A SINOGRAM RESTORATION TECHNIQUE
FOR THE HOLLOW PROJECTION
PROBLEM IN COMPUTER TOMOGRAPHY

A thesis submitted to
The Hong Kong University of Science and Technology
in partial fulfillment of the requirements for
the Degree of Master of Philosophy in
Electrical and Electronic Engineering

by

LIU, Ziyin

BACHELOR OF ENGINEERING, BACHELOR OF ARTS

TIANJIN UNIVERSITY, Tianjin, P.R.China 1996

July, 1999
A SINOGRAM RESTORATION TECHNIQUE FOR THE HOLLOW PROJECTION PROBLEM IN COMPUTER TOMOGRAPHY

by

LIU, Ziying

Approved by:

[Signature]
Dr. Sze-Fong Mark YAU
Thesis Supervisor

[Signature]
Dr. Ross D. MURCH
Thesis Supervisor

[Signature]
Prof. Philip C. H. CHAN
Thesis Examination Committee Member (Chairman)

[Signature]
Prof. Francis H. Y. CHAN
Department of Electrical and Electronic Engineering, Hong Kong University
Thesis Examination Committee Member
Prof. Philip C. H. CHAN
Head of Department

Department of Electrical and Electronic Engineering
The Hong Kong University of Science and Technology

July 1999
A SINOGRAPH RESTORATION TECHNIQUE
FOR THE HOLLOW PROJECTION PROBLEM IN
COMPUTER TOMOGRAPHY

by

LIU, Ziiying

for the Degree of
Master of Philosophy in Electrical and Electronic Engineering
at The Hong Kong University of Science and Technology
in July 1999

Abstract

Computer Tomography is a technique used to reconstruct a cross-section image of an object from its line-integral projections. A complete set of these projections is also known as a sinogram. Conventional image reconstruction algorithms implemented on existing CT systems require the collection of projection data covering the whole measurement range. Unfortunately, in many practical situations, it is not always possible to obtain all the projection data. This occurs, for example, if the tissue distributions to be measured contain X-ray opaque objects (metallic implants, screws, or clips) which attenuate the rays completely; the detector readings for these parts will be missing. This is the so-called hollow projection problem. If conventional algorithms without any restoration techniques are used in these limited-data situations, the resultant images suffer from severe streak artifacts.

In this thesis we propose an interpolation method to restore a sinogram from its available incomplete sinogram. This algorithm is based on the Ludwig-Helgason Consistency Theorem and uses the edge values of the missing part of the incomplete sinogram. In our algorithm the missing range of projection values
can be arbitrary and no *a priori* knowledge is needed. A computationally efficient algorithm was developed to solve the interpolation problem. We also assessed the effectiveness of our algorithm performed on noisy sinograms. Finally, by integrating with other restoration techniques, a hybrid algorithm was developed for fine tuning the restored image. Computer simulation results demonstrate the efficiency of the proposed method.
Acknowledgments

First and foremost, I would like to thank my former supervisor, Professor Sze-Fong Mark YAU (now with the ICAC, Hong Kong SAR Government), for his patience, guidance, inspiration and enthusiasm. I consider myself fortunate to have had the chance to work with him. He not only imparted knowledge but his way of thinking shaped my understanding of study and research. This had, and will continue to have a great impact on my future life. I am also deeply grateful to my supervisor Professor Ross Murch for his support. Without his generous help I could not have finished my thesis and taken my thesis examination. Special thanks are given to my other thesis committee members, Professor Philip Chan and Professor Francis Chan (of the Department of Electrical and Electronic Engineering, HKU), for their time and the excellent comments they made in regard to my thesis.

Secondly, I would like to thank all the professors of this university who have helped and supported me throughout my graduate study. It was from Professor Joseph Schmitt’s wonderful lectures that I formally learned Medical Imaging and became fascinated with this area. Professor Bing Zeng, Professor Kwan-Fai Cheung and Professor Li Qiu have also given me a lot of valuable instructions on signal and image processing and linear system theory. I also would like to thank Mr. Leo Y. H. Fok and Mr. Joseph S. C. Cheng for their technical support.

It is my great pleasure to acknowledge my friends. I appreciate the support and help they have given me. They made my study here more enjoyable and fruitful. I would like to mention a few names of those to which I would like to express my special thanks. I was impressed by Alexis Tourapis for his kindness and proficiency in computers. He always tried to help me solve computer prob-
lems although he was extremely busy. Shi Jiying, one of my alumni from Tianjin University, gave me a lot of help too, especially when I first arrived at HKUST. Others I would like to mention include Tan Kan, Wu Zhuang, Shuai Li, Lo Yuen Yee, Yeung Tak Keung, John C. L. Ng, Alan Lau, Yue Chung Wai, Matthew T. K. Tam, many thanks are given to them for their warm help in many ways.

Last, but not least, I would like to express my deep thanks to my dear parents, for their love, care and consistent support. I am indebted to them, and what they have given to me can never be repaid in any form.

Sincerely I thank all the people who have helped and encouraged me. They will be in my memory forever and be one motivation for my future endeavors.
Table of Contents

Abstract ..................................................... i
Acknowledgements ........................................ iii
Table of Contents ........................................ v
List of Figures ............................................ xii
List of Tables ............................................ xiii
Abbreviations ............................................. xiv

1 Introduction and Literature Review ................ 1
   1.1 Background ........................................ 1
       1.1.1 Computer Tomography ....................... 1
       1.1.2 Hollow Projection Problem .................. 3
       1.1.3 Literature Review .......................... 5
   1.2 Thesis Research ................................... 8
   1.3 Thesis Organization ............................. 10

2 Basics .................................................. 12
   2.1 The Radon Transform ............................. 12
   2.2 Why Sinogram .................................... 14
   2.3 The Projection Slice Theorem ................... 16
   2.4 Reconstruction Techniques ..................... 18
       2.4.1 Algebraic Reconstruction Techniques .... 18
       2.4.2 Direct Fourier Transform Approach ....... 18
       2.4.3 Back-Projection Methods .................. 20

3 Problem Formulation .................................. 23
   3.1 Motivation and Difficulties .................... 23
3.2 Interpolation Method ........................................ 28
3.3 Computationally Efficient Solution ......................... 34

4 Post-Processing Algorithm .................................. 36

5 Simulation Results and Discussions ......................... 39
  5.1 Results of Hollow Projection Restoration ................. 40
  5.2 Restoration Improvement After Post-Processing .......... 54
  5.3 Performance on Noisy Sinogram ........................... 56

6 Conclusions ..................................................... 65
  6.1 Conclusions ................................................. 65
  6.2 Future Research Directions ................................. 66
    6.2.1 Wavelet Applications ................................. 66
    6.2.2 Total Least Squares Solutions ....................... 67

References ....................................................... 69
List of Figures

1.1 Simple scanning system for transaxial tomography. A pencil beam of X-rays passes through the object and is detected on the far side. The source-detector assembly is scanned sideways to generate one projection. This is repeated at many viewing angles and the required set of projection data is obtained [2].................. 2

1.2 Hollow projection problem............................................. 4

1.3 ‘Naive’ reconstruction from a cross section through a patient with a metal hip-joint [3]................................................. 4

1.4 Flowchart of the research conducted in this thesis.................... 9

2.1 Projection geometry on which the Radon transform is based.......... 13

2.2 The trace of projections for a particular point located at (r, \theta)..... 15

2.3 The Radon transform pare of point (1, \pi/4)............................ 16

2.4 Illustration of projection slice theorem.................................. 17

2.5 Illustration of Fourier transform approach of reconstruction........ 19

2.6 Projections of a point at the origin back projected.................... 20

2.7 Filter-backprojection (FBP) method.................................... 22

2.8 Convolution-backprojection (CBP) method............................... 22

3.1 (a) Spatial support of function f(t_1, t_2); (b) Spectral support of F(\omega_{t_1}, \omega_{t_2}) [20].................................. 24

3.2 Spectral support of finite-length bowtie [20]......................... 25
3.3 Spectrum comparison among a complete sinogram, incomplete sinograms in missing-cone problem and hollow projection problem. (a) a complete sinogram; (b) the spectral support of (a); (c) an incomplete sinogram in missing-cone problem (120° available angular view); (d) the spectral support of (c); (e) an incomplete sinogram in hollow projection problem (a metal implant of size 20 × 20 is contained in the image); (f) the spectral support of (e).

3.4 Missing samples in hollow tomography. .......................... 28

3.5 Estimate the center of mass. ........................................ 31

3.6 Illustration of the proposed interpolation method. .............. 33

4.1 [9] Schematic diagram of Gerchberg-Papoulis iteration reconstruction algorithm. ................................. 36

4.2 Post-processing iteration algorithm. .............................. 38

5.1 Experiment 1: (a) original image; (b) incomplete sinogram; (c) 'naive' reconstruction from (b); (d) restored sinogram; (e) image reconstructed from restored sinogram. ......................... 41

5.2 Experiment 1: Comparison of the profiles of the restored sinograms with those of the original sinogram. (a) φ = 1; (b) φ = 61. The solid black line is the profile of the original sinogram which suppose it is homogeneous in the metal implant area and it contains the same mass as the original image. The dashed blue line is that of the restored sinogram. ................................. 42

5.3 Experiment 2: (a) original image; (b) incomplete sinogram; (c) 'naive' reconstruction from (b); (d) restored sinogram; (e) image reconstructed from restored sinogram. ................................. 44
5.4 Experiment 2: Comparison of the profiles of the restored sinogram with those of the original sinogram (a) $\phi = 1$; (b) $\phi = 61$. The solid black line is the profile of the original sinogram which suppose it is homogenous in the metal implant area and it contains the same mass as the original image. The dashed blue line is that of the restored sinogram.

5.5 Experiment 3: (a), (b) original image with multiple implants on it which indicate the different sizes and locations performed in this experiment.

5.6 The profiles of the original and reconstructed images at (a) $y = 50$; (b) $y = 70$; (c) $y = 105$; (d) $y = 140$. The metallic implant size is $12 \times 12$. The black line is the profile of the original image and the blue line is that of the reconstructed image from the restored sinogram. The red dotted line indicates the location of the metallic implant which the profile crosses.

5.7 The profiles of the original and reconstructed images at (a) $y = 50$; (b) $y = 70$; (c) $y = 105$; (d) $y = 140$. The metallic implant size is $24 \times 24$. The black line is the profile of the original image and the blue line is that of the reconstructed image from the restored sinogram. The red dotted line indicates the location of the metallic implant which the profile crosses.

5.8 The profiles of the original and reconstructed images at (a) $y = 50$; (b) $y = 70$; (c) $y = 105$; (d) $y = 140$. The metallic implant size is $36 \times 36$. The black line is the profile of the original image and the blue line is that of the reconstructed image from the restored sinogram. The red dotted line indicates the location of the metallic implant which the profile crosses.
5.9 The profiles of the original and reconstructed images at (a) $y = 50$; (b) $y = 70$; (c) $y = 105$; (d) $y = 140$. The metallic implant size is $12 \times 12$. The black line is the profile of the original image and the blue line is that of the reconstructed image from the restored sinogram. The red line indicates the location of the metallic implant which the profile crosses.

5.10 The profiles of the original and reconstructed images at (a) $y = 50$; (b) $y = 70$; (c) $y = 105$; (d) $y = 140$. The metallic implant size is $24 \times 24$. The black line is the profile of the original image and the blue line is that of the reconstructed image from the restored sinogram. The red line indicates the location of the metallic implant which the profile crosses.

5.11 The profiles of the original and reconstructed images at (a) $y = 50$; (b) $y = 70$; (c) $y = 105$; (d) $y = 140$. The metallic implant size is $36 \times 36$. The black line is the profile of the original image and the blue line is that of the reconstructed image from the restored sinogram. The red line indicates the location of the metallic implant which the profile crosses.

5.12 Errors after performing post-processing.

5.13 Comparison of the resultant image after the post-processing with the restored image in Experiment 2, (a) restored image before performing post-processing (b) improved image after performing post-processing.

5.14 Experiment 5: (a) original noisy image; (b) incomplete sinogram corrupted by noise (SNR=10.8dB); (c) ‘naive’ reconstruction from (b); (d) restored sinogram; (e) image reconstructed from restored sinogram.
5.15 Experiment 5: (a) original noisy image; (b) incomplete sinogram corrupted by noise (SNR=15.4dB); (c) ‘naive’ reconstruction from (b); (d) restored sinogram; (e) image reconstructed from restored sinogram. ................................. 58

5.16 Experiment 5: (a) original noisy image; (b) incomplete sinogram corrupted by noise (SNR=19.5dB); (c) ‘naive’ reconstruction from (b); (d) restored sinogram; (e) image reconstructed from restored sinogram. ................................. 59

5.17 Experiment 5: (a) original noisy image; (b) incomplete sinogram corrupted by noisy (SNR=10.8dB); (c) ‘naive’ reconstruction from (b); (d) restored sinogram; (e) image reconstructed from restored sinogram. ................................. 60

5.18 Experiment 5: (a) original noisy image; (b) incomplete sinogram corrupted by noise (SNR=15.4dB); (c) ‘naive’ reconstruction from (b); (d) restored sinogram; (e) image reconstructed from restored sinogram. ................................. 61

5.19 Experiment 5: (a) original noisy image; (b) incomplete sinogram corrupted by noise (SNR=19.5dB); (c) ‘naive’ reconstruction from (b); (d) restored sinogram; (e) image reconstructed from restored sinogram. ................................. 62

5.20 Statistic result of the error of the image reconstructed from the restored noisy sinogram. The black and blue solid lines are the errors of the reconstructed image from the restored sinogram for implant location 1 and 2 respectively. The black and blue dashed lines on the top of the graph are the corresponding errors of the ‘naive’ reconstructions. The blue dashdot line is the error of the reconstructed image from complete noisy sinogram. .......... 63
List of Tables

2.1 Properties of the Radon Transform. ............................................................. 14

5.1 Comparison of results in Experiment 1 and 2. ................................. 43

5.2 Relative Error In Experiment 3. ................................................................. 46

5.3 Comparison of Relative Errors resulted from the noisy sinograms and the clean sinogram. ................................................................. 64
<table>
<thead>
<tr>
<th>Abbreviation</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>CT</td>
<td>Computed Tomography</td>
</tr>
<tr>
<td>HPP</td>
<td>Hollow Projection Problem</td>
</tr>
<tr>
<td>FBP</td>
<td>Filtered Backprojection</td>
</tr>
<tr>
<td>CBP</td>
<td>Convolution Backprojection</td>
</tr>
<tr>
<td>ART</td>
<td>Algebraic Reconstruction Techniques</td>
</tr>
<tr>
<td>ISRA</td>
<td>Iterative Sinogram Restoration Algorithm</td>
</tr>
<tr>
<td>GPIRA</td>
<td>Gerchberg-Papoulis Iterative Reconstruction Algorithm</td>
</tr>
<tr>
<td>LTCH</td>
<td>Ludwig-Helgason Consistency Theorem</td>
</tr>
<tr>
<td>POCS</td>
<td>Projection Onto Convex Set</td>
</tr>
<tr>
<td>PPA</td>
<td>Post-Processing Algorithm</td>
</tr>
<tr>
<td>SNR</td>
<td>Signal to Noise Ratio</td>
</tr>
<tr>
<td>1DFT</td>
<td>One Dimensional Fourier Transform</td>
</tr>
<tr>
<td>1DIFT</td>
<td>One Dimensional Inverse Fourier Transform</td>
</tr>
<tr>
<td>2DFT</td>
<td>Two Dimensional Fourier Transform</td>
</tr>
<tr>
<td>2DIFT</td>
<td>Two Dimensional Inverse Fourier Transform</td>
</tr>
<tr>
<td>TLS</td>
<td>Total Least Squares</td>
</tr>
<tr>
<td>OLS</td>
<td>Original Least Squares</td>
</tr>
<tr>
<td>SVD</td>
<td>Singular Value Decomposition</td>
</tr>
</tbody>
</table>
Chapter 1

INTRODUCTION AND LITERATURE REVIEW

In this chapter, we first give an overview on the history and development of Computer Tomography (CT). Then we introduce the Hollow Projection Problem (HPP), which is the one we try to solve. A literature review is given next. Salient features of various conventional algorithms, which have been taken as reference in our research, are discussed, and their drawbacks are pointed out. Finally, we give an overview of this thesis and summarize the notations being used in this thesis.

1.1 Background

1.1.1 Computer Tomography

Computer Tomography (CT), an imaging technique developed in the early 1970s, is used to reconstruct a cross-sectional image of an object from its line-integral projections. Its development was a milestone in the area of image reconstruction from projections, and brought about a historic revolution in diagnostic radiology. The 1979 Nobel prize in medicine was awarded to G. N. Hounsfield and A. M. Cormack for their pioneering contributions to the development of computerized tomography [1]. Since then, CT has developed rapidly and is now widely used in medical imaging as well as other fields such as acoustic imaging, synthetic aperture radar and nondestructive evaluation.
Conventional X-ray transmission imaging has two main disadvantages:

(1) the loss of depth information—the three-dimensional structure of the object is collapsed, or projected, onto a two-dimensional film.

(2) poor contrast in the images.

Figure 1.1: Simple scanning system for transaxial tomography. A pencil beam of X-rays passes through the object and is detected on the far side. The source-detector assembly is scanned sideways to generate one projection. This is repeated at many viewing angles and the required set of projection data is obtained [2].

CT overcomes these limitations. Fig. 1.1 illustrates the basic concept on which CT is based. A planar slice of the body is selected and X-rays parallel to this plane are passed through it but only in directions that are contained within. Electronic detectors record the intensities of the X-rays transmitted along rays through the body; this is called one projection (it is also called a scan). Then
the source and detector planes rotate together around the body at a particular angle to take another projection. After the projections have been measured over a total viewing angle of 180°, the cross-section function of the object can be reconstructed by classical reconstruction methods such as convolution back-projection (CBP) and the algebraic reconstruction technique (ART). Since no part of the body that is outside the slice is interrogated by the X-ray beam, the problem of 'depth scrambling' is eliminated. The resulting images show a target image with a spatial resolution of about 1mm and a density (linear attenuation coefficient) discrimination of better than 1% [2].

1.1.2 Hollow Projection Problem

The image-reconstruction algorithms implemented on existing CT systems require the collection of evenly distributed projection data covering the whole measurement range. Unfortunately, in many practical situations, a complete set of projections over the 180° viewing angle is not available. For example, if parts of a body cross-section are very dense, or if the area contains X-ray opaque objects (metallic implants, screws, or clips) which attenuate the rays completely, the detector readings for this part will be missing. As illustrated in Fig. 1.2, in this kind of situation, part of the data is not available in each projection. It is either unobservable or unreliable because it exceeds the dynamic range of a detector. This is the so-called hollow projection problem (HPP). It is an inverse problem that is inherently ill-conditioned, and the inversion of the radon transform (a mathematical description of taking projection) is severely ill-posed. Without any restoration techniques, the 'naive' reconstruction\(^1\) of an incomplete sinogram is usually rife with streak artifacts and may even be unable to provide coarse structural information about the object. These artifacts are shown in Fig. 1.3. Therefore, the development of an effective image reconstruction algorithm for incomplete projection data is very important. This prompted us to conduct the research presented in this thesis.

\(^1\)A naive reconstruction assumes the missing data in a sinogram to be identically zero
Figure 1.2: Hollow projection problem.

Figure 1.3: 'Naive' reconstruction from a cross section through a patient with a metal hip-joint [3].
1.1.3 Literature Review

Many researchers have put a lot of effort into developing efficient methods to solve the hollow projection problem. They have proposed various methods either in the image space or spectral space. In the following, we give an overview of these methods as they lay the foundation of our research work.

In 1977, Lewitt and Bates [4] confirmed that unambiguous reconstruction of images was theoretically possible from hollow projections. But their results demonstrated that direct solutions for the hollow projection problem were too error sensitive to be useful. This emphasised the necessity of examining indirect methods of solution. Two possible ways of treating the hollow projection problem were presented later in the same series of papers [5]: (a) simple completion of projections — reconstruction after bridging the data gaps by polynomial interpolations representing smooth continuations of the measured data, and (b) consistent completion of projections — reconstruction after filling the gaps with data satisfying consistency criteria, which takes into account that the projections are not independent from each other because they are all derived from the same cross section. The consistency of all projections is enforced by simultaneous operations on their Fourier coefficients. The simple completion of projections can only be effective when one or more of the following conditions are satisfied: (i) A small fraction of the projection is missing; (ii) The bulk of the density is circularly symmetric; (iii) The complete projection varies slowly over the missing part of the data. Although consistent completion of projections is more applicable to the hollow projection problem, the task of meeting consistency criteria requires a great deal of computation time. Also, the missing parts that Lewitt and Bates dealt with explicitly were only in the central portion of each projection.

Hinderling et al. [6] implemented the above simple completion of projection approach in a real case which involved the examination of the long-term mechanical stability of total hip arthroplasties. It was found that even the simplest completion of the hollow projections (e.g., a linear interpolation between the
edges of the implant) provided acceptable reconstructed images. However, the reconstructed CT values obtained with this procedure were slightly distorted due to inconsistencies in the gap data.

Glover and Pele [7] introduced a nonlinear, multistep algorithm for removing streak artifacts caused by small metal clips. They showed impressive results with real clinical data when the missing line integrals represented about 1% of the complete set of measurements.

The Gerchberg-Papoulis iterative frequency-domain extrapolation algorithm [8, 9] has been applied to limited angle reconstruction problems in several papers [10, 11]. This algorithm specifically assumes that the missing data corresponds to a missing region in the frequency domain. This is a restrictive assumption because it applies only to parallel-beam geometry when complete projections are missing [12]. Therefore this algorithm is not applicable to the hollow projection problem.

Iterative algorithm based on the more general concept of projections onto convex sets has also been developed [13, 14]. This approach has the important theoretical advantage of providing convergence proofs for algorithms involving certain types of nonlinear constraints. But, here again, a parallel-beam geometry with complete missing projections is assumed in order to obtain the required mathematical properties.

Other restoration techniques may need to incorporate a priori knowledge into the restoration algorithms. In [12], an operator framework was developed for the design and analysis of algorithm for image reconstruction from limited data. The approach is not based on a frequency-domain interpretation, as are the Gerchberg-Papoulis types of algorithms. From this framework, iterative convolution backprojection algorithms were derived that explicitly incorporate all available a priori information. However, due to the inaccurate computation of projection and backprojection, additional artifacts will appear with repeated iterations.

Seitz and Rüegsegger [15] proposed a consistency condition for the completion of incomplete projection data, and this led to a system of linear equations
which can be solved to obtain the missing projection values. It was shown that
the system was ill-posed, and regularizing methods were presented for its so-
lution. However, the Tikhonov regularization method requires a large amount
of computation and the symmetry-preserving regularization method introduces
ripple patterns a at certain distance away from the X-ray opaque object.

Kudo and Saito [16] proposed another algorithm for the completion of hollow
projections based on Cormack’s inversion formula [17]. This algorithm utilizes
the regularized numerical evaluation of the inverse Circular Harmonic Transform
(CHT) [18, 19]. However the missing range is restricted to the center of each
projection.

More recent research about the missing data problem in CT can be found
in [20]. The proposed iterative sinogram restoration algorithm (ISRA) does not
require any a priori knowledge of the underlying object and is guaranteed to
converge. It was believed that this method was applicable to any incomplete
sinogram. However, our experiments show that the ISRA is not applicable to
the hollow projection problem. A detailed analysis will be given in Section 3.1
of this thesis.

From the above discussion it can be seen that, although the problem of re-
construction from incomplete projection data has been investigated by a number
of researchers with different methods, no algorithm has emerged that combines
a fast computation time, moderate storage requirements, an insensitivity to
photon noise in the projections, an absence of reconstruction artifacts, and an
arbitrary range of the missing projection values.
1.2 Thesis Research

The aim of this study is to develop an efficient algorithm to solve the hollow projection problem. Our main objective is to develop reliable and computationally efficient procedures which can be applied to arbitrary missing data in HPP. Since the ‘naive’ reconstruction cannot provide even a coarse structure of the image, method must be found to interpolate the missing data before applying the standard reconstruction techniques. Because of the difficulties and limitations as mentioned in the previous section, we can not simply transplant some other techniques developed for other problems which may appear to have a similar structure as HPP. After careful study of the special characteristics of HPP, we developed an interpolation method for HPP. By integrating other restoration algorithms, a hybrid method was designed for fine tuning the reconstructed image. We tested the effectiveness of our algorithm on sinograms which had been corrupted by Poisson noise. Note that in real applications, it is almost impossible to get a noise free sinogram. The dominant noise in a X-ray system is Poisson distributed as the emission of photons from the X-ray source is a Poisson process. Therefore robustness to noise is an indispensable requirement for a good restoration technique. Fig. 1.4 illustrates the procedures conducted during our research.

The contributions of this thesis can be grouped into several categories,

1. Real head CT data were used to test the effectiveness of our algorithm; other researchers usually use simple computer generated images with a circularly symmetric property.

2. Our algorithm improves other interpolation methods developed for HPP. Although researchers have proposed various interpolation algorithms, none of them is beyond using linear interpolation or using the average of the surroundings to fit the gap created by the surgical metal clips or screws.

3. Our proposed algorithm does not require any a priori knowledge and is applicable to an arbitrary range of the missing data in HPP.
Figure 1.4: Flowchart of the research conducted in this thesis.
4. Our algorithm does not need a huge capacity of computation; it can be implemented easily with lower computational requirements.

5. We also give a detailed analysis of the performance of our algorithm on noisy sinograms. This is very important since, in real situations, the addition of photon noise to X-ray transmissions is almost unavoidable.

1.3 Thesis Organization

The organization of this thesis is as follows. In Chapter 2 we review the fundamentals of computer tomography. The essential concepts include Radon transform, projection slice theorem and backprojection techniques. Introduced in Chapter 3 are the hollow projection problem and the difficulties we met. An interpolation method is proposed and a theoretical solution is given. Due to the ill-conditioned property of this problem and because of concern about the huge size of the matrix, the result cannot be computed directly from the formula. Therefore, a computational efficient algorithm is presented to solve this problem. In Chapter 4, we present a post-processing algorithm which can further improve the restored image. Simulation results are given in Chapter 5. These can be used to judge the restored image both visually and numerically. Finally, in Chapter 6, our work is summerized and a conclusion and future research directions are presented.

Throughout this thesis, we use the following notations:

\[ I = \sqrt{-1} \]

\[ \delta(\cdot) \quad \text{– Dirac delta function} \]

\[ (\cdot)^T \quad \text{– transpose} \]

\[ A \odot B \quad \text{– Schur-Hadamard (element by element) matrix product} \]

\[ A_{ij} \quad \text{– the } i, j \text{ element of } A \]

\[ \|A\|_F = \sqrt{\sum(A_{ij})^2} \quad \text{(Frobenious norm of } A) \]

\[ ** \quad \text{– 2-D convolution} \]
$\mathcal{R}(\cdot)$ – the Radon transform operator

$\mathcal{F}_1(\cdot)$ – 1-D Fourier transform operator

$\mathcal{F}_2(\cdot)$ – 2-D Fourier transform operator

$I_{n \times n}$ – $n \times n$ identity matrix

$(t_1, t_2)$ – Cartesian (rectangular) coordinates in spatial domain

$(r, \theta)$ – polar coordinates in spatial domain

$(\rho, \phi)$ – projection domain coordinates

$(\omega_1, \omega_2)$ – Cartesian (rectangular) coordinates in frequency domain

$(R, \psi)$ – polar coordinates in frequency domain
Chapter 2

BASICS

Computer Tomography (CT) is a mature technique and has its own theory, algorithms, and system structure. In this chapter, we briefly review the basic concepts of CT, some essential theorems, and the standard reconstruction techniques. Due to limited space in this thesis, we are not able to give all the detailed derivations of theorems and well-known results. Interested readers can refer to the references listed at the end of this thesis. [1] [2] and [21] are particularly useful for understanding the basic concepts of CT.

2.1 The Radon Transform

The Radon transform of a function on $\mathbb{R}^2$ arises naturally in the tomographic imaging problem which is concerned with reconstructing a cross section of an object from measurements which are integrals of the ‘cross-section function’ over lines [22]. Physically, this line integral is obtained in transmission tomography in the manner illustrated by Fig. 2.1. A new coordinate system $(\rho, \eta)$ that is rotated by an angle $\phi$ with respect to $(t_1, t_2)$ needs to be introduced, and the relationship between $(t_1, t_2)$ and $(\rho, \eta)$ are

\begin{align*}
\rho &= t_1 \cos \phi + t_2 \sin \phi, \quad (2.1) \\
\eta &= -t_1 \sin \phi + t_2 \cos \phi, \quad (2.2) \\
t_1 &= \rho \cos \phi - \eta \sin \phi, \quad (2.3) \\
t_2 &= \rho \sin \phi + \eta \cos \phi. \quad (2.4)
\end{align*}
Figure 2.1: Projection geometry on which the Radon transform is based.

Consider a 2D function $f(t_1, t_2)$ which represents a cross-section of an object. The parallel-ray projection, or the Radon transform, of $f(t_1, t_2)$ is defined as

$$x(\phi, \rho) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(t_1, t_2) \delta(\rho - t_1 \cos \phi - t_2 \sin \phi) dt_1 dt_2 \quad (2.5)$$

where $-\infty < \rho < \infty, 0 \leq \phi < 2\pi$. This equation can be interpreted as that each sample of the Radon transform of $f$ is the integral of $f(t_1, t_2)$ over the line

$$t_1 \cos \phi + t_2 \sin \phi = \rho. \quad (2.6)$$

These lines are perpendicular to the $\eta$ axis (the projection axis). Fig. 2.1 gives a simple geometric interpretation of the line integrals in (2.5). Owing to the sifting property of delta functions, (2.5) can be written as

$$x(\phi, \rho) = \int_{-\infty}^{\infty} f(\rho \cos \phi - \eta \sin \phi, \rho \sin \phi + \eta \cos \phi) d\eta \quad (2.7)$$
which brings out its nature as a line integral better. (2.7) indicates that Radon transform maps a 2-D object or density function \( f(t_1, t_2) \) in the spatial domain to a new 2-D function, the projection function \( p(\rho, \eta) \), in the projection domain. In X-ray CT, the integrations that define the Radon transform are performed along the lines which represent the accumulated attenuation of the rays connecting the X-ray source and detectors.

Although the Radon transform was introduced by Radon as early as in 1917, its real value was not realized until the early seventies by Cormack[26]. Some properties of the Radon transform are listed in Table 2.1. Here we denote \( x(\rho, \phi) \), \( x_1(\rho, \phi) \) and \( x_2(\rho, \phi) \) as the projections of \( f(t_1, t_2) \), \( f_1(t_1, t_2) \) and \( f_2(t_1, t_2) \) respectively, and \( f(r, \theta) \) is the object function \( f(t_1, t_2) \) expressed in polar coordinates.

Table 2.1: Properties of the Radon Transform.

<table>
<thead>
<tr>
<th>Properties</th>
<th>Function (In Object Domain)</th>
<th>Radon Transform</th>
</tr>
</thead>
<tbody>
<tr>
<td>Range</td>
<td>( f(t_1, t_2) = 0 ) for (</td>
<td>x</td>
</tr>
<tr>
<td>Linearity</td>
<td>( a_1 f_1(t_1, t_2) + a_2 f_2(t_1, t_2) )</td>
<td>( a_1 x_1(\rho, \phi) + a_2 x_2(\rho, \phi) )</td>
</tr>
<tr>
<td>Symmetry</td>
<td>( f(t_1, t_2) )</td>
<td>( x(\rho, \phi) = x(-\rho, \phi + \pi) )</td>
</tr>
<tr>
<td>Periodicity</td>
<td>( f(t_1, t_2) )</td>
<td>( x(\rho, \phi) = x(\rho, \phi + 2\pi) )</td>
</tr>
<tr>
<td>Shift</td>
<td>( f(t_1 - c_1, t_2 - c_2) )</td>
<td>( x(\rho - c_1 \cos \phi - c_2 \sin \phi, \phi) )</td>
</tr>
<tr>
<td>Rotation</td>
<td>( f(r, \theta + \theta_0) )</td>
<td>( x(\rho, \phi + \theta_0) )</td>
</tr>
<tr>
<td>Scaling</td>
<td>( f(at_1, at_2) )</td>
<td>( \frac{1}{a} x(a\rho, \phi) )</td>
</tr>
</tbody>
</table>

2.2 Why Sinogram

A complete set of projections \( x(\rho, \phi) \) (taken over the entire range from 0 to \( \pi \)) of a cross-section image is also referred to as a sinogram. This is because the point spread function of the Radon transform is a sinusoidal function.

Consider a point located at \((r, \theta)\) as shown in Fig. 2.2. Its projections are all located on the circle whose center is \((\frac{1}{2}r, \theta)\) and diameter is \(r\). In other words, the projection \((\rho, \phi)\) of a point \((r, \theta)\) satisfies
\[ \rho = r \cos(\theta - \phi). \]  

(2.8)

Figure 2.2: The trace of projections for a particular point located at \((r, \theta)\).

Therefore, if \(\rho\) is taken as abscissa and \(\phi\) as ordinate, i.e. in the projection space \((\rho, \phi)\), when \((r, \theta)\) is fixed, (2.8) draws a sinusoidal curve. Alternatively, if we substitute the two-dimensional impulse function \(\delta(t_1 - c_1, t_2 - c_2)\) into (2.5), where \((c_1, c_2)\) are the Rectangular coordinates of \((r, \theta)\), the same result is reached. In Fig. 2.3 a Radon transform pair is shown. It can be seen that the sinogram corresponding to the point \(r = 1, \theta = \pi/4\) in the spatial domain is a cosine curve.

Since the Radon transform operation is linear, the Radon transformation of any function can be obtained by the superposition of sinusoids. This is the so-called sinogram.
2.3 The Projection Slice Theorem

The Projection Slice Theorem (also called the central-section theorem) gives a beautiful mathematic interpretation to the reconstruction problem in CT which makes it possible to avoid using the computationally intensive post-backprojection filtering method to reconstruct the original image. Both Filtered Backprojection (FBP) and Convolution Backprojection (CBP) reconstruction techniques, which have been implemented on real CT scanners, were developed based on this theorem.

The Projection Slice Theorem states that the one-dimensional Fourier transform with respect to $\rho$ of the projection $x(\rho, \phi)$ is equal to the two-dimensional Fourier transform of $f(t_1, t_2)$ evaluated along the line through the origin at the angle $\phi$. That is,

$$\mathcal{S}_1 \{ x(\rho, \phi) |_{\phi = \text{const}} \} = F(\omega_1, \omega_2) |_{\omega_1 = R \cos \phi, \omega_2 = R \sin \phi}$$  \hspace{1cm} (2.9)

where $F(\omega_1, \omega_2) = \mathcal{S}_2 \{ f(t_1, t_2) \}$, and $\mathcal{S}_1, \mathcal{S}_2$ denote 1-D and 2-D Fourier transform respectively. This relationship can be best illustrated in Fig. 2.4.
In order to verify this theorem, we first calculate the 1-D Fourier transform of the projection function,

\[
X_\phi(R) = \mathfrak{F}_1\{x(\rho, \phi)|_{\phi=const}\} = \int_{-\infty}^{\infty} x(\rho, \phi)e^{-I2\pi \rho R}d\rho \\
= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(\rho \cos \phi - \eta \sin \phi, \rho \sin \phi + \eta \cos \phi)e^{-I2\pi \rho R}d\eta d\rho,
\]

(2.10)

and then convert \((\rho, \eta)\) to \((t_1, t_2)\) using the coordinate transformations in (2.2)—(2.4),

\[
X_\phi(R) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(t_1, t_2)e^{I2\pi(t_1 R \cos \phi + t_2 R \sin \phi)}dt_1 dt_2 \\
= F(R \cos \phi, R \sin \phi),
\]

(2.11)

which verifies (2.9).

Figure 2.4: Illustration of projection slice theorem.
2.4 Reconstruction Techniques

2.4.1 Algebraic Reconstruction Techniques

The Algebraic Reconstruction Techniques (ART) are iterative approaches which attempt to find a two-dimensional distribution that matches all the projections. An initial distribution is assumed and it is compared with the measured projections. Using a variety of iterative algorithms, the initial distribution can be successively modified. In the following, we use the additive ART to illustrate the fundamental concepts.

The ART system is based on the very general premise that the resultant reconstruction should match the measured projections. The iterative process starts with all reconstruction elements \( f_i \) set to a constant such as the mean \( \bar{f} \) or zero. In each iteration the difference between the measured data for a projection \( x_i \) and the sum of the reconstructed elements along that ray \( \sum_{i=1}^{N} f_{ij} \) is calculated. Here \( f_{ij} \) represents an element along the \( j \)th line forming the projection ray \( x_j \). This difference is then evenly divided among the \( N \) reconstruction elements. The iterative algorithm is defined as

\[
  f_{ij}^{q+1} = f_{ij}^{q} + \frac{x_j - \sum_{i=1}^{N} f_{ij}^{q}}{N}
\]  

(2.12)

where the superscript \( q \) indicates the iterations. The algorithm recursively relates the values of the elements to those of the previous iteration.

Although the iterative methods were the most popular methods in the early days of computerized tomography, they have become almost completely supplanted by direct methods due to problems such as computation time and convergence accuracy in the presence of noise.

2.4.2 Direct Fourier Transform Approach

According to the projection slice theorem, the Fourier transform of a projection at angle \( \phi \) forms a line in the two-dimensional Fourier plane at this same angle. Since the projection angle \( \phi \) and the resultant polar angle in the
Fourier transform plane $\psi$ are identical, we can use the same symbol, $\phi$, for both. Therefore, after filling the entire $X(R, \phi)$ plane with projections at all angles, the reconstructed density is provided by the two-dimensional inverse Fourier transform as

$$f(t_1, t_2) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} X(\omega_1, \omega_2) \exp[i2\pi(\omega_1 t_1 + \omega_2 t_2)] d\omega_1 d\omega_2$$

$$= \int_0^{2\pi} d\phi \int_0^{\infty} X(R, \phi) \exp[i2\pi R(t_1 \cos \phi + t_2 \sin \phi)] R dR.$$ (2.13)

Although this reconstruction method appears simple, it is not practical to implement in X-ray computed tomography. Besides the computational intensity needed for the 2-D inverse Fourier transform, another problem is that in the periphery of $X(R, \phi)$, the spacing between the data points becomes large, as shown in Fig. 2.5. This makes interpolation on a rectangular grid very difficult.

Figure 2.5: Illustration of Fourier transform approach of reconstruction.
2.4.3 Back-Projection Methods

**BACK-PROJECTION**

Since the high computational cost and the need for various interpolations and coordinate transforms impede the application of the direct FT reconstruction method, back-projection was developed as an alternative method to perform the reconstruction. It has great computational advantages.

The basic idea about back-projection is to 'smear' the measured values across the unknown density function. Fig. 2.6 illustrates this idea by a simple case.

![Diagram of back-projection](image)

**Figure 2.6: Projections of a point at the origin back projected.**

Mathematically, the back projection of a single measured projection along the unknown density is given by

\[ b_{\phi}(t_1, t_2) = \int x_{\phi}(R)\delta(t_1 \cos \phi + t_2 \sin \phi - R) dR \]  

(2.14)

where \( b_{\phi}(t_1, t_2) \) is the back projection of \( x(\rho, \phi) \) at angle \( \phi \).

Adding up these densities at all angles, we obtain a crude reconstruction \( f_b(t_1, t_2) \) of the original image,

\[
  f_b(t_1, t_2) = \int_0^\pi b_{\phi}(t_1, t_2) d\phi \\
  = \int_0^\pi d\phi \int_{-\infty}^{\infty} x_{\phi}(R)\delta(t_1 \cos \phi + t_2 \sin \phi - R) dR. 
\]  

(2.15)
By calculating the impulse response of back projection operator, we get $1/r$. Therefore, the reconstructed image from back projection is

$$f_b(t_1, t_2) = f(t_1, t_2) \ast \ast \frac{1}{r}$$

(2.16)

where ** denotes two dimensional convolution.

As we can see, the back-projection operator is not the inverse of the Radon transform. It brings star artifacts to the reconstructed image. Some modified back-projection methods are developed to remove this $1/r$ blurring, as discussed next.

It may seem that it would be straightforward to apply an inverse filter on the backprojected image. This approach, however, lacks computational efficiency since it involves 2DFT. This implies that, in order to undo this $1/r$ effect, special consideration should be taken before the backprojection operation.

**FILTERED BACK-PROJECTION (FBP)**

Recall the projection slice theorem, $x(\rho, \phi)$ can be written as

$$x(\rho, \phi) = \mathcal{F}_1^{-1}\{F(R, \phi)\} = \int_{-\infty}^{\infty} F(R, \phi)e^{i2\pi \rho R} dR.$$  

(2.17)

Plug (2.17) into (2.15) and, after a few steps of deduction, we can obtain

$$f_b(t_1, t_2) = \int_0^{\pi} d\phi \int_{-\infty}^{\infty} \mathcal{S}_1^{-1}\{x(\rho, \phi)\} e^{i2\pi r(t_1 \cos \phi + t_2 \sin \phi)} |R| dR,$$

(2.18)

then the ideal reconstruction can be expressed as

$$f(t_1, t_2) = \int_0^{\pi} d\phi \int_{-\infty}^{\infty} \mathcal{S}_1^{-1}\{x(\rho, \phi)\} \mathcal{F}_1^{-1}\{H(R)\} \delta(t_1 \cos \phi + t_2 \sin \phi - \rho) d\rho.$$  

(2.19)

The blurring effect of the $1/|R|$ can be removed by multiplying each transformed projection by the compensating filter $H(R) = |R|$ before backprojection. Fig. 2.7 shows the procedures of FBP.
CONVOLUTION BACK-PROJECTION (CBP)

Alternatively, it is possible to filter the projection directly in the spatial domain. This method is referred to as the convolution backprojection method. Now the filtered projection term in (2.19) becomes

$$\mathcal{F}^{-1}\{\mathcal{F}[x(\rho, \phi)][|\rho|]\} = x(\rho, \phi) * * \mathcal{F}^{-1}\{|\rho|\}. \quad (2.20)$$

The detailed steps are illustrated in Fig. 2.8.

As the FBP method involves 1DFT and 1DIFT, the CBP method is more computationally efficient than the FBP method. In the software used in commercial X-ray CT scanners, algorithms based on the CBP method are the most popular. In all our simulation experiments, we used the CBP method to reconstruct images from their sinograms.
Chapter 3

Problem Formulation

The hollow projection problem occurs very frequently in practical applications, especially in medical imaging. If the scanned anatomical section contains a metal object, conventional CT images usually exhibit severe artifacts. These artifacts arise because the prosthesis attenuates the X-ray beam beyond tolerable limits, thereby inducing localized suppressions of information in the projections [6]. The most common cases are metal hip-joints, and tantalum clips in sections of the brain.

In this chapter, we first discuss our motivation to conduct this research, and outline the difficulties we encountered throughout this study. Then we propose an interpolation method for solving the HPP. A computationally efficient algorithm which reduces the computation intensity is also presented.

3.1 Motivation and Difficulties

In a recent publication by Yau and Yu [20], an iterative sinogram restoration algorithm (ISRA) was presented for the limited angle problem in CT. In [20], Yau and Yu used the results of Rattey and Lindgen [22] which showed that the spectral support of a sinogram was bowtie-shaped, to develop a matrix formulation. The essence of this algorithm is to force the spectrum of an incomplete sinogram, which is no longer bounded into the bowtie shape, back into the bowtie shaped area. Although the target problem of this algorithm is supposed to be the limited-angle problem, we did not see any limitations in
applying this algorithm to other kinds of missing projection data problems, such as the HPP, considering that the spectral support of each kind of incomplete sinogram is also corrupted and no longer limited itself to the bowtie area. We simply changed the indicator matrix $Z$ to make the algorithm fit our problem. However, our simulation results revealed that ISRA does not work for the HPP. The improvement achieved by a comparable number of iterations performed on the limited-angle problem was too limited to be considered useful. The reason has been explored and is presented in the following.

Consider a cross-section function $f(t_1, t_2)$ of an object. Let $F(\omega_1, \omega_2)$ be its two-dimensional Fourier transform. Suppose that $f(t_1, t_2)$ is spatially limited to a disk with radius $R_m$ in the $t_1 - t_2$ plane and is essentially bandlimited\(^1\) in the frequency domain to a disk with radius $W_M$ as shown in Fig. 3.1. The continuous sinogram, or Radon transform $x(\phi, \rho)$ of $f(t_1, t_2)$, is a continuous two-dimensional signal. This was defined in Chapter 2.

\[ X(\omega_1, \omega_2) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x(\phi, \rho) \exp(-J2\pi(\phi \omega_1 + \rho \omega_2)) d\phi d\rho, \]

\(^1\)It is well known that a spatially limited signal cannot be bandlimited. A spatially limited function is essentially bandlimited to $S$ if there exists a spectral support $S$ in the frequency domain such that most of the signal energy (say 99%) is concentrated in $S$.

Figure 3.1: (a) Spatial support of function $f(t_1, t_2)$; (b) Spectral support of $F(\omega_1, \omega_2)$ [20].
then approximately 99% of the total energy of \( X(\omega_p, \omega_p) \) is within the finite-length bowtie, as illustrated in Fig. 3.2. Hence, \( X(\omega_p, \omega_p) \) is essentially band-limited to the finite-length bowtie.

![Figure 3.2: Spectral support of finite-length bowtie [20].](image)

If the sinogram is not a complete one, this property will not be strictly conformed any more. In other words, the spectral content will spread out of the bowtie shaped support. In Fig. 3.3, we illustrate the differences of spectral support among a complete sinogram, incomplete sinograms in the limited-angle problem and in the hollow projection problem respectively. All the spectra of the sinograms displayed are in a logarithm scale to reduce the contrast of the image. Fig. 3.3(a) shows a complete sinogram with 240 views (from 0° to 360°) and 270 raysums per view projected from an image, the size of which is 190 × 190. Fig. 3.3(b) shows the spectral support of (a). It can be seen that most of the energy is within the bowtie shaped area. The numerical computation shows that this bowtie area contains 99.99% of the whole sinogram energy. Fig. 3.3(c) shows an incomplete sinogram in the limited-angle problem — only 160 views out of 240 views are available. It is very clear that there is a bright horizontal band in the middle of the bowtie shaped area in its spectrum, as shown in Fig. 3.3(d). Only 95.66% energy is contained in the bowtie area. Fig. 3.3(e)
shows a sinogram in HPP which is taken from the same image but containing an x-ray opaque object, the size of which is 20 × 20. Fig. 3.3(f) depicts its spectral support. Although the area outside the bowtie shaped area became a little bit brighter, the bowtie shaped area still contains as much as 99.77% of the energy.

We can thus reach the conclusion that, although the spectral support of an incomplete sinogram does not restrict itself in the bowtie shaped area as a complete one, differences exist between these two problems. In the limited-angle problem, more energy gets out and it concentrates in a horizontal band in the middle of the spectrum. However in the HPP, very little energy gets out which can be neglected. Since the ISRA mainly takes advantage of this property to restore the complete sinogram, it is not surprising to see that not much improvement can be achieved by applying the ISRA on the HPP.

During the period of testing the ISRA on the HPP, we tried to choose a good initial sinogram so that a better restoration image could be obtained through less iterations. We discovered that even a very simple compensation (e.g. assign the average value of the sinogram homogeneously on the missing part) applied to the missing data gave better results than the ‘naive’ reconstruction although the ISRA could not improve the image quality further. Lewitt and Bates had also proposed the use of linear interpolation to restore the image. Their idea was subsequently tested on a real case [6], and the improvement was impressive. This inspired us to try to find a better way to interpolate the missing data since only part of the data is missing in each projection.
Figure 3.3: Spectrum comparison among a complete sinogram, incomplete sinograms in missing-cone problem and hollow projection problem. (a) a complete sinogram; (b) the spectral support of (a); (c) an incomplete sinogram in missing-cone problem (120° available angular view); (d) the spectral support of (c); (e) an incomplete sinogram in hollow projection problem (a metal implant of size 20 × 20 is contained in the image); (f) the spectral support of (e).
3.2 Interpolation Method

As defined in Section 2.1, the parallel-ray projection, or the Radon transform of $f(t_1,t_2)$, is a continuous two-dimensional signal $x(\phi, \rho)$. In practice, only the discretized version of $x(\phi, \rho)$ is measured. Let the discretized sinogram be $p(m,n)$, $m = 0, \ldots, M - 1$, $n = 0, \ldots, 2N - 1$, where $m$ is the index indicating individual raysum of each projection and $n$ is the index for direction of projection between 0° and 360°. In complete data tomography, $p(m,n)$ is available for all $m,n$ within their respective ranges and standard reconstruction methods such as CBP can be used to reconstruct the original image with little artifacts. However, in hollow tomography, $p(m,n)$ is available only for $n = 0, \ldots, 2N - 1$, $m = \{0, \ldots, l_n\} \cup \{h_n, \ldots, M - 1\}$, as illustrated in Fig. 3.4, where solid dots indicate the locations where $p(m,n)$ is available, and circles indicate the locations where $p(m,n)$ is not available.

![Figure 3.4: Missing samples in hollow tomography.](image)

We observed that, in each projection, i.e. for fixed values of $n$, only the middle part of the data was missing. To restore the complete sinogram we propose to interpolate the missing data by a polynomial of order $d$ based on some constraints extracted from the available incomplete sinogram and the characteristics of the Radon transform. Thus, we let
\[ p(m, n) = a_{nd}m^d + a_{n(d-1)}m^{d-1} + \cdots + a_{n1}m + a_{n0}. \]  \hspace{1cm} (3.2)

Our goal is to find the coefficients \( a_{nd}, \ldots, a_{n0} \) in (3.2) by using the available sinogram and the Ludwig-Helgason consistency theorem. First, the polynomial should be consistent with the available data at points \( m = l_n \) and \( m = h_n \). This leads to the following two constraints:

\[ a_{nd}l_n^d + a_{n(d-1)}l_n^{d-1} + \cdots + a_{n1}l_n + a_{n0} = p(l_n, n), \]  \hspace{1cm} (3.3)

\[ a_{nd}h_n^d + a_{n(d-1)}h_n^{d-1} + \cdots + a_{n1}h_n + a_{n0} = p(h_n, n). \]  \hspace{1cm} (3.4)

Further, it is well known that a sinogram must satisfy the following Ludwig-Helgason consistency conditions.

**Theorem 3.1 : Ludwig-Helgason Consistency Theorem (LHCT)**

Let \( L \) be the space of rapidly decreasing \( C^\infty \) functions on \( \mathbb{R}^2 \), and let \( S^1 \) be the unit circle. Then, in order for \( x(\phi, \rho) \) to be the two-dimensional Radon transform of a function \( f \in L \), it is necessary and sufficient that

(a) \( x \in L(\mathbb{R}^1 \times S^1) \)

(b) \( x(\phi + \pi, \rho) = x(\phi, -\rho), \) and

(c) the integral

\[ \int_{-\infty}^{\infty} x(\phi, \rho)\rho^k d\rho \]  \hspace{1cm} (3.5)

be a homogeneous polynomial of degree \( k \) in \( \cos \phi \) and \( \sin \phi \) for all \( k \geq 0 \).

The proof of Theorem 3.1 can be found in [23] and [25]. This theorem specifies an infinite set of constraints on the moments of \( x(\phi, \rho) \). Another interpretation of the consistency theorem can be stated as [3]

\[ \int_0^{2\pi} \int_{-\infty}^{\infty} P(\rho)x(\phi, \rho)\exp(ik\phi)d\rho d\phi = 0 \]  \hspace{1cm} (3.6)
for \( k = -l + 1, -l + 3, \ldots, l - 3, l - 1 \) or \(|k| > l\), where \( I = \sqrt{-1} \) and \( P_l(\rho) \) is the normalized Legendre polynomial of degree \( l \). We can write (3.6) directly into the following two equations:

\[
\int_0^{2\pi} \int_{-\infty}^{\infty} P_l(\rho)x(\phi, \rho)\cos(k\phi)d\rho d\phi = 0 \tag{3.7}
\]

and

\[
\int_0^{2\pi} \int_{-\infty}^{\infty} P_l(\rho)x(\phi, \rho)\sin(k\phi)d\rho d\phi = 0. \tag{3.8}
\]

It is worth noting that not all constraints are active since, for some \( k \) and \( l \), the incomplete sinogram also satisfies (3.6). Therefore we have to select those active constraints in our experiments.

Two important relationships that exist between an object and its 2-D Radon transform can follow from the direct use of condition (c) in Theorem 3.1. The first relationship results from setting \( k = 0 \) in (3.5), where it follows that the integral of any projection is a constant. Substituting (2.5) in (3.5), we get

\[
\int_{-\infty}^{\infty} x(\phi, \rho)d\rho = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(t_1, t_2)dt_1 dt_2 \quad \forall \phi. \tag{3.9}
\]

Intuitively we also can reach this conclusion because the summation of each projection should be the same and it is equal to the summation of the whole image. Applying the above constraint to our problem, we get

\[
\sum_{i=0}^{d} \sum_{m=l_n+1}^{h_n-1} a_{ni}m^i = c - c_n, \tag{3.10}
\]

where \( c = \sum_{m=0}^{M-1} p(m, n) \) and \( c_n = \sum_{m=0}^{l_n} p(m, n) + \sum_{m=h_n}^{M-1} p(m, n) \). Note that \( c \) must be estimated since we do not have a complete sinogram, and the estimated value is denoted by \( \hat{c} \). After simple computation, (3.10) can be written as

\[
\sum_{i=0}^{d} a_{ni}f_{ni} = \hat{c} - c_n, \tag{3.11}
\]

where \( f_{ni} \) are constants. We estimate \( \hat{c} \) by using a linear estimator given by
\[
\hat{c} = \frac{1}{N} \sum_{n=0}^{N-1} \left( s_n \frac{l_n + h_n}{2} + c_n \right)
\]  

(3.12)

where \( s_n \) is the number of the missing data for each projection.

The second relationship results from setting \( k = 1 \) in (3.5), which leads to

\[
\int_{-\infty}^{\infty} \rho x(\phi, \rho) d\rho = \int_{-\infty}^{\infty} f(t_1, t_2)(t_1 \cos \phi + t_2 \sin \phi) dt_1 dt_2.
\]  

(3.13)

This relationship reveals that the center of the mass of the projection at angle \( \phi \) is equal to the projection of the center of the object mass onto the \( \phi \)-axis.

![Diagram of metal implant and projection](image)

Figure 3.5: Estimate the center of mass.

It is necessary to estimate the weight center \((\hat{X}_w, \hat{Y}_w)\), and the procedure is illustrated in Fig. 3.5. An affine function is used to make up the missing part, which is consistent with the two edges of the missing data. In this way, \((\hat{X}_w, \hat{Y}_w)\) is calculated and (3.13) becomes
\begin{equation}
\sum_{i=0}^{d} \sum_{m=l_{n}+1}^{h_{n}-1} a_{ni} i^{i+1} = \dot{X}_{w} \cos \phi + \dot{Y}_{w} \sin \phi + \text{const},
\end{equation}

which can be simplified to the following linear equation

\begin{equation}
\sum_{i=0}^{d} a_{ni} g_{ni} = \dot{X}_{w} \cos \phi + \dot{Y}_{w} \sin \phi + \text{const}.
\end{equation}

Additional constraints can be obtained from (3.7) and (3.8) for other values of \( l \). These constraints can be expressed in the form of linear equation

\begin{equation}
\sum_{n=0}^{2N-1} \sum_{i=0}^{d} a_{ni} q_{ni}^{(v)} = w^{(v)}
\end{equation}

where \( q_{ni} \), \( w^{(v)} \) are constants and the superscript \((v)\) indicates it is the \( v \)-th constraint coming from (3.6). Notice that these constraints involve all the 360° projection data. We, therefore, cannot compute the unknown coefficients for each projection individually. After we have a sufficient number of constraints, the value of \( a = [a_{1d} \ldots a_{10}, \ldots, a_{(2N-1)d} \ldots a_{(2N-1)0}]^T \) can be determined by solving the linear equation

\begin{equation}
X a = p
\end{equation}

where

\[
X = \begin{bmatrix}
U_0 \\
U_1 \\
\vdots \\
U_{2N-1}
\end{bmatrix} = \begin{bmatrix}
U \\
V
\end{bmatrix},
\]

\[
U_i = \begin{bmatrix}
l_i^d & l_i^{d-1} & \cdots & l_i & 1 \\
h_i^d & h_i^{d-1} & \cdots & h_i & 1 \\
f_{i(d-1)} & f_{i(d-1)} & \cdots & f_{i1} & f_{i0} \\
g_{i(d-2)} & g_{i(d-2)} & \cdots & g_{i1} & g_{i0}
\end{bmatrix},
\]

\[
V_i = \begin{bmatrix}
q_{i1}^{(1)} & q_{i1(d-1)}^{(1)} & \cdots & q_{i10}^{(1)} \\
\vdots & \vdots & \ddots & \vdots \\
q_{i1}^{(2N(d-2))} & q_{i1(d-1)}^{(2N(d-2))} & \cdots & q_{i10}^{(2N(d-2))}
\end{bmatrix},
\]

32
and \( \mathbf{p} \) is a known vector. This interpolation method is summarized in Fig. 3.6. Note that the dimension of \( \mathbf{X} \) is \( 2N(d+1) \times 2N(d+1) \) and is, therefore, very large for most practical situations. Thus, we must have a computationally efficient algorithm to estimate \( \mathbf{a} \) from (3.17).

![Diagram](image)

**Figure 3.6: Illustration of the proposed interpolation method.**
3.3 Computationally Efficient Solution

It can be seen that the upper part of the matrix $X$ is a block diagonal matrix and that the lower part of the matrix $X$ is a full matrix. Since the inverse of $X = \begin{bmatrix} U \\ V \end{bmatrix}$ may not exist, we minimize

$$\| \begin{bmatrix} U \\ V \end{bmatrix} a - p \|^2_F$$

with respect to $a$. The optimal solution is given by

$$\hat{a} = \left( \begin{bmatrix} U \\ V \end{bmatrix}^T \begin{bmatrix} U \\ V \end{bmatrix} \right)^{-1} \begin{bmatrix} U \\ V \end{bmatrix}^T p$$

$$= (U^T U + V^T V)^{-1} [U^T V^T] p. \quad (3.18)$$

**Theorem 3.2** [38] Let $A$ and $C$ be invertible matrices, and $B$ be a matrix such that $BCB^T$ is of the same dimension as $A$. Then

$$[A + BCB^T]^{-1}$$

$$= A^{-1} - A^{-1} B [B^T A^{-1} B + C^{-1}]^{-1} B^T A^{-1}. \quad (3.19)$$

By using Theorem 3.2 we can rewrite (3.18) as

$$\hat{a} = \{ (U^T U)^{-1} - (U^T U)^{-1} V^T [V (U^T U)^{-1} V^T + I]^{-1} V^T (U^T U)^{-1} \} [U^T V^T] p, \quad (3.20)$$

where $I$ is the identity matrix.

Since matrix $U$ is a block diagonal matrix, therefore $(U^T U)^{-1}$ can be computed by computing the inverse of each diagonal block individually. If we use the fourth order polynomial in our interpolation, the dimension of $[V (U^T U)^{-1} V^T + I]$ is one fifth of the original coefficient matrix $X$. Thus, by using (3.20) the computational requirement is reduced significantly. From our experiments, it was observed that using the fourth order polynomial gave us nearly optimal results, and that there was not much gain if we increased the order. In view of this, our simulations were all based on fourth order polynomials.
Our algorithm need only one time projection and convolution back-projection process for restoring each image. The dominant computation is to compute the inverse of the matrix $X$ whose dimension is $2N(d + 1) \times 2N(d + 1)$. The computational intensity is reduced by the computational efficient algorithm, and now the computational complexity is proportional to $N^3$. 


Chapter 4

POST-PROCESSING ALGORITHM

As discussed in Section 1.1.3, there are some image restoration techniques which can achieve good performances. They either incorporated available a priori knowledge or used iteration approaches. In this chapter, we take advantage of some of these methods to propose a hybrid post-processing algorithm which can achieve further improvement on the restored image.

We propose two post-processing algorithms based on iteration processes. The first one iterates between the sinogram and the spectral space, and the second iterates between the object and the sinogram space. Both algorithms originate from the Gerchberg-Papoulis iterative reconstruction algorithm (GPIRA) [8][9]. As illustrated in Fig. 4.1, this algorithm imposes space-limiting and band-limiting constraints on the object function and applies them iteratively.

Figure 4.1: [9] Schematic diagram of Gerchberg-Papoulis iteration reconstruction algorithm.

As discussed before (Section 1.1.3 and Section 3.1), the frequency-domain
approach does not work well on the hollow projection problem because most energy of the missing data in the spectral domain is also within the bowtie area. However, after using our proposed interpolation algorithm, the resulting sinogram was not a perfect sinogram yet, and to be more specific, its spectral support was not strictly limited in the bowtie area. By suppressing the components outside the bowtie area, we expect to achieve a further improvement in our restored sinogram, hence in the restored image.

The first algorithm mentioned above consists of the following operations:

(a) Take the 2-D Fourier transform of the given initial sinogram \( x_0(\phi, \rho) \) to obtain \( X_0(\omega_\phi, \omega_\rho) \).

(b) Revise \( X_0 \) by setting all the values outside the bow-tie area to zeros.

(c) Take the inverse 2-D Fourier transform of \( X_0 \), resulting in \( x_1 \).

(d) Revise \( x_1 \) by replacing the values of \( x_1 \) with the available incomplete sinogram.

These four steps are repeated until the process is terminated by using some stopping criterion.

Suppose the implant is convex and compact, then its location and can be calculated from the incomplete sinogram. Further, since the implant is opaque to X-rays, by restraining the values of this part in each reconstructed image to zero we get one more constraint in the object domain. Similarly, iterations are done between the object and the sinogram space (like the iterations between the sinogram and the spatial domain), where FFT and IFFT need to be substituted by projection and the CBP operation. When these two procedures are combined together we have the following post-processing iteration algorithm, the flow chart of which is shown in Fig. 4.2. \( I_1 \) and \( I_2 \) are the iteration numbers of these two iteration processes respectively.
Figure 4.2: Post-processing iteration algorithm.

The overall post-processing algorithm (PPA) consists of the following computation steps: image reconstruction (e.g., by CBP), 2-D Fourier and inverse Fourier Transform, and numerical tomography projection of the reconstructed image to go back to the sinogram domain. Because numerical and systematic errors are invariably introduced in each reconstruction and numerical tomography projection, the convergence of the PPA may not be achieved in practice and, in some cases, the reconstruction quality may even deteriorate with more iterations. This phenomena was observed in our simulation results (Fig. 5.12).
Chapter 5

SIMULATION RESULTS AND DISCUSSIONS

In this chapter, we first present some simulation results to demonstrate the performance of the proposed interpolation method in solving the hollow projection problem. Then we show the improvement of the restored image after the post-processing algorithm. Finally, we show the results of testing our algorithm on a much more difficult situation in CT, which is when the sinogram is corrupted by Poisson noise.

To evaluate the performance of the proposed sinogram restoration technique, visual evaluation is, of course, the most straightforward method. To this end, we display the available incomplete sinograms and the sinograms restored by the interpolation method. Corresponding reconstructions from sinograms are displayed as well. We also plotted profiles along certain lines of the original and restored sinograms and the original and restored images in order to see the differences between them. Two separate implant locations were tested in order to examine the efficiency of our algorithm for different kinds of images or when the implant is located at various surroundings consisting of smooth regions or small structures.

The object used for all simulations presented in this thesis is a real CT head cross section image. This is a big advancement since most previous researchers chose to use a computer generated image which is much simpler than a real one and some images even have the property of circular symmetry.
5.1 Results of Hollow Projection Restoration

In this section, we show the simulation results of the proposed interpolation algorithm applied to the hollow projection problem. To evaluate the performance of the proposed method quantitatively, we defined the relative error of an estimated image in object-space as

\[
RE_o = \frac{1}{\lambda} \frac{\| (f - \hat{f}) \odot Z \|_F}{\| f \odot Z \|_F} \times 100\% \tag{5.1}
\]

where \( Z \) is an indicator matrix such that

\[
Z_{i,j} = \begin{cases} 
0 & \text{if } (i, j) \text{ is within the implant} \\
1 & \text{otherwise.}
\end{cases}
\]

Notation \( \odot \) represents for the Schur-Hadamard (element-by-element) matrix product, \( f \) is the original image, \( \hat{f} \) is the CBP reconstruction from the restored sinogram and \( \lambda \) is a normalization parameter, which is defined as,

\[
\lambda = 1 - \frac{S_{\text{implant}}}{S_{\text{image}}} \tag{5.2}
\]

where \( S_{\text{implant}} \) is the size of the implant and \( S_{\text{image}} \) is the size of the image.

**Experiment 1: Hollow projection reconstruction of the CT image of the head (The implant is at location 1)**

The object used in this simulation is cross-section CT data from a real human head. Fig. 5.1(a) shows a 190 \times 190 pixel square image of the cross-section of the head, and the black block indicates a metal implant. The available incomplete sinogram which has 120 views (from 0 to \( \pi \)) and 190 raysums per view is shown in Fig. 5.1(b). Fig. 5.1(c) shows the ‘naive’ CBP reconstruction in which it is assumed that the entries in the missing part are equal to zero. Not only does the surrounding of the implant suffer, but the whole image is heavily distorted. Hardly any information can be obtained from the reconstructed image. By using the proposed interpolation method, the sinogram was restored, as shown in Fig. 5.1(d). Fig. 5.1(e) displays a reconstruction of the cross-section of the head from the restored sinogram. Visual examination shows that this reconstruction
is a dramatic improvement on that of the hollow projection data. Fig. 5.2 shows the profiles of the sinogram restored in Experiment 1.

Figure 5.1: Experiment 1: (a) original image; (b) incomplete sinogram; (c) 'naive' reconstruction from (b); (d) restored sinogram; (e) image reconstructed from restored sinogram.
Figure 5.2: Experiment 1: Comparison of the profiles of the restored sinograms with those of the original sinogram. (a) $\phi = 1$; (b) $\phi = 61$. The solid black line is the profile of the original sinogram which suppose it is homogeneous in the metal implant area and it contains the same mass as the original image. The dashed blue line is that of the restored sinogram.
Experiment 2: Hollow projection reconstruction of the CT image of the head (The implant is at location 2)

The second experiment is the same as the first one except that the location of the implant is different. Although the size of the implant is the same in the two experiments, the relative error in the second experiment is much larger than the first one. Corresponding profiles of the restored sinogram in this experiment are shown in Fig. 5.4. Table 5.1 lists the numerical values of the relative error in these two experiments.

<table>
<thead>
<tr>
<th></th>
<th>Experiment 1</th>
<th>Experiment 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>$RE_o$</td>
<td>5.6930%</td>
<td>10.9581%</td>
</tr>
</tbody>
</table>

After doing the previous two experiments we noticed that the location of the implant plays a very important role. As in Experiment 1 and 2, although the implant size is the same, the error of the restored image in Experiment 1 is much smaller than that in Experiment 2. This is because the implant is located at a smooth area in Experiment 1 while in Experiment 2 it is at a area full of small structures. From the profiles of the restored sinogram (see Fig. 5.2 and Fig. 5.4), it can be seen that the amplitudes of the restored sinogram closely trace those of the original sinogram when the missing part of data are smoother, i.e., when they do not fluctuate very fast.

Experiment 3: Different implant size (at location 1 and 2)

Experiment 3 illustrates how the size and location influence the quality of the restored image. We enlarged the missing part step by step and displayed it in different gray values. Two locations were tested as shown in Fig. 5.5(a) and (b) respectively. The simulation results are given in Table 5.2. The results of these experiments demonstrate that an image can be recovered with a higher quality if the implant is located within a smooth area. If the implant is in an area of small structures, it is harder to recover the image.
Figure 5.3: Experiment 2: (a) original image; (b) incomplete sinogram; (c) 'naive' reconstruction from (b); (d) restored sinogram; (e) image reconstructed from restored sinogram.
Figure 5.4: Experiment 2: Comparison of the profiles of the restored sinogram with those of the original sinogram (a) $\phi = 1$; (b) $\phi = 61$. The solid black line is the profile of the original sinogram which suppose it is homogenous in the metal implant area and it contains the same mass as the original image. The dashed blue line is that of the restored sinogram.
Figure 5.5: Experiment 3: (a),(b) original image with multiple implants on it which indicate the different sizes and locations performed in this experiment.

Table 5.2: Relative Error In Experiment 3.

<table>
<thead>
<tr>
<th>Missing Size</th>
<th>( RE_o(a) )</th>
<th>( RE_o(b) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>6( \times ) 6</td>
<td>3.6418%</td>
<td>4.4787%</td>
</tr>
<tr>
<td>12( \times ) 12</td>
<td>4.0407%</td>
<td>8.2432%</td>
</tr>
<tr>
<td>18( \times ) 18</td>
<td>5.2577%</td>
<td>10.2066%</td>
</tr>
<tr>
<td>24( \times ) 24</td>
<td>6.4304%</td>
<td>11.9826%</td>
</tr>
<tr>
<td>30( \times ) 30</td>
<td>7.8509%</td>
<td>15.0368%</td>
</tr>
<tr>
<td>36( \times ) 36</td>
<td>9.2759%</td>
<td>15.5149%</td>
</tr>
<tr>
<td>42( \times ) 42</td>
<td>12.3146%</td>
<td>16.7522%</td>
</tr>
<tr>
<td>48( \times ) 48</td>
<td>16.2116%</td>
<td>17.5778%</td>
</tr>
</tbody>
</table>

In order to present a clear view of the differences between the restored image and the original image, and show how they vary in different parts of the image, profiles of several restored images in Experiment 3 are shown in Fig. 5.6 -- Fig. 5.11. In these figures, four profiles of the restored image are plotted along the lines \( y = 50 \), \( y = 70 \), \( y = 105 \) and \( y = 140 \) respectively, corresponding profiles in the original image are also plotted for comparison. The black and blue lines distinguish the profiles of the original and restored images. We selected these four lines for the purpose of covering all the features of the image. Line \( y = 70 \) and line \( y = 140 \) cross the metallic implant in location 1 and 2 respectively. While line \( y = 50 \) and line \( y = 105 \) do not go through any missing data area.
in either case, one is located in an area full of small features, and the other is located in a rather smooth area. Fig. 5.6 to Fig. 5.8 show the profiles on the restored images with metallic implants in location 1, and Fig. 5.9 to Fig. 5.11 shows the profiles on the restored images with metallic implants in location 2. It can be seen that if the implant is small, or if the implant is located in a smooth area, the quality of the restored image is much higher. It can also be observed that the surrounding area of the implant is usually heavily corrupted, and sometimes it is almost impossible to restore.
Figure 5.6: The profiles of the original and reconstructed images at (a) $y = 50$; (b) $y = 70$; (c) $y = 105$; (d) $y = 140$. The metallic implant size is $12 \times 12$. The black line is the profile of the original image and the blue line is that of the reconstructed image from the restored sinogram. The red dotted line indicates the location of the metallic implant which the profile crosses.
Figure 5.7: The profiles of the original and reconstructed images at (a) $y = 50$; (b) $y = 70$; (c) $y = 105$; (d) $y = 140$. The metallic implant size is $24 \times 24$. The black line is the profile of the original image and the blue line is that of the reconstructed image from the restored sinogram. The red dotted line indicates the location of the metallic implant which the profile crosses.
Figure 5.8: The profiles of the original and reconstructed images at (a) $y = 50$; (b) $y = 70$; (c) $y = 105$; (d) $y = 140$. The metallic implant size is $36 \times 36$. The black line is the profile of the original image and the blue line is that of the reconstructed image from the restored sinogram. The red dotted line indicates the location of the metallic implant which the profile crosses.
Figure 5.9: The profiles of the original and reconstructed images at (a) $y = 50$; (b) $y = 70$; (c) $y = 105$; (d) $y = 140$. The metallic implant size is $12 \times 12$. The black line is the profile of the original image and the blue line is that of the reconstructed image from the restored sinogram. The red line indicates the location of the metallic implant which the profile crosses.
Figure 5.10: The profiles of the original and reconstructed images at (a) $y = 50$; (b) $y = 70$; (c) $y = 105$; (d) $y = 140$. The metallic implant size is $24 \times 24$. The black line is the profile of the original image and the blue line is that of the reconstructed image from the restored sinogram. The red line indicates the location of the metallic implant which the profile crosses.
Figure 5.11: The profiles of the original and reconstructed images at (a) $y = 50$; (b) $y = 70$; (c) $y = 105$; (d) $y = 140$. The metallic implant size is $36 \times 36$. The black line is the profile of the original image and the blue line is that of the reconstructed image from the restored sinogram. The red line indicates the location of the metallic implant which the profile crosses.
5.2 Restoration Improvement After Post-Processing

Before applying the post-processing algorithm, we had to decide the extent of the bowtie-shaped spectral support of the sinogram. This region specifies the positions where the Fourier transform of a sinogram are known to be zeros. For all experiments in this study, we assumed that the largest spatial extent of the object was equal to the width of the sampling window. Thus, we set the edges of the bowtie (see Fig. 3.2) to have a slope of unity. Consequently, this region occupies approximately 50% of the whole spectrum and contains more than 99.99% of the spectral energy of the sinogram.

Experiment 4: Post-processing

In our post-processing experiment we used the restored image from Experiment 2 as the initial image. The post-processing algorithm includes two iterative processes as described in Chapter 4. The iteration numbers of these two iterative processes are referred to as $I_1$ and $I_2$ (see Fig. 4.2). Fig. 5.12 illustrates the errors obtained after post-processing with various $I_1, I_2$; it can be seen that $I_1 = 4$ and $I_2 = 3$ gave the optimal result. More iterations may degrade the quality of the restored image because numerical errors are unavoidably introduced and accumulated in each projection and backprojection in the iteration, and system noise will also be introduced in a real X-ray system. After post-processing the relative error drops from 10.9581% to 10.1697%. Visually the image becomes sharper and more details can be distinguished, as shown by a comparison between Fig 5.13(b) and Fig 5.13(a).
Figure 5.12: Errors after performing post-processing.

Figure 5.13: Comparison of the resultant image after the post-processing with the restored image in Experiment 2, (a) restored image before performing post-processing (b) improved image after performing post-processing.
5.3 Performance on Noisy Sinogram

In an X-ray imaging system, the dominant noise is Poisson distributed because the emission of photons from the X-ray source is a Poisson process. Its probability density is

\[ P_k = \frac{N_0^k e^{-N_0}}{k!} \quad (5.3) \]

where \( P_k \) is the probability, in a given time interval, of emitting \( k \) photons, and \( N_0 \) is the average number of photons emitted during that interval. As we reduce the dose of X-rays in order to minimize the possibility of radiation damage to the human body, the noise will become relatively larger and the signal-to-noise ratio (SNR) decreases correspondingly. Therefore, whether or not the restoration algorithm is robust to noise is a very important factor which should be considered.

**Experiment 5: Noisy hollow projected sinogram**

The two cases used in Experiment 1 and 2 were tested, but the available incomplete sinograms were corrupted by noise. We simulated the real situation and generated each element in the sinogram by a Poisson process. The mean of the Poisson process was the value of each element in the noise-free sinogram. The SNR is defined as

\[ SNR = 10\log_{10} \left( \frac{\|x\|_F}{\|x - \tilde{x}\|_F} \right) \quad (5.4) \]

where \( x \) is the noise-free sinogram and \( \tilde{x} \) is the noisy sinogram. By adjusting the parameter (the mean) of the Poisson process, we generated a series of sinograms with different SNR. Three simulations for each implant location are displayed in Fig. 5.14 to Fig. 5.19. The SNR of the incomplete sinograms are around 10dB, 15dB and 20dB respectively.
Figure 5.14: Experiment 5: (a) original noisy image; (b) incomplete sinogram corrupted by noise (SNR=10.8dB); (c) 'naive' reconstruction from (b); (d) restored sinogram; (e) image reconstructed from restored sinogram.
Figure 5.15: Experiment 5: (a) original noisy image; (b) incomplete sinogram corrupted by noise (SNR=15.4dB); (c) 'naive' reconstruction from (b); (d) restored sinogram; (e) image reconstructed from restored sinogram.
Figure 5.16: Experiment 5: (a) original noisy image; (b) incomplete sinogram corrupted by noise (SNR=19.5dB); (c) 'naive' reconstruction from (b); (d) restored sinogram; (e) image reconstructed from restored sinogram.
Figure 5.17: Experiment 5: (a) original noisy image; (b) incomplete sinogram corrupted by noisy (SNR=10.8dB); (c) 'naive' reconstruction from (b); (d) restored sinogram; (e) image reconstructed from restored sinogram.
Figure 5.18: Experiment 5: (a) original noisy image; (b) incomplete sinogram corrupted by noise (SNR=15.4dB); (c) 'naive' reconstruction from (b); (d) restored sinogram; (e) image reconstructed from restored sinogram.
Figure 5.19: Experiment 5: (a) original noisy image; (b) incomplete sinogram corrupted by noise (SNR=19.5dB); (c) 'naive' reconstruction from (b); (d) restored sinogram; (e) image reconstructed from restored sinogram.
In the presence of noise, the elements in the sinogram generated each time (for fixed parameters in the simulation program) were not the same. As a result, the SNR of each noisy sinogram also varied. Because only the statistical properties in random processes are meaningful, we did each simulation several times. The resultant SNR, as well as the error of the reconstructed image restored from noisy sinogram, are plotted in Fig. 5.20. The least squares method was used to find out the curves (as shown in Fig. 5.20) which fitted in those dots best.

![Graph showing error of reconstructed image vs SNR](image)

**Figure 5.20**: Statistic result of the error of the image reconstructed from the restored noisy sinogram. The black and blue solid lines are the errors of the reconstructed image from the restored sinogram for implant location 1 and 2 respectively. The black and blue dashed lines on the top of the graph are the corresponding errors of the ‘naive’ reconstructions. The blue dashdot line is the error of the reconstructed image from complete noisy sinogram.

From the above graph, it can be seen that at low SNR, the error caused by noise dominates the quality of the reconstructed image, and that the difference between the reconstructions of the two implant locations becomes smaller and sometimes the two lines merge. Also the error of the restored image is close to that of the ‘naive’ reconstruction. The effectiveness of our sinogram restoration
technique is severely affected at low SNR. As the SNR increases, the error of the ‘naive’ reconstruction remains very large, but the error of the reconstruction from the restored sinogram drops quickly. When SNR is above 15dB, the error due to the missing data becomes important. The blue dashdot line in Fig. 5.20 shows the error due to the noise on the sinogram. If SNR is less than 15dB, our algorithm is corrupted heavily by noise but, in this situation, even the image reconstructed from complete sinogram will have lost most of the information. However, above 18dB our algorithm works well. We can conclude, that as long as the noise is not unreasonably large, the relative error increases only slightly. This means that, in normal noisy environments, our algorithm still can achieve satisfactory results.

The LS solution of the resultant errors corresponding to SNRs are listed in Table 5.3. They are also compared with the errors resulting from the clean sinogram (SNR = ∞).

Table 5.3: Comparison of Relative Errors resulted from the noisy sinograms and the clean sinogram.

<table>
<thead>
<tr>
<th>SNR (dB)</th>
<th>REo(a)</th>
<th>REo(b)</th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
<td>63.8742%</td>
<td>60.8470%</td>
</tr>
<tr>
<td>12</td>
<td>47.6591%</td>
<td>44.7560%</td>
</tr>
<tr>
<td>14</td>
<td>32.5723%</td>
<td>32.0720%</td>
</tr>
<tr>
<td>16</td>
<td>21.5953%</td>
<td>22.0354%</td>
</tr>
<tr>
<td>18</td>
<td>15.2066%</td>
<td>14.5001%</td>
</tr>
<tr>
<td>20</td>
<td>12.3427%</td>
<td>9.5327%</td>
</tr>
<tr>
<td>22</td>
<td>11.3593%</td>
<td>7.0122%</td>
</tr>
<tr>
<td>24</td>
<td>10.9917%</td>
<td>6.2293%</td>
</tr>
<tr>
<td>∞</td>
<td>10.9581%</td>
<td>5.4861%</td>
</tr>
</tbody>
</table>
Chapter 6

CONCLUSIONS

In this chapter we summarize the results and the contributions made by our research, and give a conclusion. Further, we discuss some ideas which we are now studying or will study in our future research.

6.1 Conclusions

We proposed an interpolation method and a computationally efficient algorithm for the hollow projection problem in computer tomography. It was demonstrated, via computer simulations, that this method gives much better reconstructed images than the 'naive' reconstructions. It is a very useful and practical method for the HPP from the perspective of the quality of the reconstructed image, the computational complexity, and the stability when there is measurement noise. It should be noted that the resulting relative error is highly dependent on the characteristics of the image and the location of the implant.

Our algorithm treats the HPP directly and individually, unlike other restoration techniques such as those proposed in [12], [15]. These other techniques are general methods dealing with any kind of missing data problems. Our algorithm takes advantage of the characteristics of the missing data in this problem, and we do not assume any a priori knowledge. In many practical applications in medical imaging, for example in arthroplastic surgery, in which we want to determine, with the help of CT, the amount of bone that has grown around metallic implants, we do not have enough a priori information about the object.
to be reconstructed. In this situation, our algorithm shows great advantages.

6.2 Future Research Directions

Two research directions are presented in the following two subsections. The first one is wavelet applications, which are important research topics nowadays. Many researchers are endeavouring to apply this newly developed theory to medical imaging. The second research area of interest is applying the total least square approach to our problem. It has great potential in giving a better result because the collected data are not accurate but corrupted by system noise.

6.2.1 Wavelet Applications

Wavelets have received much attention from the engineering community in the past few years. Wavelets can be used in a multiresolution representation using orthogonal basis functions. In medical imaging, such an algorithm can help reduce the dose of radiation to the patient is exposed to by reconstructing the targeted regions at a high resolution, and the surrounding area at a lower resolution. This algorithm can also be used interactively to precisely locate a particular organ (e.g., a blood vessel in the liver). That is, by creating a series of reconstructions that progressively increase resolution over a localized region of decreasing size, the algorithm allows the zooming into the organ. All in all, the advantage we can gain from applying wavelets in CT technology is that we can reconstruct different parts of an image with different resolutions.

A multiresolution tomographic reconstruction algorithm was proposed in [39]. This algorithm is similar to the conventional filtered backprojection algorithm, except that the filters are now angle dependent, and the backprojection gives the wavelet coefficients of the reconstruction which are then used to synthesize the reconstruction at various resolution levels.

Using this technique, we would first analyze the effect of different parts of the sinogram on the reconstructed image. Further, we expect to get a better
reconstruction of some parts of the image by reconstructing just that part, since, by partially reconstruction, we may minimize the adverse effects caused by the missing data. For the part of the image that is near the opaque object, which is severely deteriorated by the missing data or inaccurate interpolation and cannot be restored, we may want to use a lower resolution in order to reduce the computation complexity.

6.2.2 Total Least Squares Solutions

In the least squares (LS) problem we are given an $m \times n$ ‘data matrix’, a ‘vector of observations’ $b$ having $m$ components, and a nonsingular weighting matrix $D = \text{diag}(d_1, \cdots, d_m)$, and try to find a vector $x$ to minimize

$$\|D(b - Ax)\|_2.$$  

For the convenience of further discussion, the ordinary LS is recast as follows:

$$\min_{b+r \in \text{Range}(A)} \|D r\|_2.$$  

In this problem there is an underlying assumption that all the errors are confined to the observation vector $b$. When error is present in the ‘data’ $A$, it is natural to consider perturbations of both $b$ and $A$ which can be formulated as:

$$\min_{b+r \in \text{Range}(A+E)} \|D[E, r]T\|_F, \quad E \in \mathbb{R}^{m \times n}, r \in \mathbb{R}^n$$

where $D = \text{diag}(d_1, \cdots, d_m)$ and $T = \text{diag}(t_1, \cdots, t_{n+1})$ are nonsingular weighting matrices. This problem, discussed in Golub and Van Loan [41], is referred to as the total least square (TLS) problem. Fig. 6.1 is the geometric interpretation of the difference between OLS and TLS.

A theoretical solution is also given by [41]. We are considering using TLS to solve the HPP because of two reasons. The first is that after our interpolation procedure, errors will be introduced into our least square problem, not only into
Figure 6.1: Ordinary and total least-squares approaches.

$b$ but also into $A$. The second reason is that noise is always a factor which must be considered. By using the TLS solution, we expect that our algorithm will be more robust to noise.

Although the TLS solution exists in theory, the required conditions are difficult to satisfy in our problem. This is mainly because the size of our matrix is too big to be effectively manipulated. When we take SVD in order to find the singular values, the singular values vary so slowly that it is very difficult to set a threshold after which we can claim the singular values are the same. Finding an efficient way to break down this huge matrix becomes an issue when endeavouring to make the TLS solution feasible in our problem.
REFERENCES


