\[
\beta(t) = \frac{G_r(t)\gamma + [\bar{G}_r(t) - G_r(t)]I_{\Gamma_1} (k_3) \gamma_r}{G_r(t)I_{\Gamma_1} + (D_0)_{\Gamma_1} + G_r[I_{\Gamma_2} - I_{\Gamma_1}] + (D_0)_{2\Gamma_2} - (D_0)_{2\Gamma_1}} \tag{5.141}
\]

\[
= \frac{5.435 \times 800 + [5.844 - 5.435] \times 0.00159 \times 359424}{5.435 \times 5.844 \times 26990748.779 + 76.9 \times 1519077.427}
\]

\[
= 3.07 \times 10^{-6} \text{ Rad/mm}
\]

\[
\theta_{\text{total}}(t) = 3.07 \times 10^{-6} \times 1000 = 3.07 \times 10^{-3} \text{ Rad}
\]

The maximum shear stress acting on the edge of the concrete, \(\tau_e(t)\) and the inner boundary are the following:

\[
\tau_r(t) = (\tau_r)_{\Gamma_1} + \Delta \tau_r(t) = 1.59 - 0.36 = 1.23 \text{ MPa}
\]

\[
\tau_e(t) = \frac{G_r[I_{\Gamma_2} - I_{\Gamma_1}] + (D_0)_{2\Gamma_2} - (D_0)_{2\Gamma_1}\beta(t)}{(k_3) \gamma_r} = \frac{76.9 \times 1519077.427 \times 3.07 \times 10^{-6}}{365268.29} = 0.98 \text{ MPa}
\]

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At loading \((t - \tau_0)\) in days = 100, \(\phi(t - \tau_0) = 2.0\), \(\chi(t - \tau_0) = 0.8\).

\[
G_e(t, \tau) = 3.623 \text{GPa}, \quad \bar{G}_e(t, \tau) = 4.181 \text{GPa}, \quad \beta(t) = 3.866 \times 10^{-6} \text{ Rad/mm}, \\
\theta_{\text{total}}(t) = 3.866 \times 10^{-3} \text{ Rad}, \quad \tau_v(t) = 0.97 \text{MPa}, \quad \tau_s(t) = 1.24 \text{MPa}
\]

At loading \((t - \tau_0)\) in days = 10000, \(\phi(t - \tau_0) = 3.0\), \(\chi(t - \tau_0) = 0.8\).

\[
G_e(t, \tau) = 2.717 \text{GPa}, \quad \bar{G}_e(t, \tau) = 3.197 \text{GPa}, \quad \beta(t) = 4.435 \times 10^{-6} \text{ Rad/mm}, \\
\theta_{\text{total}}(t) = 4.435 \times 10^{-3} \text{ Rad}, \quad \tau_v(t) = 0.78 \text{MPa}, \quad \tau_s(t) = 1.42 \text{MPa}
\]
5.4 Cracked Analysis by EMM Method

5.4.1 Short Term Analysis

The Rausch’s space truss analogy is mainly applied for analysing the reinforced concrete subjected to torsion after cracking. The cross section of the member has an arbitrary shape and is assumed to be hollow. There would be a series of 45° cracks appear in the concrete after cracking. The helical concrete members are assumed to interact with the longitudinal bars and stirrups to form a space truss as shown in Fig 5.13. The 45° helical concrete members are idealized into a series straight struts resisting compression force and producing outward radial force at each joint. The stirrups are also idealized as chains of short straight bars connected to the concrete to the concrete struts at the joints. The longitudinal bars are assumed to be a chain of short bars connecting at the joints to the diagonal concrete struts and the chains of stirrups in order to form a mechanism that will lengthen under an infinitesimal external torque.

![Diagram](image)

Fig 5.13

Thus, a space truss is formed that consists of 45° concrete struts in compression and longitudinal bars and stirrups in tension. There are four assumptions for forming a space truss:

1) The space truss is made up of 45° diagonal concrete struts, longitudinal bars and stirrups connected at the joints by hinges.

2) A diagonal concrete member carries only axial compression, i.e. shear resistance is neglected.

3) Longitudinal and lateral bars carry only axial tension, i.e. dowel resistance is neglected.
4) For a solid section, the concrete core does not contribute to the ultimate torsional resistance.

Referring to Chapter 3, the post-cracking torsional rigidity can be derived by original method and simplified method. Both methods assume a reinforced concrete tube with uniform wall thickness, \( h \). But the difference is in the expression of post-cracking torque-twist behaviour. The summaries for both methods are shown as follows:

**Summary of Torsional Rigidities**

Before cracking \((T \leq T_{cr})\):

\[
(AG_c + BG_t) = \frac{T}{\beta} \tag{5.144}
\]

where,

- \( G_c = \) shear modulus of concrete = \( E_c / 2(1 + \nu) \)
- \( G_t = \) shear modulus of steel

A and B are the torsional constants calculated by the alternatives from Method 1 to Method 3.

After cracking \((T \geq T_{cr})\):

\[
G_{cr}J_{cr} = \frac{T - \eta T_c}{\beta} \tag{5.145}
\]

where:

\[
G_{cr} = \frac{E_s}{\left(4n + \frac{uh_t}{A_s\rho_t} + \frac{uh_h}{A_s\rho_h}\right)} \tag{5.146}
\]

\[
J_{cr} = \frac{4A_t^2h_t}{u} \tag{5.147}
\]

and:
Time Dependent Effects of Reinforced Concrete Subjected to Torsional Loading

\[ h_c = 1.4 (\rho_l + \rho_h) x \]  \hspace{1cm} (5.148) \\
\[ \eta = 0.57 + 2.86 \frac{h}{x} \]  \hspace{1cm} (5.149)

According to Rausch's theory, the torque carried by concrete, \( T_c \), is expressed as follows equation:

\[ T_c = \frac{x^2 y}{3} \left( 2.4 \sqrt{f_c} \right) \]  \hspace{1cm} (5.150)

*for dimensions in pounds and inches.*

The cracking torque \( T_{cr} \) may be calculated by the skew-bending theory or by thin-tube theory:

Skew-bending theory (applicable to solid as well as hollow sections):

\[ T_{cr} = 6 (x^2 + 10) y \sqrt{f_c} \left( \frac{4h}{x} \right) \text{ where } h \leq \frac{x}{4} \left( \text{use } h = \frac{x}{4} \text{ when } h \geq \frac{x}{4} \right) \]

Thin-tube theory (applicable only to thin hollow sections where \( h \leq \frac{x}{4} \)):

\[ T_{cr} = 2x_1 y_1 h f_t \]  \text{ where } \ f_t \text{ may be taken as } 5 \sqrt{f_c}.

In the case of non-uniform wall thickness, the minimum wall thickness can be conservatively used for \( h \) in the above two equations.

**Summary of Simplified Method for Determination of Torsional Rigidity**

The post-cracking torsional rigidity can be wholly simplified using two assumptions:

1) Take \( \eta = 0 \) and the torque-twist curve passes through the origin.

\[ G_{cr} J_{cr} = \frac{T}{\beta} \]

2) The concrete struts do not contribute any portion to the torsional rigidity.
\[ G_{cr} = \frac{E_c}{h_c \left( \frac{u}{z} + \frac{s}{A_t} \right)} \]

So that,

\[ G_{cr} J_{cr} = \frac{4E_c A_t^2}{u \left( \frac{u}{z} + \frac{s}{A_t} \right)} \]

Therefore, the angle of twist of the reinforced concrete member after cracking can be determined by directly substituting the dimensions and material properties into the above equations. The section of numerical example can provide a comparison for the difference of using these two methods of determining angle of twist.

### 5.4.2 Time Dependent Analysis – EMM Approach for Cracked Analysis

The time-dependent deformation of the concrete increases the twist of the beam with time. The effect of creep can cause increase of deformation of the concrete struts by the creep coefficient, \( \phi \). The strains in the steel members are assumed to remain constant. Thus, the reinforced concrete member would behaviour as a series model that the longitudinal bars and stirrups act as springs while the concrete struts act as dashpot. The connection of them can be considered as in series condition.

The shear distortion of the cell is represented by the distortion angle \( \gamma_s \):

\[ \gamma_s = 2\varepsilon_c \]  \hspace{1cm} (5.151)
\[ \gamma_s (t) = 2\varepsilon_c (t) \]  \hspace{1cm} (5.152)

The distortion \( \gamma_l \) due to the lengthening of the longitudinal bars and the distortion \( \gamma_h \) due to the lengthening of the stirrups are expressed as follows:

\[ \gamma_l = \frac{\varepsilon_l s}{s} = \varepsilon_l \]  \hspace{1cm} (5.153)
\[ \gamma_h = \frac{\varepsilon_h s}{s} = \varepsilon_h \]  \hspace{1cm} (5.154)
The total shear distortion $\gamma$ and $\gamma(t)$ after time, $t$ are:

$$\gamma = \gamma_e + \gamma_t + \gamma_h = 2\varepsilon_e + \varepsilon_t + \varepsilon_h \quad (5.155)$$

$$\gamma(t) = \gamma_e(t) + \gamma_t + \gamma_h = 2\varepsilon_e(t) + \varepsilon_t + \varepsilon_h \quad (5.156)$$

Let us define $G_{cr} = \tau / \gamma$ = post-cracking shear modulus, where $n = E_s/E_c$.

$$G_{cr} = \frac{\tau}{\gamma} = \frac{E_s}{4n + \frac{uh_e}{A_c\rho_l} + \frac{uh_e}{A_c\rho_h}} \quad (5.146)$$

$$J_{cr} = \frac{4A_l^2h_e}{u} \quad (5.147)$$

$G_{cr}$ describes the material property of a torsional member after cracking. $J_{cr}$ is the polar moment of inertia of the rectangular reinforced member after cracking.

The EMM approach (Effective Modulus Method) can be applied for cracked analysis. If the instantaneous and creep components of strain are combined, a reduced or effective modulus for concrete $E_e(t, \tau)$ is the same as the uncracked analysis, which can be defined as follows:

$$\gamma(t, \tau) = \frac{\tau}{E_e(t, \tau)} \left[1 + \phi(t, \tau)\right]$$

$$= \frac{\tau}{E_e(t, \tau)}$$

where

$$E_e(t, \tau) = \frac{E_e(\tau)}{1 + \phi(t, \tau)}$$

Therefore, the post-cracking shear effective modulus can be expressed as follows:

$$G_{cr}(t) = \frac{\tau}{\gamma(t)} = \frac{E_s}{4n^* + \frac{uh_e}{A_c\rho_l} + \frac{uh_e}{A_c\rho_h}} \quad (5.157)$$

where

$$n^* = \frac{E_s}{E_e(t, \tau)}$$
A constant sustained torque, \( T \) is acting on the reinforced concrete member. The external torsional load, \( T \) is mainly resisted by internal torque of concrete and steel. The redistribution of internal torques due to the gradual development of creep strains is to be examined. The time-dependent stress and strain in the concrete is to be calculated.

The rate of twist, \( \beta_{\text{total}}(t) \) after time, \( t \) can be calculated in the following method. For the original method, the rate of twist for creep can be expressed as follows:

\[
G_{cr}(t)J_{cr} = \frac{T - \eta T_c}{\beta(t)} \Rightarrow \beta_{cr}(t) = \frac{T - \eta T_c}{G_{cr}(t)J_{cr}} \quad (5.158)
\]

Shrinkage strains are not dependent on load and shrinkage in a truss with 45° struts leads to a shear angle \( \gamma_{sh}(t) = -2 \varepsilon_{sh}(t, \tau) \). Therefore the rate of twist due to shrinkage can be stated as follows:

\[
\beta_{sh}(t) = \frac{h}{2A_r} \gamma_{sh}(t) = -\frac{h}{A_r} \varepsilon_{sh}(t, \tau) \quad (5.159)
\]

Therefore, the total rate of twist, \( \beta_{\text{total}}(t) \) and total angle of twist, \( \theta_{\text{total}}(t) \) after time, \( t \) can be simply expressed as the following equations:

\[
\beta_{\text{total}}(t) = \beta_{cr}(t) + \beta_{sh}(t) \quad (5.160)
\]

\[
\theta_{\text{total}}(t) = \beta_{\text{total}}(t)xL
\]

However, there is no contribution of elastic modulus due to concrete for the simplified torsional rigidity, it cannot model the creep analysis for simplified method.

Simultaneously, the post-cracking torsional case cannot apply gradual stress increment \( \Delta \tau(t) \) for doing AEMM analysis to find rate of twist of the reinforced member. It is because only concrete struts have shear deformations and the shear deformations of longitudinal bars and stirrups are kept constant, the stress increment \( \Delta \tau(t) \) will become zero for analysis. This would give the same results as the EMM approach.
5.4.2.1 Numerical Example

In order to illustrate the time-dependent effect of reinforced concrete member after cracking under a pure torque, a numerical example is presented. The angle of twist are to be determined at selected times for the pure torsion loaded. The external pure torque $T$ is 1500 Nm. Cross sectional, material properties and dimensions are as follows, which are the same as the uncracked examples:

\[
\begin{align*}
2a & = 120 \text{mm} \quad 2b = 120 \text{mm} \\
x_1 & = 80 \text{mm} \quad y_1 = 80 \text{mm} \\
d_{\nu} & = 6 \text{mm} \quad d_s = 6 \text{mm} \quad (4 \text{ longitudinal bars}) \\
E_c (\tau_o) & = 25 \text{GPa} \quad \nu_c = 0.15 \quad (\text{Poisson ratio for concrete}) \\
E_s & = 200 \text{GPa} \quad \nu_s = 0.3 \quad (\text{Poisson ratio for steel}) \\
L & = 1000 \text{mm}, \quad (\text{Length of member}) \quad h_e = h = 17 \text{mm} \\
\rho_l & = 0.0611 \quad \rho_h = 0.0153 \quad \eta = 1.8 \quad (\text{for solid section}) \\
n & = 13, (\text{No of stirrups}) \quad s = 80 \text{mm} \quad (\text{Spacing of stirrups}) \\
f' & = 30 \text{MPa} = 4258 \text{psi}
\end{align*}
\]

At first loading $(t - \tau_0)$ in days = 0, $\phi(t - \tau_0) = 0$,

\[
E_s(t, \tau) = E_c(\tau_o) = 25 \text{GPa}
\]

\[
G_s(\tau_o) = \frac{E_s(\tau_o)}{2(1 + \nu_s)} = 10.87 \text{GPa}
\]

\[
G_s = \frac{E_s}{2(1 + \nu_s)} = 76.9 \text{GPa}
\]

\[
n = n^* = \frac{E_s}{E_c} = \frac{200}{25} = 8
\]

The polar moment of the rectangular reinforced member after cracking,

\[
J_{cr} = \frac{4A_t^2 h_e}{u}
\]

\[
= \frac{4 \times (80 \times 80)^2 \times 17}{2 \times (80 + 80)}
\]

\[
= 8704000 \text{mm}^4
\]
$G_{cr}$, the material property of a torsional member after cracking, can be determined as follows:

$$G_{cr} = G_{cr}(t) = \frac{E_s}{\left(\frac{u h_c}{A, \rho_l} + \frac{u h_c}{A, \rho_h}\right) \left(4n + \frac{u h_c}{A, \rho_l} + \frac{u h_c}{A, \rho_h}\right)}$$

\[= \frac{200\times1000}{4\times8 + \frac{320\times17}{120\times120\times0.0611} + \frac{320\times17}{120\times120\times0.0153}}\]

\[= 3180MPa\]

The torque carried by concrete, $T_c$, is expressed as follows:

$$T_c = \frac{x^2 y}{3} (2.4\sqrt{f_c})$$

for dimensions in pounds and inches.

\[= \frac{(120/25.4)^2(120/25.4)(2.4\sqrt{4258})}{3}\]

\[= 5504lb - in = 622Nm\]

Hence, the rate of twist for creep can be determined as follows:

$$\beta_{cr}(t) = \frac{T - \eta T_c}{G_{cr} J_{cr}}$$

\[= \frac{1500 - 1.8\times622}{(3180/1000)\times870400}\]

\[= 1.374\times10^{-5} \text{ Rad} / \text{mm}\]

Shrinkage generates a rate of twist of zero at initial. Therefore, the total rate of twist, $\beta_{total}(t)$ is $1.374 \times 10^{-5} \text{ Rad} / \text{mm}$ and total angle of twist, $\theta_{total}(t)$ is $1.374 \times 10^2$ at initial.
At loading \( t - \tau_o \) in days = 25, \( \phi(t - \tau_o) = 1.0 \), \( \varepsilon_{sh}(t, \tau) = 200 \times 10^{-6} \)

\[
E_e(t, \tau) = \frac{E_e(\tau)}{[1 + \phi(t, \tau)]} = \frac{25}{1+1} = 12.5 \text{ GPa}
\]

\[
n^* = \frac{E_e}{E_e(t, \tau)} = \frac{200}{12.5} = 16
\]

\( G_{cr}(t) \), the material property of a torsional member after cracking at time \( t \), can be calculated as follows:

\[
G_{cr}(t) = \frac{E_e}{4n + \frac{uh_e}{A_e \rho_i} + \frac{uh_e}{A_e \rho_h}} \left( \frac{200 \times 1000}{4 \times 16 + \frac{320 \times 17}{120 \times 120 \times 0.0611} + \frac{320 \times 17}{120 \times 120 \times 0.0153}} \right)
\]

\[
= 2108 \text{ MPa}
\]

Hence, the rate of twist for creep can be determined:

\[
\beta_{cr}(t) = \frac{T - \eta T_e}{G_{cr}J_{cr}} = \frac{1500 - 1.8 \times 622}{(2108/1000) \times 870400} = 2.073 \times 10^{-5} \text{ Rad/mm}
\]

Shrinkage generates a rate of twist of:

\[
\beta_{sh}(t) = -\frac{u}{A_e} \varepsilon_{sh}(t, \tau) = -\frac{320}{120 \times 120} \times 200 \times 10^{-6} = 4.44 \times 10^{-6} \text{ Rad/mm}
\]
Thus, this results in a total rate of twist, $\beta_{\text{total}}(t)$ and total angle of twist, $\theta_{\text{total}}(t)$ are after 25 days:

$$\beta_{\text{total}}(t) = \beta_{cr}(t) + \beta_{sh}(t) \quad (5.160)$$

$$= 2.073 \times 10^{-5} + 4.44 \times 10^{-6}$$

$$= 2.52 \times 10^{-5} \text{ Rad/mm}$$

$$\theta_{\text{total}}(t) = \beta_{\text{total}}(t) x L$$

$$= 2.52 \times 10^{-5} \times 1000$$

$$= 2.52 \times 10^{-2} \text{ Rad}$$

At loading $(t - \tau_o)$ in days = 100, $\phi(t - \tau_o) = 2.0$, $\varepsilon_{\text{sh}}(t, \tau) = 400 \times 10^{-6}$,

$$E_c(t, \tau) = 8.33 \text{GPa}, \quad G_{cr}(t) = 1576 \text{MPa}, \quad n^* = 24,$$

$$\beta_{cr}(t) = 2.772 \times 10^{-5} \text{ Rad/mm,} \quad \beta_{sh}(t) = 8.89 \times 10^{-6} \text{ Rad/mm}$$

$$\beta_{\text{total}}(t) = 3.66 \times 10^{-5}, \quad \theta_{\text{total}}(t) = 3.66 \times 10^{-2} \text{ Rad}$$

At loading $(t - \tau_o)$ in days = 10000, $\phi(t - \tau_o) = 3.0$, $\varepsilon_{\text{sh}}(t, \tau) = 600 \times 10^{-6}$,

$$E_c(t, \tau) = 6.25 \text{GPa}, \quad G_{cr}(t) = 1259 \text{MPa}, \quad n^* = 32,$$

$$\beta_{cr}(t) = 3.471 \times 10^{-5} \text{ Rad/mm,} \quad \beta_{sh}(t) = 1.33 \times 10^{-5} \text{ Rad/mm}$$

$$\beta_{\text{total}}(t) = 4.80 \times 10^{-5}, \quad \theta_{\text{total}}(t) = 4.80 \times 10^{-2} \text{ Rad}$$
Chapter Six

Time Dependent Behaviour of Reinforced Concrete under Pure Torsional Loading – Experimental Study

6.1 Introduction

The analytical model and method of short-term and long-term analysis for uncracked and cracked reinforced concrete members has been discussed in previous section. An experimental investigation was conducted to consider the validity of the analytical model and analysis. In the research, tests on reinforced concrete subjected to pure torsion, were done at the concrete and structural laboratory. Fig 6.1 shows the conceptual diagram of the experiment.

Fig 6.1 Schematic Diagram of long-term tests for cracked and uncracked RC.

The instrumentation of each specimen includes the concrete and reinforcement strains and the deflection of the RC member. So that the experimental data can be compared with numerical result by different analytical methods.

In this experimental justification, the reinforced concrete columns were subjected to pure torque. Each column was 1m long, fixed ended, with 120mm square cross section containing stirrups and four reinforcing bars in two layers with diameter from 6mm to10mm, details are indicated in Fig 6.2.
Fig 6.2 Column cross section used in experiment.

In each column tested, strain gauges were attached to the reinforcement, stirrups and the surface of concrete. In order to obtain independent reliable results, LVDT were also placed on the surface of the concrete. These readings together with those electronics data provide both accurate and reliable results.
6.2 Methods of Data Analysis

As previous section mentioned, strain gauges were attached to the longitudinal bars, stirrups and the concrete surface. Hence, at these sections strains in both the reinforcing steel and the concrete were obtain. For each surface with strain gauges installed, one to three sets of strain data were collected. The following sections will generally discuss some of the methods of analysis that were utilized in this work.

The simplest type data analysis curve fitting, suppose that it has data set of N points \((x_i, y_i)\). The main propose is to fit this data to a function \(Y(x; \{a_j\})\) where \(\{a_j\}\) is a set of M adjustable parameters. The objective of this is to find the values of these parameters for which the function best fit the data. Intuitively, it is expected that if the curve fit is good, then a graph of data set \((x_i, y_i)\) and the function \(Y(x; \{a_j\})\) will show the curve passing near the points. Using the concept of this plane fitting, values of \(\gamma\), \(\beta\) and \(\tau\) can be determined from the experimental data.
6.3 Procedure for Experimental Tests

The reliable results depends on several factors, the quality of the specimen may be one of the most important one. The detail of the specimen preparation and setup procedure is discussed in the following sections.

6.3.1 Preparation of Specimens

Specification of test column

Dimensions of column: L x W x D = 1000mm x 120mm x 120mm (For long-term tests)
Dimensions of column: L x W x D = 600mm x 60mm x 60mm (For short-term tests)
Diameter of reinforcement: 6mm to 10mm
Water cement ratio: w: c = 1:0.7
Size of strain gauge: 5mm and 10mm

The 6mm diameter reinforcement was used for small size of specimens for short-term tests. For long-term tests, 6mm to 10mm diameter of reinforcement was used. 5mm strain gauges were attached on the surface of the steel bar at different levels. A small portion of surface of steel is removed so that a flat surface on which to attach each gauge.

Curing of the concrete is an important creep factor, which can be significant to the creep coefficient and aging coefficient. The columns were placed into the humidity and temperature control room for 28 days at temperature of 30°C and humidity of 100%. When the column was removed from the control room, strain gauges were attached to the concrete surface.

Finally, the strain gauges were connected to the data logger for recording the data automatically at each particular interval of time, programmable on a computer program system. Two loading caps were attached to supports of the top and bottom of each column before the experiment setup commences.

6.3.2 Procedure of Setups

For short-term tests, the specimens were placed to the automatically controlled machine for testing as shown in Fig 6.3. The machine is a strain-controlled machine, i.e.
the rate of twist increase for each specific interval and the applied torque increase or decrease as the rate of twist increase. Hence, the whole T-β curve can be obtained.

Fig 6.3 Setup for short-term torsional test.

For long-term tests for uncracked RC, the column was subjected to sustained torque, which is provided by dead weight connecting to the pulley attached at the top of the specimen. Fig 6.4 shows the setup for performing the creep tests of uncracked plain concrete and RC. Torque range is about 200 - 500 Nm.

Fig 6.4 Setup for performing creep tests of uncracked RC.
The torsional load cell is connected to the applying torque from the pulley to the steel plate connected to the specimen as shown in Fig 6.5. The LVDT (Fig 6.6) were then put at different levels of the surface of concrete specimen so that the displacement of the concrete movement can be measured. Hence, the rate of twist of the sample can be calculated.

Fig 6.5 Torsional load cell for measuring the applying torque.

Fig 6.6 LVDT for measuring the displacement of movement of concrete.

The data logger (Fig 6.7) was initialized such that all the readings on the strain gauges and LVDTs were zero. At the same time, time interval for taking the taking the data was set so that it can record the data at each time interval.
Fig 6.7 Data logger connecting strain gauges and LVDTs.

For long-term tests for cracked RC, the setup is similar to the one of uncracked RC, which is provided by dead weight connecting to larger pulley attached at the top of the specimen. Fig 6.8 shows the setup for performing the creep tests of cracked RC. Applying torque can be up to 2000 Nm.

Fig 6.8 Setup for performing creep tests of cracked RC.

6.3.3 Other Instrumentation

Apart from those instruments mentioned above, some supportive instruments are also needed in this test series and those included the determine the elastic modulus of the
materials and magnitude of the loading. The universal testing machine MTS, which has been used to measure the elastic modulus or strength of the materials.

The elastic modulus of concrete is very important for experimental analysis, as creep coefficient and aging coefficient are directly affected by these properties. The size of the test cylinders is 150mm diameter and 300mm in height. The elastic modulus is then obtained by the ELE machine, which is commonly used in experiments and sites.
6.4 Experimental Investigation Results and Comparison with Theoretical and Numerical Results

For the experiment, short-term tests for small specimens of plain concrete and reinforced concrete have been done for determining the strength and elastic behaviour under torsional loadings. Time dependent tests on plain concrete have been done for testing the validity of applying axial creep coefficient to predict rate of twist under sustained torsional loading. Also, time dependent tests on uncracked reinforced concrete have been performed in order to compare the experimental results to the three analytical methods for predicting the rate of twist under an uncracked sustained torsional load. In addition, time dependent tests on cracked reinforced concrete have been carried out so that the Rausch's space truss analogy together with EMM approach can be used for comparing the experimental results.

Table 6.1 shows the completion of different kinds of short-term and long-term tests.

<table>
<thead>
<tr>
<th>Series of Test</th>
<th>Short-term Tests</th>
<th>Long-term Tests</th>
</tr>
</thead>
<tbody>
<tr>
<td>Plain Concrete</td>
<td>1</td>
<td>3</td>
</tr>
<tr>
<td>Reinforced Concrete</td>
<td>8</td>
<td>3 (Uncracked)</td>
</tr>
<tr>
<td></td>
<td></td>
<td>3 (Cracked)</td>
</tr>
</tbody>
</table>

Table 6.1 Summary of number of tests for different conditions.

6.4.1 Failure Tests for Plain Concrete and Reinforced Concrete

One series of test has been done for testing the behaviour of plain concrete. Two series of tests have been carried out for reinforced concrete with longitudinal bars only. Six series of tests have been performed for reinforced concrete with stirrups and longitudinal bars. For each series of tests, three same specimens have been used in order to obtain more accurate and consistent results.

1) Plain Concrete

The followings are the results of plain concrete under short-term tests. From the graphs, we found that the plain concrete broke suddenly as the applying torque increase. The slope of the T-\( \beta \) curve from experimental results are close to the calculated results that the strain of concrete can show that it increase linearly as rate of twist increase.
Fig 6.9 T-β curve for plain concrete (Test 1)

Fig 6.10 $\varepsilon_r$ Vs β for plain concrete (Test 1)

Fig 6.11 T-β curve for plain concrete (Test 2)
Fig 6.12 \( \varepsilon_r \) Vs \( \beta \) for plain concrete (Test 2)

Fig 6.13 T-\( \beta \) curve for plain concrete (Test 3)

Fig 6.14 \( \varepsilon_r \) Vs \( \beta \) for plain concrete (Test 3)
2) Reinforced concrete with longitudinal bars only

The following shows the results the short-term tests results of reinforced concrete with longitudinal bars only. Two series of tests have been done. One contained longitudinal bars with distance of 30mm from center to center. Another one contained longitudinal bars with distance of 15mm from center to center. From the graphs, we discovered that the behaviour of RC with longitudinal bars only was similar to plain concrete. It still broke suddenly as the applying torque increase.

![Graph T vs β for reinforced concrete with longitudinal bars only](image1)

Fig 6.15 T-β curve for RC with longitudinal bars only (x1=y1=30mm, Test 1)

![Graph Max Shear strain ε vs Rate of twist β for reinforced concrete with longitudinal bars only](image2)

Fig 6.16 $\varepsilon_c$ vs β curve for RC with longitudinal bars only (x1=y1=30mm, Test 1)
Fig 6.17 $T - \beta$ curve for RC with longitudinal bars only ($x_1 = y_1 = 30\text{mm}$, Test 2)

Fig 6.18 $\varepsilon_c$ Vs $\beta$ curve for RC with longitudinal bars only ($x_1 = y_1 = 30\text{mm}$, Test 2)
Fig 6.19 $T$-$\beta$ curve for RC with longitudinal bars only ($x_1=y_1=30\text{mm}$, Test 3)

Fig 6.20 $\varepsilon$ Vs $\beta$ curve for RC with longitudinal bars only ($x_1=y_1=30\text{mm}$, Test 3)

Fig 6.21 $T$-$\beta$ curve for RC with longitudinal bars only ($x_1=y_1=15\text{mm}$, Test 1)
Fig 6.22 $\varepsilon_c$ Vs $\beta$ curve for RC with longitudinal bars only ($x_1=y_1=15\text{mm}$, Test 1)

Fig 6.23 $T$-$\beta$ curve for RC with longitudinal bars only ($x_1=y_1=15\text{mm}$, Test 2)

Fig 6.24 $\varepsilon_c$ Vs $\beta$ curve for RC with longitudinal bars only ($x_1=y_1=15\text{mm}$, Test 2)
3) Reinforced concrete with longitudinal bars and stirrups

The following shows the results the short-term tests results of reinforced concrete with longitudinal bars and stirrups. Six series of tests have been performed. The main difference of the different series of tests was the number of stirrups, hence the total amount of reinforcement. For reinforcement more than 2%, the T-\(\beta\) curve can show the existence of before and after cracking behaviour. The graphs of strain of concrete, longitudinal bars and stirrups after cracking are also included.
Fig 6.27 $T$-$\beta$ curve for RC ($\rho t=0.8\%$, Test 1)

Fig 6.28 $\epsilon_c$ Vs $\beta$ curve for RC ($\rho t=0.8\%$, Test 1)
Fig 6.29 \( \varepsilon_l \) and \( \varepsilon_h \) Vs \( \beta \) curve for RC (\( \rho_t=0.8\% \), Test 1)

Fig 6.30 T-\( \beta \) curve for RC (\( \rho_t=0.8\% \), Test 2)
Fig 6.31 $\varepsilon_r$ Vs $\beta$ curve for RC ($\rho_t=0.8\%$, Test 1)

Fig 6.32 $\varepsilon_l$ and $\varepsilon_h$ Vs $\beta$ curve for RC ($\rho_t=0.8\%$, Test 2)

Fig 6.33 T-$\beta$ curve for RC ($\rho_t=0.8\%$, Test 3)
Fig 6.34 $\varepsilon_c$ Vs $\beta$ curve for RC ($\rho_t=0.8\%$, Test 3)

Fig 6.35 $\varepsilon_l$ and $\varepsilon_h$ Vs $\beta$ curve for RC ($\rho_t=0.8\%$, Test 3)
Fig 6.36 $T$-$\beta$ curve for RC ($\rho_t=1.3\%$, Test 1)

Fig 6.37 $\varepsilon_r$ Vs $\beta$ curve for RC ($\rho_t=1.3\%$, Test 1)

Fig 6.38 $\varepsilon_l$ and $\varepsilon_h$ Vs $\beta$ curve for RC ($\rho_t=1.3\%$, Test 1)
Fig 6.39 $T-\beta$ curve for RC ($\rho_t=1.3\%$, Test 2)

Fig 6.40 $\varepsilon$ vs $\beta$ curve for RC ($\rho_t=1.3\%$, Test 2)
Fig 6.41 $\varepsilon_i$ and $\varepsilon_h$ Vs $\beta$ curve for RC ($\rho_t=1.3\%$, Test 2)

Fig 6.42 $T$-$\beta$ curve for RC ($\rho_t=1.3\%$, Test 3)

Fig 6.43 $\varepsilon_c$ Vs $\beta$ curve for RC ($\rho_t=1.3\%$, Test 3)
Max Shear strain $\varepsilon$ of longitudinal bars and stirrups after cracking for measured value Vs Rate of twist $\beta$ (RC with stirrups, n=3, $\rho_t=1.3\%$) Test 3

Fig 6.44 $\varepsilon_i$ and $\varepsilon_h$ Vs $\beta$ curve for RC ($\rho_t=1.3\%$, Test 3)

T Vs $\beta$ (RC with stirrups, n=4, $\rho_t=2\%$) Test 1

$\rho_t=2\%$, Total % of reinforcement (including equal volume longitudinal steel and stirrups)

Fig 6.45 T-$\beta$ curve for RC ($\rho_t=2\%$, Test 1)
Fig 6.46 $\varepsilon_c$ Vs $\beta$ curve for RC ($\rho_t=2\%$, Test 1)

Fig 6.47 $\varepsilon_l$ Vs $\beta$ curve for RC ($\rho_t=2\%$, Test 1)
Fig 6.48 $\varepsilon_h$ Vs $\beta$ curve for RC ($\rho_t=2\%$, Test 1)

Fig 6.49 $T$-$\beta$ curve for RC ($\rho_t=2\%$, Test 2)
Fig 6.50 $\varepsilon_c$ Vs $\beta$ curve for RC ($\rho_t=2\%$, Test 2)

Fig 6.51 $\varepsilon_l$ Vs $\beta$ curve for RC ($\rho_t=2\%$, Test 2)
Fig 6.52 $\varepsilon_h$ Vs $\beta$ curve for RC ($\rho_t=2\%$, Test 2)

Fig 6.53 T-$\beta$ curve for RC ($\rho_t=2\%$, Test 3)
Fig 6.54 $\varepsilon_c$ Vs $\beta$ curve for RC ($\rho_t=2\%$, Test 3)

Fig 6.55 $\varepsilon_l$ Vs $\beta$ curve for RC ($\rho_t=2\%$, Test 3)
Fig 6.56 $\varepsilon_h$ Vs $\beta$ curve for RC ($\rho_t=2\%$, Test 3)

Fig 6.57 $T-\beta$ curve for RC ($\rho_t=3\%$, Test 1)
Fig 6.58 $\epsilon_c$ Vs $\beta$ curve for RC ($\rho_t=3\%$, Test 1)

Fig 6.59 $\epsilon_l$ Vs $\beta$ curve for RC ($\rho_t=3\%$, Test 1)
Fig 6.60 $\varepsilon_h$ Vs $\beta$ curve for RC ($\rho t=3\%$, Test 1)

Fig 6.61 T-$\beta$ curve for RC ($\rho t=3\%$, Test 2)
Fig 6.62 $\varepsilon_c$ Vs $\beta$ curve for RC ($\rho t=3\%$, Test 2)

Fig 6.63 $\varepsilon_l$ Vs $\beta$ curve for RC ($\rho t=3\%$, Test 2)
Fig 6.64 $\varepsilon_h$ Vs $\beta$ curve for RC ($\rho t=3\%$, Test 2)

Fig 6.65 $T$-\(\beta\) curve for RC ($\rho t=3.5\%$, Test 1)
Fig 6.66 $\varepsilon_c$ Vs $\beta$ curve for RC ($\rho_t=3.5\%$, Test 1)

Fig 6.67 $\varepsilon_l$ Vs $\beta$ curve for RC ($\rho_t=3.5\%$, Test 1)
Fig 6.68 $\varepsilon_h$ Vs $\beta$ curve for RC ($\rho_t=3.5\%$, Test 1)

Fig 6.69 $T-\beta$ curve for RC ($\rho_t=3.5\%$, Test 2)
Fig 6.70 $\varepsilon_c$ Vs $\beta$ curve for RC ($\rho_t=3.5\%$, Test 2)

Fig 6.71 $\varepsilon_i$ Vs $\beta$ curve for RC ($\rho_t=3.5\%$, Test 2)
Fig 6.72 $\varepsilon_h$ Vs $\beta$ curve for RC ($\rho_t=3.5\%$, Test 2)

Fig 6.73 T-$\beta$ curve for RC ($\rho_t=3.5\%$, Test 3)
Fig 6.74 $\varepsilon_c$ Vs $\beta$ curve for RC ($\rho_t=3.5\%$, Test 3)

Fig 6.75 $\varepsilon_l$ Vs $\beta$ curve for RC ($\rho_t=3.5\%$, Test 3)
Fig 6.76 $\varepsilon_h$ Vs $\beta$ curve for RC ($\rho_t=3.5\%$, Test 3)

Fig 6.77 T-\(\beta\) curve for RC ($\rho_t=4\%$, Test 1)
Fig 6.78 $\varepsilon_c$ Vs $\beta$ curve for RC ($\rho t=4\%$, Test 1)

Fig 6.79 $\varepsilon_l$ Vs $\beta$ curve for RC ($\rho t=4\%$, Test 1)
Fig 6.80 $\varepsilon_h$ Vs $\beta$ curve for RC ($\rho_t=4\%$, Test 1)

Fig 6.81 $T$-$\beta$ curve for RC ($\rho_t=4\%$, Test 2)
Fig 6.82 $\varepsilon_c$ Vs $\beta$ curve for RC ($\rho_t=4\%$, Test 2)

Fig 6.83 $\varepsilon_l$ Vs $\beta$ curve for RC ($\rho_t=4\%$, Test 2)
Fig 6.84 $\varepsilon_h$ Vs $\beta$ curve for RC ($\rho_t=4\%$, Test 2)

Fig 6.85 T-$\beta$ curve for RC ($\rho_t=4\%$, Test 3)
Fig 6.86 $\epsilon_c$ Vs $\beta$ curve for RC ($\rho t=4\%$, Test 3)

Fig 6.87 $\epsilon_l$ Vs $\beta$ curve for RC ($\rho t=4\%$, Test 3)
Fig 6.88 $\varepsilon_h$ Vs $\beta$ curve for RC ($\rho t=4\%$, Test 3)
6.4.2 Time Effect on Plain Concrete

For the time dependent experiment of plain concrete under torsional loading, three sets of experiments have been conducted. The specimens of each test were the same but with different sustained applying torque. The applying torques were 190 Nm, 275 Nm and 360 Nm. Each test has been done for about 60 to 70 days. The torsional creep coefficient Vs time of each tests were plotted in order to be compared with the axial compression creep coefficient by different predictions such ACI method, power function, exponential function and hyperbolic function.

1) Applying torque = 190 Nm.

The following is the result of plain concrete under long-term test of T = 190 Nm. From the graph, we can find that the creep coefficient, \( \phi \) increase as the time increases. The increase of \( \phi \) was quite rapid from 0 to 30 days. The increase of \( \phi \) settled after 30 days. The performance of the test is not too stable as the curve is not smooth enough.

![Graph showing creep coefficient Vs time for T = 190 Nm](image)

Fig 6.89 Creep coefficient of plain concrete under T = 190 Nm.

2) Applying torque = 275 Nm.

The following is the result of plain concrete under long-term test of T = 275 Nm. The graph of \( \phi \) Vs t shows that the creep coefficient, \( \phi \) still increase as the time increases. The increase of \( \phi \) was quite rapid from 0 to about 25 days, which is similar to the previous case. The increase of \( \phi \) settled after 30 days. The curve of the experimental result is smoother this time.
Fig 6.90 Creep coefficient of plain concrete under $T = 275$ Nm.

3) Applying torque = 360 Nm.

The following is the result of plain concrete under long-term test of $T = 360$ Nm. The graph of $\phi$ Vs $t$ shows that the creep coefficient, $\phi$ still increase as the time increases. The increase of $\phi$ was similar to previous cases, which increased rapidly from 0 to about 25 days. The increase of $\phi$ settled after 30 days. The curve of the experimental result is smoother in earlier of time.

Fig 6.91 Creep coefficient of plain concrete under $T = 360$ Nm.
6.4.3 Time Effect on Uncracked Reinforced Concrete

There were three sets of experiments have been performed for the time dependent experiment of uncracked RC under torsional loading. The size of the specimens were of each test were the same but with different sustained applying torque and total steel amount. The applying torques were 200 Nm, 345 Nm and 400 Nm. Each test has also been loaded for about 60 to 70 days. The rate of twist, shear strains of concrete acting at section with and without stirrups and shear strains of longitudinal bars and stirrups of each tests were plotted in order to be compared with the EMM and AEMM approaches given by the three analytical methods for uncracked RC under torsional loads.

1) Applying torque = 200 Nm, diameter of longitudinal bars and stirrups, 
\( ds = dsv = 8 \), no of stirrups, \( n = 4 \).

The following is the result of RC under long-term test of \( T = 200 \) Nm, 
\( ds = dsv = 8 \), and \( n = 4 \). From the graph of \( \beta \) Vs t, we can find that the rate of twist, \( \beta \) increase as the time increases. The rate of twist \( \beta \) increased rapidly from 0 to 20 days. The increase of \( \beta \) settled after 20 days. The shear strains of concrete decrease sharply from 0 to 15 days and become nearly constant afterward while the shear strains of the reinforcement had reverse behaviour for initial change, i.e. increase initially and become stable afterward.

![Comparison between Lab result (n=4) and 3 Theo Method by EMM and AEMM of \( \beta \) Vs Time](image)

Fig 6.92 \( \beta \) Vs t of RC under \( T = 190 \) Nm.
Comparison of lab Result and Theo Method for Shear Stress to the section Concrete without stirrups Vs time

Fig 6.93 $\gamma_c$ Vs t of RC for the part without stirrups under $T = 190$ Nm.

Comparison of lab Result and Theo Method for Shear Stress to the section Concrete with stirrups Vs time

Fig 6.94 $\gamma_c$ Vs t of RC for the part with stirrups under $T = 190$ Nm.
Fig 6.95 $\gamma_1$ Vs t of RC under $T = 190$ Nm.

Fig 6.96 $\gamma_2$ Vs t of RC under $T = 190$ Nm.
2) Applying torque = 400 Nm, diameter of longitudinal bars and stirrups, 
\( ds = dsv = 6 \), no of stirrups, \( n = 13 \).

The following is the result of RC under long-term test of \( T = 400 \) Nm, 
\( ds = dsv = 6 \), and \( n = 13 \). From the graph of \( \beta \) Vs t, it can be seen that the rate of twist, 
\( \beta \) still increase as the time increases. The rate of twist \( \beta \) increased rapidly from 0 to 20 
days. The increase of \( \beta \) reduced after 20 days. The shear strains of concrete decrease 
sharply from 0 to 10 days and become stable after 10 days while the shear strains of 
the reinforcement had reverse behaviour for initial change, where the increase were 
similar to the increase of rate of twist.

![Comparison between Lab result (n=13) and 3 Theo Method by EMM and AEMM of \( \beta \) Vs Time](image)

Fig 6.97 \( \beta \) Vs t of RC under \( T = 400 \) Nm.
Fig 6.98 $\gamma_c$ Vs t of RC for the part without stirrups under $T = 400$ Nm.

Fig 6.99 $\gamma_c$ Vs t of RC for the part with stirrups under $T = 400$ Nm.
Comparison of lab Result and Theo Method for Max Shear Stress of longitudinal bars Vs time

Fig 6.100 $\gamma_i$ Vs t of RC under $T = 400$ Nm.

Comparison of lab Result and Theo Method for Max Shear Stress of stirrups Vs time

Fig 6.101 $\gamma_h$ Vs t of RC under $T = 400$ Nm.
3) Applying torque = 345 Nm, diameter of longitudinal bars and stirrups, 
   \( ds = 10, dsv = 4 \), no of stirrups, \( n = 25 \).

   The following is the result of RC under long-term test of \( T = 345 \) Nm, 
   \( ds = 10, dsv = 4 \), and \( n = 25 \). From the graph of \( \beta \) Vs \( t \), it can be discovered that the 
   rate of twist, \( \beta \) still increase as the time increases, which is similar to previous cases. 
   The rate of twist \( \beta \) still increased rapidly from 0 to 20 days. The increase of \( \beta \) become 
   lower after 20 days. The shear strains of concrete decrease sharply from 0 to 10 days 
   and become stable after 10 days while the shear strains of the reinforcement had 
   reverse behaviour for initial change, but the increase were quite different from the 
   increase of rate of twist.

![Comparison between Lab result (n=25) and 3 Theo
Method by EMM and AEMM of \( \beta \) Vs Time](image_url)

Fig 6.102 \( \beta \) Vs \( t \) of RC under \( T = 345 \) Nm.
Fig 6.103 $\gamma_c$ Vs t of RC for the part without stirrups under $T = 345$ Nm.

Fig 6.104 $\gamma_c$ Vs t of RC for the part with stirrups under $T = 345$ Nm.
Comparison of lab Result and Theo Method for Max Shear Stress of longitudinal bars Vs time

Fig 6.105 $\gamma_i$ Vs t of RC under $T = 345$ Nm.

Comparison of lab Result and Theo Method for Max Shear Stress of stirrups Vs time

Fig 6.106 $\gamma_h$ Vs t of RC under $T = 345$ Nm.
6.4.4 Time Effect on Cracked Reinforced Concrete

There were also three sets of experiments have been carried out for the time dependent experiment of cracked RC under torsional loading. The size of the specimens were of each test were the same but with different sustained applying torque and total steel amount. Two of the specimens were loaded until crack occurred and kept the sustained constant torque for a period of time. The other one of specimen was loaded until it cracked and reloaded it with a smaller sustained torque, which was an uncracked one. The applying sustained torques was 1300 Nm, 1400 Nm and 360 Nm. Each test has also been loaded for about 60 to 70 days. The rate of twist, strain of concrete struts acting at section without stirrups and strains of longitudinal bars and stirrups of each tests were plotted in order to be compared with the EMM approach given by the Rausch’s space truss analogy.

1) Applying torque = 1300 Nm, diameter of longitudinal bars and stirrups, \( ds = dv = 8 \), no of stirrups, \( n = 13 \).

The following is the result of RC under long-term test of \( T = 1300 \) Nm, \( ds = dv = 8 \), and \( n = 13 \). From the graph of \( \beta \) Vs t, we can find that the rate of twist, \( \beta \) increase as the time increases. The rate of twist \( \beta \) increased more from 0 to 20 days. The increase of \( \beta \) becomes smaller after 20 days. The strain of concrete increase with the similar rate as the rate of twist while the strains of the reinforcement nearly kept constant.

![Comparison between Lab result cracked RC (ds=8, T=1300Nm) and Theo Method by EMM of \( \beta \) Vs Time](image)

Fig 6.107 \( \beta \) Vs t of RC under \( T = 1300 \) Nm.
Fig 6.108 $\varepsilon_c$ Vs t of RC under $T = 1300$ Nm.

Fig 6.109 $\varepsilon_l$ Vs t of RC under $T = 1300$ Nm.
Comparison between Lab result cracked RC (ds=8, T=1300Nm) and Theo Method of $\varepsilon$ h Vs Time

Fig 6.110 $\varepsilon_s$ Vs t of RC under T = 1300 Nm.
2) Applying torque = 1400 Nm, diameter of longitudinal bars and stirrups, 
\( ds = dsv = 10 \), no of stirrups, \( n = 13 \).

The following is the result of RC under long-term test of \( T = 1400 \) Nm, 
\( ds = dsv = 10 \), and \( n = 13 \). From the graph of \( \beta \) Vs \( t \), we can notice that the rate of 
twist, \( \beta \) increase as the time increases, which is the same as before. The rate of twist \( \beta \) 
increased promptly from 0 to 30 days. The increase of \( \beta \) becomes slighter after 30 
days. The strain of concrete increase with the similar amount as the rate of twist. The 
strains of the reinforcement can fix with a constant value but fluctuated when after 55 
days.

Fig 6.111 \( \beta \) Vs \( t \) of RC under \( T = 1400 \) Nm.
Fig 6.112 $\varepsilon_c$ Vs t of RC under $T = 1400$ Nm.

Fig 6.113 $\varepsilon_l$ Vs t of RC under $T = 1400$ Nm.
Comparison between Lab result cracked RC (ds=10, T=1400Nm) and Theo Method of $\varepsilon_h$ Vs Time

Fig 6.114 $\varepsilon_h$ Vs t of RC under T = 1400 Nm.
3) Applying torque = 360 Nm, diameter of longitudinal bars and stirrups, $d_s = d_{sv} = 10$, no of stirrups, $n = 13$.

The following is the result of RC under long-term test of $T = 360$ Nm, $d_s = d_{sv} = 10$, and $n = 13$, where the RC member was loaded until it cracked and applied a smaller torque to it. The $\beta$ Vs t graph can be observed that the rate of twist, $\beta$ still go up as the time increase, which is like before. However, the rate of twist $\beta$ increased slightly than previous cases. The strain of concrete increase with the similar amount as the rate of twist but fluctuated a lot initially. The strains of the reinforcement can fix with a constant value but the there is difference of strain for longitudinal bar at initial time. The strain of stirrups had great fluctuation in the process, in particular day 40 to 60.

![Comparison between Lab result cracked RC, with uncracked torque (ds=10, T=360Nm) and Theo Method by EMM of $\beta$ Vs Time](image)

Fig 6.115 $\beta$ Vs t of RC under $T = 360$ Nm.
Comparison between Lab result cracked RC, with uncracked torque (ds=10, T=360Nm) and Theo Method by EMM of $\varepsilon_c$ Vs Time

Fig 6.116 $\varepsilon_c$ Vs t of RC under T = 360 Nm.

Comparison between Lab result cracked RC, with uncracked torque (ds=10, T=360Nm) and Theo Method by EMM of $\varepsilon_l$ Vs Time

Fig 6.117 $\varepsilon_l$ Vs t of RC under T = 360 Nm.
Comparison between Lab result cracked RC, with uncracked torque (ds=10, T=360Nm) and Theo Method by EMM of $\varepsilon_h$ Vs Time

Fig 6.118 $\varepsilon_h$ Vs t of RC under T = 360 Nm.
6.5 Discussion

1) Failure Tests for Plain Concrete and Reinforced Concrete

The purpose of this test was to investigate the determining the strength and elastic behaviour of small specimens of plain concrete and RC. By the result, it can be found that the samples of plain concrete collapsed suddenly as the applying torque increase. In general, the T-β curve can show that the experimental results of the elastic part of plain concrete were quite close to the theoretical prediction. The shear strain of experimental results gave a little bit higher than the theoretical prediction. This may be due to the imperfect connection of strain gauge on the concrete surface, since the surface of the concrete was not smooth enough, which caused the error from the reading of apparatus. For the short-term tests of RC with longitudinal bars only, the experimental results of the elastic part of the specimens were still near to the theoretical values, but the torsional strength of the RC with longitudinal bars only had a small amount of increase. For the tests of longitudinal bars farer away from the centre of the concrete section, the torsional resistance was greater than the one with longitudinal bars closer to the centre of the concrete section. This is because the torsional resistance can be increased with longer level arm for internal torque by putting the longitudinal bars more separate from each other. The torsional strength tests for RC with longitudinal bars and stirrups can mainly be divided into two types of behaviour. One is that the torsional strength with similar behaviour as plain concrete, i.e. it broke suddenly and did not has too much ductility.

Another is the RC has post-cracking rigidity, which means the RC has before cracking polar moment of inertia and after cracking polar moment of inertia. The RC member would not collapse suddenly and it can have further development of torsional strength after cracking until reaching ultimate torsional strength. Usually, the total steel ratio of RC samples higher than 2% can exist post-cracking torsional behaviour for resistance the applying torque. The shear strain of longitudinal bars and stirrups of the experimental results can show that it still has the linear increase after cracking, which quite match the theoretical values. However, due to the limitation of the applying torque of the machine, the complete T-β curve cannot be plotted for higher steel ratio of RC members. It can still show part of the post-cracking behaviour of RC from the T-β curve.
2) Time Effect on Plain Concrete

The typical results of plain concrete under creep in torsion are shown from Fig 6.89 to Fig 6.91. The creep coefficient-time relation was found to be similar in shape to that in compression. From the three experimental results, the creep coefficient increases as time increase with the shape more similar to the prediction of ACI of creep coefficient under compression. The result given by the smallest applying torque has very little increase or nearly no increase. This may be due to the difficulty of measuring the displacement from the low applying. In fact, the effect of measurement error existed in the three tests of plain concrete under torsional creep. Indeed, the smallest displacement would have greatest influence caused by the error of measurement. In addition, the drying before loading also has a similar to that effect that caused by the error of measurement. Definitely, the experiments were in the conditions of linear creep, i.e. the stress level caused by the applying torque was smaller than 50% of yield stress of concrete. Comparison between the relation of non-elastic deformations in torsion and in compression did not include in this research. Since the analysis of uncracked RC under creep in torsion were based on linear creep analysis of plain concrete. The subsequent discussion would show the validity of applying axial compression creep coefficient for determining the effective shear modulus of concrete by using the experimental results and FEM analysis.

3) Time Effect on Reinforced Concrete under Uncracked Loads

Among all of the tests, it can be discovered that the rate of twist, $\beta$ increased with time and the rapid increase of $\beta$ started from day 0 to day 10 or day 15. This period of rapid increase of $\beta$ reduced when it was compared with the period of rapid increase of $\beta$. This is because the existence of reinforcement, which took part the stress, i.e. stress redistribution took place, which can reduce stress acting on the concrete so that the creep of the concrete decrease and the increase of rate of twist, $\beta$ can be condensed. In addition, the shape of the $\beta$-t curves was fairly close to the prediction given by analytical method 2, both EMM and AEMM approaches. The analytical method 1 tends to provide overestimate results while the analytical method 2 gives underestimate results. Actually, the finite element analysis also provides evidence that the analytical method 2 had the similar result. In fact, the specimens of the experiments were square sections, as section 5.2.4 discuss that Method 1 cannot
take account the effects of position of longitudinal bars and the square sections more overestimate results and as the sections become rectangular section, the prediction of rate of twist become underestimate. Method 3 always gives underestimate values, which is due to the expression of mapping function is provided by expansion of mathematical series for the integral. The error can be caused by the fluctuation of the curve of the expanded function.

Usually, the experimental results of the strains of longitudinal bars and stirrups had variation to the calculated values given by the analytical methods and with great fluctuation. This is because the strain calculated by the rate of twist from the experimental results was so small that the fluctuation of the rate of twist would be amplified after obtaining the strain from the $\beta$. Generally, the specimens subjected to small applying torque would have give smaller value of rate of twist and it turn to stable value for time after a month. By this research, the samples had mainly been tested for 60 to 70 days and the sustained applying torques were checked by using a torsional load cell. However, the there was trendily of slightly increase for the rate of twist even after two months of time for normally reinforced concrete, i.e. the second case of the experiment ($d_s = d_{sv} = 6, n = 13$). This increase of rate of twist can be settled by adding more reinforcement to the sample, i.e. the third case of the experiment ($d_s = 10, d_{sv} = 4, n = 25$).

4) Time Effect on Reinforced Concrete under Cracked Loads

The control of this test was relatively difficult, as the cracks were not perfectly distributed as the idealized space truss model. However, the whole RC can behave well, especially for the first two tests, without reloading the samples. The prediction of rate of twist can match the experimental results. For the first test ($d_s = d_{sv} = 8, n = 13$), the initial rate of twist was little higher than the prediction of the EMM method and the final value of experimental result can reach the analytical prediction. Maybe due to the inaccurate location of the sample, i.e. the center of the pulley cannot completely overlap the center of sample, which leaded to error of measurement. Better performance was presented for the second test ($d_s = d_{sv} = 10, n = 13$), since the test was started after 2 months of completion of the first test. The management was highly developed after finishing the first test. In case of the third test, the sample was reloaded to a small torque, which cannot cause
cracking for the RC sample and the RC has been cracked already. The torsional
rigidity of the RC was assumed with the post-cracking behaviour and polar moment
of inertia is $G_{cr} J_{cr}$. This may be a factor for causing the overestimation for the rate of
twist. Since the crack of the specimen may not be propagated well after applying the
cracking torque to the specimen as the specimen was subjected to a torque just higher
than the theoretical cracking torque. This can lead to the space truss cannot be
smoothly performed in the test.

The experimental concrete strains were normally close to the theoretical value,
in particular the second test ($d_s = d_n = 10, n = 13$). The fluctuation of the concrete
strain of the first test ($d_s = d_n = 8, n = 13$) was quite stable, i.e. the shape of the $\varepsilon - t$
curve can still match to the prediction of the EMM approach. For the third test, the
concrete strain had great difference in the period of 10 to 25 days. This may be caused
by the surrounding disturbance such as humidity and temperature effect. This effect
can be also disturbed the experimental results for strains of longitudinal bars and
stirrups. The strains of the longitudinal bars and stirrups were abnormal in the period
of day 0 to day 10. Maybe the reinforcement needed time to response for stress
redistribution from the stress released from the concrete struts. In fact, the imperfect
quality of the reinforcement can be a factor causing the error of the reading for strains.
The unsatisfactory connection of electrical wires and attachment to the surface on the
reinforcement for strain gauges on the reinforcement was also sources of errors.
Chapter Seven

Conclusions

There are many reinforced concrete structures that are subjected to resist torsional load. Creep and shrinkage of concrete are common problems that can increase large deformation and redistribution of stresses. The existence of both torsion and creep can magnify the destructive effects on the structures.

Based on the short-term and long-term tests, some relationships have been determined which can reflect the time dependent behaviour plain concrete and reinforced concrete members; both uncracked and cracked specimens under the torsional loading has been investigated. It is believed that much more information has been established regarding the mechanisms of how plain concrete creeps under pure torsion, how torsional creep affects the uncracked reinforced concrete members and the influence of torsional creep on the cracked reinforced concrete.

For the short-term test of plain concrete and reinforced concrete, the relation of torsion to rate of twist has been determined. The torsional rigidity for plain concrete of the experimental results showed that it complies with the prediction of homogenous material behaviour. The failure behaviour of plain concrete is brittle and it breaks suddenly when the applying torque reaches the ultimate strength. The strength of the reinforced concrete with longitudinal bars can only increase a small amount. The experiment has shown that it also collapsed suddenly. The experiments for reinforced concrete specimens with both longitudinal bars and stirrups have proved that the post-cracking behaviour can be developed. It can match the prediction given by the theoretical equations. The total steel amount larger than 2% of the whole volume for reinforced concrete members can present the post-cracking behaviour for resisting torsional loading. This behaviour would also provide ductility for the reinforced concrete member so that it would not has sudden collapse. The space truss model is applicable to determine for the torsional rigidity and rate of twist of reinforced concrete after cracking. Reinforced concrete specimens with higher than 2% of total steel ratio can give reliable results to verify the space truss model.

For long-term tests, it is required to determine the torsional creep effect on both the uncracked and cracked reinforced concrete. The rate of twist, concrete strains and strains of longitudinal bars and stirrups have been tested in order to validate the analytical methods. The analytical method 1 (By composite materials properties) that
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predicts the rate of twist and strains of strains of reinforcement always provides overestimate values, which is greater than experimental results. Conversely, the analytical method 3 (By conformal mapping) often gives underestimate results. Generally, the analytical method 2 (Modified analytical method) can proper results, which is close to the experimental results. The drawback of Method 1 is that it cannot take account for the effect of the position of longitudinal bars for determining the internal torques. The shortcoming of Method 3 is that the determination of the mapping function would cause error, which is due to the fluctuation of expression of mapping function by binominal expansion from the conformal integral. In general, the fluctuation of strains measured from the reinforcements was quite serious, especially for the initial stage. This may be due to the improper connection of electrical wires to the strain gauges. In fact, the error of strain gauges for concrete was not too severe and the results were relatively acceptable.

The experimental results for cracked reinforced concrete under torsional creep were satisfactory. In particular, the reinforced concrete specimens loaded until they cracked and kept the sustained torques for a period time. However, the cracked reinforced concrete specimen loaded until it cracked and reloaded to the uncracked load for long-term test could not meet the prediction well. This could be because the torsional rigidity of this test was assumed equal to the post-cracking torsional rigidity but the specimen was not cracked thoroughly. The sample may have some uncracked behaviour, which caused the error for the results.
Appendix A Complex Variable Method and Conformal Mapping

The complex variable method can be an alternative means for solving the homogenous elastic torsional problems. Instead of solving the Laplace's equation,

$$\nabla^2 \Psi = 0$$

And satisfying the boundary conditions, in terms of warping function $\psi$,

$$\bar{X} = \sigma_x l + \tau_{xy} m + \tau_{xz} n = 0$$
$$\bar{Y} = \sigma_y m + \tau_{xy} n + \tau_{yz} l = 0$$
$$\bar{Z} = \sigma_x n + \tau_{xy} l + \tau_{xz} m = 0 = \tau_{xz} l + \tau_{yz} m$$

One may set the problem in terms of a complex analytic function $F(z)$ defined by

$$F(z) = \Psi + i\varphi$$

Where $\Psi$ and $\varphi$ are complex conjugates which satisfy the Cauchy-Riemann conditions namely,

$$\frac{\partial \Psi}{\partial x} = -\frac{\partial \varphi}{\partial y} \quad \text{and} \quad \frac{\partial \varphi}{\partial y} = \frac{\partial \Psi}{\partial x}$$

This can enable one to express the equations of boundary in terms of $\varphi$ and the complex conjugate of $\Psi$. By using Laplace's equation and equations of boundary, one finds

$$\frac{d\varphi}{dy} = \frac{d}{ds} \left( \frac{y^2 + x^2}{2} \right)$$

By integrating of the above equation, then the boundary conditions for any multiply connected cross section would be in terms of the complex conjugate as

$$\varphi = \frac{1}{2} (x^2 + y^2) + C_k \quad \text{on} \quad \Gamma_k$$

or

$$\varphi = \frac{1}{2} (r^2) + C_k = \frac{1}{2} \bar{z} \bar{z} + C_k \quad \text{on} \quad \Gamma_k \quad (A.1)$$

If one finds an analytic function $F(z)$ whose imaginary part satisfies equations on the boundaries, the torsion problem can be solved.

In general, the conjugate function $\varphi$ of the single-valued function $\Psi$ can be multi-valued, in the case of the torsion problem the condition that $\varphi$ must return to its original
value on the boundaries requires $\varphi$ to be single valued. The constants $C_k$ in equation (A.1) are determined to ensure the torsion function $\varphi$ can be single value.

However, a number of boundary curves can occur in multiply connected regions. The constant for only one of these boundaries can be selected arbitrarily.

Therefore, the torsional rigidity, $GJ$ or $D$ is defined. It can be expressed in terms of $\Psi$ as follows:

$$GJ = D = G \iint_{S} \left[ x^2 + y^2 + x \frac{\partial \Psi}{\partial y} - y \frac{\partial \Psi}{\partial x} \right] dxdy = Gl + GD_0$$

Where $I$ is the polar moment of inertia of the cross section about the origin. The shear stresses can be calculated and the external torque from the following equation.

$$T = GJ\beta = D\beta$$

By using the complex variable theory and the conformal mapping technique, the solution of the torsion problem for singly connected prism can be determined.

If the cross section of the prism is mapped onto the interior of a unit circle, $|\zeta| < 1$, the boundary conditions for a homogenous prism, can be expressed in terms of one variable. When mapped onto the $\zeta$-plane, the complex torsion function $F(z)$ in the $z$-plane becomes a function of $\zeta$, i.e.

$$F(z) = \Psi + i\varphi = F[\omega(\zeta)] = f(\zeta)$$

When mapped onto the $\zeta$-plane, the boundary conditions become

$$\varphi = \frac{1}{2} (x^2 + y^2) = \frac{1}{2} \bar{z}\bar{z} = \frac{1}{2} \omega(\zeta)\overline{\omega(\zeta)} \quad \text{on} \quad \gamma \quad (A.2)$$

Where $z = \omega(\zeta)$ is the mapping function.

Where $\gamma$ denotes the boundary of the unit circle. Since

$$\overline{f(\zeta)} = \Psi - i\varphi$$

It follows that
\[ f(\zeta) - \overline{f(\zeta)} = 2i\varphi \]

Hence,

\[ f(\zeta) - \overline{f(\zeta)} = i\omega(\zeta)\overline{\omega(\zeta)} \quad \text{on} \quad \gamma \]

If the mapping function \( z = (\zeta) \) is known for a singly connected cross section, then on the boundary of the unit circle, \( \omega(\zeta) \) becomes a function of \( \theta \) only. Thus,

\[ \omega(\zeta) = \omega(\sigma) \quad \text{on} \quad \gamma \]

Where \( \sigma = e^{i\theta} \), the expansion of \( \omega(\sigma) \) in a power series in \( \sigma \) may be expressed as

\[ \omega(\sigma) = \sum_{-\infty}^{\infty} A_n \sigma^n \]

Since \( f(\zeta) \) and \( \overline{f(\zeta)} \) are rational in the interior of the unit circle, one can spread out them in power series in \( \sigma \) in the following way:

\[ f(\sigma) = \sum_{n=0}^{\infty} B_n \sigma^n \quad ; \quad \overline{f(\sigma)} = \sum_{n=0}^{\infty} \overline{B_n} \sigma^n \]

Substitution of the above equations,

\[ \sum_{n=0}^{\infty} B_n \sigma^n - \sum_{n=0}^{\infty} \overline{B_n} \sigma^n = i \left[ \sum_{n=0}^{\infty} A_n \sigma^n \left( \sum_{n=0}^{\infty} \overline{A_n} \sigma^n \right) \right] \]

By comparing the coefficients on both sides of the \( \sigma^n \) terms of the equation, the coefficients \( B_n \) can be expressed. \( f(\sigma) \) is known and warping function \( \Psi \) is simply the real part of \( f(\zeta) \).

The torsional rigidity \( D \) or \( GJ \) of equation can be expressed in terms of \( f(\sigma) \) and \( \omega(\sigma) \) as follows:
\[ D = GJ = GI + GD_0 \]

where,

\[ D_0 = \oint_R \left[ \frac{\partial(x\Psi)}{\partial y} - \frac{\partial(y\Psi)}{\partial x} \right] dxdy \]

Green's formula allows \( D_0 \) as a line integral around the boundary of the unit circle \( \gamma \). Thus,

\[ D_0 = -\oint_r \Psi \cdot (x\, dx + y\, dy) = -\oint_r \Psi \cdot d\left( \frac{1}{2} r^2 \right) \]

Noting that

\[ r^2 = z\bar{z} = \omega(\sigma)\overline{\omega(\sigma)} \quad \text{on} \quad \gamma \]

And

\[ \Psi = \frac{1}{2} \left[ f(\sigma) + \overline{f(\sigma)} \right] \quad \text{on} \quad \gamma \]

So that

\[ D_0 = -\frac{1}{4} \oint_r \left[ f(\sigma) + \overline{f(\sigma)} \right] d\left[ \omega(\sigma)\overline{\omega(\sigma)} \right] \]

The polar moment of inertia can be easily expressed in terms of the mapping function \( \omega(\zeta) \). The polar moment of inertia, \( I \), for any singly connected cross section can be expressed in the form

\[ I = \oint_R \left( x^2 + y^2 \right) dxdy \]

In the case above, \( D_0 \), Green's formula allows the integral of the equation as a line integral.
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\[ I = -\int_{r} xy(adx - ydy) \]

By complex variable theory, the variables \( x \) and \( y \) can be written in terms of the complex variable \( z \) as

\[ x = \frac{z + \bar{z}}{2} \quad ; \quad y = \frac{z - \bar{z}}{2i} \]

The polar moment of inertia, \( I \), becomes:

\[ I = -\frac{1}{8i} \int_{r} \left( z^2 - \bar{z}^2 \right)(zd\bar{z} + \bar{z}d\bar{z}) \]

The equation can be simplified,

\[ \int_{r} z^3 d\bar{z} = \int_{r} \bar{z}^3 d\bar{z} = 0 \]

By using integration by parts, the remaining terms become

\[ \int_{r} z^2 d\bar{z} = \int_{r} \bar{z}^2 d\bar{z} = 0 \]

Thus, it can be written as

\[ I = \frac{1}{4i} \int_{r} z^{-2} zd\bar{z} = \frac{1}{4i} \int_{r} \omega^2(\sigma) \rho(\sigma) d\omega(\sigma) \]

The conformal mapping function can be used to map the polar coordinates \( \rho \) and \( \theta \) in the \( \zeta \)-plane onto the \( z \)-plane, these coordinates will appear as a set of \( \rho = \) constant and \( \theta = \) constant curves on the \( z \)-plane. Due to the angle preserving properties of conformal mapping functions, the curves, \( \rho = \) constant and \( \theta = \) constant, are orthogonal. As a result, \( \rho \) and \( \theta \) can be taken as orthogonal curvilinear coordinates. Accordingly, the relation can transform a vector \( A \) in the \( z \)-plane from one set of coordinate axes to another form.

\[ A_\rho + iA_\theta = \rho^{-i\alpha}(Ax + iAy) \]

Where \( \alpha \) is the angle between the axes. By using the mapping function \( z = \omega(\zeta) \).

It can be expressed in the following form

\[ A - 5 \]
\[ A_\rho + iA_\theta = \begin{bmatrix} \zeta \\ \rho \end{bmatrix} \begin{bmatrix} \omega'(\zeta) \\ \omega(\zeta) \end{bmatrix} (Ax + iAy) \]

where,

\[ \rho^{-\alpha} = \frac{\zeta}{\rho} \begin{bmatrix} \omega'(\zeta) \\ \omega(\zeta) \end{bmatrix} \]

Then it becomes,

\[ \tau_\rho - i\tau_\theta = \frac{\zeta}{\rho} \frac{\omega'(\zeta)}{\omega'(\zeta)} (\tau_x - i\tau_y) \]

One can show that

\[ \tau_x - i\tau_y = G\beta \left[ \frac{\partial \Psi}{\partial x} + i\frac{\partial \phi}{\partial x} - i(x - iy) \right] \]

In terms of the complex torsion function can be written

\[ \tau_x - i\tau_y = G\beta \left( F'(\zeta) - i\zeta \right) \]

Or

\[ \tau_x - i\tau_y = G\beta \left( f'(\zeta) - i\omega(\zeta)\omega'(\zeta) \right) \]

And

\[ \tau_\rho - i\tau_\theta = \frac{G\beta \zeta}{\rho \omega'(\zeta)} \left( f'(\zeta) - i\omega(\zeta)\omega'(\zeta) \right) \]

The conformal mapping method can be applied for compound prisms. It can be used in the case of doubly connected homogenous prisms subjected to torsion provided that the shear modulus of elasticity of material in the inner region is set equal to zero.
The previous theory for homogenous prisms subjected to pure torsion can be spread out to the problem of prisms consisted several different materials. It is assumed that each of the component materials is prismatic, homogenous and isotropic. Also, it is assumed that the various materials are arranged in prisms which do not touch each other but are surrounded by an elastic matrix. With these assumptions, the problem of compound prisms differs from that of homogenous prisms only in the boundary conditions as

1) The lateral exterior surface is stress equal zero.

\[ (\tau_{xx}, \tau_{yy} m) = 0 \text{ on } \Gamma_0 \]

2) The forces acting on surfaces separating different materials equal and opposite.

\[ (\tau_{xx}, l + \tau_{yy} m) = (\tau_{xx}, l + \tau_{yy} m) \]

3) There is no slip between the various members. This assumption implies that \( u, v \) and \( w \) are continuous sorting out the various materials at the boundaries. Therefore, \( \Psi \) must be continuous across these boundaries; that is

\[ \Psi_1 = \Psi_2 \]

The expression for torsional rigidity, are in the following form:

\[ GJ = D = \sum_{j=0}^{m} \iint_{\Gamma_j} G_j \left( x^2 + y^2 + x \frac{\partial \Psi}{\partial y} - y \frac{\partial \Psi}{\partial x} \right) dx\, dy \]

In the case of a homogenous prism, it is shown that if \( \Psi \) satisfies the boundary condition (3) above that the resultant forces are zero and only a torque \( T \) acts on the ends of the shaft. The shear stresses in each of the regions can be calculated.

The conformal mapping method is also valid to doubly connected compound prisms if the mapping function. It can map the interior of the cross section \( S_0 \) onto the ring \( \rho_1 < |\zeta| < \rho_2 \) and the interior of the cross section \( S_1 \) onto the unit circle \( |\zeta| < \rho_1 \) may be determined. This mapping function, \( z = \omega(\zeta) \) is shown in. The method of solution is
basically the same as for a singly connected homogenous prism but with more complicated boundary conditions and with more than one interior region. When the mapping function \( z = \omega(\zeta) \) is known, the torsion boundary value problem can be solved for regions bounded by concentric circles. For such regions, the boundary conditions are simplified noticeably.

For the case of homogenous prisms, let,

\[
F(z) = \Psi + i\phi \quad \text{in the } z \text{-plane}
\]

\[
f(\zeta) = \Psi + i\phi \quad \text{in the } \zeta \text{-plane}
\]

\[
F(z) = \Psi + i\phi \quad \text{in the } z \text{-plane}
\]

\[
f_2(\zeta) = \Psi_2 + i\phi_2 \quad \text{in } s_2 \quad \text{where } \quad \rho_1 < |\zeta| < \rho_2
\]

The complex torsion function \( f_1(\zeta) \) can be extended in terms of power series in \( S_1 \) and the function \( f_2(\zeta) \) in terms of a Laurent series in \( S_2 \). Hence, these functions can be expressed as:

\[
f_1(\zeta) = \sum_{k=0}^{\infty} (a_k^* + ib_k) \zeta^k \quad \text{in } s_1 \quad (A.3)
\]

\[
f_2(\zeta) = \sum_{k=-\infty}^{\infty} (a_k^* + ib_k) \zeta^k \quad \text{in } s_2 \quad (A.4)
\]

The complex conjugate \( \phi \) can be written as the imaginary parts,

\[
\phi_1 = b_0^* + \sum_{k=1}^{\infty} \left[ (a_k^* \sin(k\theta) + b_k^* \cos(k\theta)) \right] \rho^k
\]

\[
\phi_2 = b_0^* + \sum_{k=1}^{\infty} \left[ (\rho^k a_k^* - \rho^{-k} a_{-k}^*) \sin(k\theta) + (\rho^k b_k^* + \rho^{-k} b_{-k}^*) \cos(k\theta) \right]
\]

As the problem is to be put in terms of the complex conjugate the boundary conditions must also be in terms of \( \phi \). From the above equations and substituting the expressions for \( \tau_{xz} \) and \( \tau_{yz} \), it may express the boundary conditions as:

\[
\frac{d\Psi}{dn} = \frac{\partial \Psi}{\partial x} l - \frac{\partial \Psi}{\partial y} m = yl - xm \quad \text{on } \Gamma_2
\]

\[\Psi_1 = \Psi_2 \quad \text{on } \Gamma_1
\]

A-8
where \( n \) is the normal in outward direction. It is convenient to express the equation in terms of the normal derivatives of \( \varphi_1 \) and \( \varphi_1 \) on \( \Gamma_1 \), which in turn can be expressed as derivatives of the radius \( \rho_1 \). It is necessary to assume that the following equations validate on the inner boundary, in order to complete this reformulation.

\[
\frac{d\Psi_1}{ds} = \frac{d\Psi_2}{dn} \quad ; \quad \frac{d\Psi_2}{ds} = \frac{d\Psi_2}{dn}
\]

Hence,

\[
\varphi_2 = \frac{1}{2}(x^2 + y^2) + k = \frac{1}{2}zz \quad \text{on} \quad \Gamma_2
\]

\[
\frac{d\varphi_1}{dn} = \frac{d\varphi_2}{dn} \quad \text{on} \quad \Gamma_1
\]

\[
G_2\varphi_2 - G_1\varphi_1 = \frac{1}{2}(G_2 - G_1)(x^2 + y^2) + c = \frac{1}{2}(G_2 - G_1)zz + c \quad \text{on} \quad \Gamma_1
\]

The complex conjugates \( \Psi \) and \( \varphi \) can be solved and then comparing the coefficients of \( \sigma^n \) terms.

The torsional rigidity is determined by the summation

\[
GJ = D = \sum_{j=0}^{m}(G_1 + GD_0)j
\]

Where,
\[ D_0 = \sum_{j=0}^{m} \left[ \int_{\tau_j} \left[ f(\sigma) + \overline{f(\sigma)} \right] d\omega(\sigma) \bar{\omega}(\sigma) \right] \quad \text{and} \quad I = \sum_{j=0}^{m} \frac{1}{4} \int_{\tau_j} \overline{\omega^2(\sigma)} \omega(\sigma) d\omega(\sigma) \]

Therefore, the shear stresses can be determined in the separate regions. The torque is determined from the above equation.
Appendix B Typical worksheet for analyzing the data for Idealized Reinforced Concrete Beam Method (Conformal Mapping)

<table>
<thead>
<tr>
<th>13</th>
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<tbody>
<tr>
<td>120</td>
<td>b/a</td>
</tr>
<tr>
<td>60</td>
<td>60</td>
</tr>
<tr>
<td>0.141</td>
<td>0.208</td>
</tr>
<tr>
<td>(By table)</td>
<td>(By table)</td>
</tr>
<tr>
<td>110</td>
<td>110</td>
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<tr>
<td>y1/x1</td>
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<tr>
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</tr>
<tr>
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<tr>
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</tr>
<tr>
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<td>Asv 56.54866776</td>
</tr>
<tr>
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<td>As1 28.27433388</td>
</tr>
<tr>
<td>6</td>
<td>As2 28.27433388</td>
</tr>
<tr>
<td>6</td>
<td>As3 28.27433388</td>
</tr>
<tr>
<td>6</td>
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</tr>
<tr>
<td>As2 28.27433388</td>
<td>As3 28.27433388</td>
</tr>
<tr>
<td>x2s (right bottom) 28.27433388</td>
<td>x3s (left top) 28.27433388</td>
</tr>
<tr>
<td>49</td>
<td>-49</td>
</tr>
<tr>
<td>y2s (right bottom) 49</td>
<td>y3s (left top) 49</td>
</tr>
<tr>
<td>104</td>
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<td>Nmm</td>
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<tr>
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<td>mm^3</td>
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<tr>
<td>274826.5253</td>
<td>mm^3</td>
</tr>
<tr>
<td>Steel Ratio (long bars) 1.123119374 %</td>
<td></td>
</tr>
<tr>
<td>Steel Ratio (stirrups) 0.785398163 %</td>
<td></td>
</tr>
<tr>
<td>Total Steel Ratio 1.908517537 %</td>
<td></td>
</tr>
</tbody>
</table>

| 64.8 | \(\omega(\zeta)\) 59.42468653 |
| -6.265 | \(\omega(\zeta)\) 35.62792722 |
| 0.910 | \(\zeta n+1\) -0.683924327 |
| -1.128 | a2 59.42468653 |
| 0.566 | k3 81.34615385 |
| -0.265 | |
| 0.098 | |
Time Dependent Effects of Reinforced Concrete Subjected to Torsional Loading

<table>
<thead>
<tr>
<th>( \theta )</th>
<th>( \beta )</th>
</tr>
</thead>
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<td>0.000975063</td>
<td>0.001258646</td>
</tr>
<tr>
<td>9.75063E-07</td>
<td>1.25865E-06</td>
</tr>
</tbody>
</table>

\[
\begin{align*}
\theta (\text{Plain concrete}) &= 0.001258646 \\
\beta (\text{Plain concrete}) &= 1.25865E-06
\end{align*}
\]
REFERENCES


