On the oscillation of $f''' + e^{-z}f' + B(z)f = 0$
and $f''' + gf' + hf = 0$ where $B(z), g$ and $h$ are entire, and $\rho(h) < \rho(g) \leq \frac{1}{2}$

By

M.H. Lo, B.Sc. (Mathematics)

A Thesis Presented to
The Hong Kong University of Science and Technology
in Partial Fulfillment of the Requirements for
the Degree of Master of Philosophy in Mathematics

Hong Kong, January 1997

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\[ \rho(h) < \rho(g) \leq \frac{1}{2} \]

By

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_Soli Deo gloria._
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Abstract

In 1982, Steven B. Bank and Ilpo Laine had written a paper entitled "On the oscillation theory of $f'' + Af = 0$ where $A$ is entire". Since then, quite a number of research articles have been concerned with the exponent of
convergence of zeros and the order of the non-trivial solutions $f$.

One of the reasons why people are very interested in studying the above equation is because of the existence of the so-called Bank-Laine equation. Unfortunately, such equation does not seem to exist for third or even higher order differential equations. However, can we still generalize the results that were obtained in the second order case to higher order case? In this thesis, we will provide some answers to the third order case.
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Introduction

In 1982, Steven B. Bank and Ilpo Laine had written a paper entitled "On the oscillation theory of $f'' + Af = 0$ (1) where $A$ is entire"[1]. Since then, quite a number of research articles have been concerned with the exponent of convergence of zeros $\lambda(f)$ and the order $\rho(f)$ of the non-trivial solutions $f$.

If we let $f_1$ and $f_2$ be two linearly independent solutions of the equation (1) and we also let $E = f_1 f_2$. By considering the Wronskian of $f_1$ and $f_2$, we obtain that

$$-4A = (c/E)^2 - (E'/E)^2 + 2(E''/E)$$

where $c$ is a non-zero constant. Many results were and can be obtained by using the above relationship between $A$ and $E$. Two natural questions are: 1. Can we obtain a similar formula for the third order, or even higher order differential equations? 2. Can we generalize the results that were obtained in the second order case to higher order case?

The answer to the first question above seems to be NO. For the second one, I will give some answers to the third order case in this thesis.

For the convenience of the reader, this thesis is divided into three parts: Chapter 1 gives briefly an account of Nevanlinna Theory, its properties and some definitions such as order, exponent of convergence of zeros, which will be used throughout the entire thesis.

Chapter 2 introduces some lemmas and theorems which will be needed to prove our results in Chapter 3.

Chapter 3 will give the proofs of our theorems and propositions. The theorems we are going to prove can be divided into two groups: Theorem 3.1,
Propositions 3.1 to 3.3, Theorem 3.2 and Theorem 3.3 deals with the equation \( f''' + e^{-z} f' + B(z)f = 0 \) where \( B(z) \) is entire. Theorem 3.4 deals with the equation \( f''' + gf' + hf = 0 \) where \( g, h \) are entire and \( \rho(h) < \rho(g) \leq \frac{1}{2} \).
Chapter 1

Nevanlinna Theory

The theory of the distribution of values of meromorphic function developed by R. Nevanlinna, now known as Nevanlinna Theory, was one of the most outstanding achievements in function theory this century. A function is said to be meromorphic in a region if it is analytic in the region except at a finite number of poles. An entire function may be considered as a meromorphic function which does not take the value $\infty$. In this thesis, unless explicitly stated, by a meromorphic(entire) function we always mean functions which are meromorphic(entire) in the complex plane. Only a brief account of the Theory is presented here; for a more comprehensive treatment, the reader is referred to the excellent monograph of Hayman[9].

The starting point of the theory is the Poisson-Jensen formula:

**Theorem 1.1** Suppose $f(z)$ is meromorphic in $|z| < r \leq \infty$ and have zeros $a_1, a_2, \ldots, a_N$, and poles $b_1, b_2, \ldots, b_M$ in $|z| < r$, repeated according to multiplicity. If $f(z) \neq 0, \infty$ and $z = \rho e^{i\theta}$ is a point in $|z| < r$, then
\[
\log |f(z)| = \frac{1}{2\pi} \int_0^{2\pi} \log |f(re^{i\phi})| \frac{r^2 - \rho^2}{r^2 - 2r\rho \cos(\theta - \phi) + \rho^2} d\phi
\]

\[
+ \sum_{i=1}^{N} \log \frac{r(z - a_i)}{r^2 - a_iz} - \sum_{j=1}^{M} \log \frac{r(z - b_j)}{r^2 - b_jz}
\]

When \( z = 0 \), this yields the Jensen’s formula:

\[
\log |f(0)| = \frac{1}{2\pi} \int_0^{2\pi} \log |f(re^{i\phi})| d\phi
\]

\[
- \sum_{i=1}^{N} \log \frac{r}{|a_i|} + \sum_{j=1}^{M} \log \frac{r}{|b_j|}.
\]

Define \( n(t, \frac{1}{f}) \) and \( n(t, f) \) as the number of zeros and poles in \( |z| \leq t \) respectively, counted according to multiplicity. If \( f(z) \) is as in Theorem 1.1 and \( t_1, t_2, \ldots, t_N \) are the moduli of the zeros \( a_1, a_2, \ldots, a_N \) respectively, then

\[
\sum_{i=1}^{N} \log \frac{r}{|a_i|} = \sum_{i=1}^{N} \log \frac{r}{t_i} = \int_0^{r} \log \frac{r}{t} \, dc(t, \frac{1}{f})
\]

\[
= \left[ \log \frac{r}{t} \, n(t, \frac{1}{f}) \right]_0^{r} - \int_0^{r} n(t, \frac{1}{f}) \, d \log \frac{r}{t}
\]

\[
= \int_0^{r} n(t, \frac{1}{f}) \, dt = N(r, \frac{1}{f}),
\]

where we have applied integration by parts to the Stieltjes integral. Similarly,

\[
\sum_{j=1}^{M} \log \frac{r}{|b_j|} = \int_0^{r} \frac{n(t, f)}{t} \, dt = N(r, f)
\]
where \( N(r, f) \) is called the counting function of \( f \).

Therefore, the Jensen’s formula becomes

\[
\log |f(0)| = \frac{1}{2\pi} \int_0^{2\pi} \log |f(re^{i\phi})| d\phi - N(r, \frac{1}{f}) + N(r, f). 
\]

We further define

\[
m(r, f) = \frac{1}{2\pi} \int_0^{2\pi} \log^+ |f(re^{i\phi})| d\phi
\]

where \( m(r, f) \) is called the proximity function of \( f \) and

\[
\log^+ x = \max(\log x, 0)
\]

Note that \( \log x = \log^+ x - \log^+ \frac{1}{x} \) for all \( x > 0 \). This gives,

\[
\frac{1}{2\pi} \int_0^{2\pi} \log |f(re^{i\phi})| d\phi = \frac{1}{2\pi} \int_0^{2\pi} \log^+ |f(re^{i\phi})| d\phi - \int_0^{2\pi} \log^+ \frac{1}{|f(re^{i\phi})|} d\phi
\]

\[
= m(r, f) - m(r, \frac{1}{f}).
\]

Putting this into the Jensen’s formula, we get

\[
m(r, f) + N(r, f) = m(r, \frac{1}{f}) + N(r, \frac{1}{f}) + \log |f(0)|
\]

Define \( T(r, f) = m(r, f) + N(r, f) \) to be the characteristic function of the function \( f(z) \). Then the above equation becomes

\[
T(r, f) = T(r, \frac{1}{f}) + \log |f(0)|
\]
By some basic calculation, we can show that $T(r, f)$ is a continuous, increasing and logarithmic convex function of $r$.

By using the elementary inequalities:

$$\log^+ \left( \sum_{i=1}^{p} x_i \right) \leq \sum_{i=1}^{p} \log^+ x_i + \log p$$

and

$$\log^+ (x_1 \cdot x_2 \cdots x_p) \leq \sum_{i=1}^{p} \log^+ x_i$$

We deduce immediately for meromorphic functions $f_i (i = 1, 2, \ldots, p)$:

$$m(r, \sum_{i=1}^{p} f_i) \leq \sum_{i=1}^{p} m(r, f_i) + \log p,$$

$$m(r, f_1 \cdot f_2 \cdots f_p) \leq \sum_{i=1}^{p} m(r, f_i),$$

$$N(r, \sum_{i=1}^{p} f_i) \leq \sum_{i=1}^{p} N(r, f_i),$$

$$N(r, f_1 \cdot f_2 \cdots f_p) \leq \sum_{i=1}^{p} N(r, f_i).$$

Therefore
\[ T(r, \sum_{i=1}^{p} f_i) \leq \sum_{i=1}^{p} T(r, f_i) + \log p , \]

\[ T(r, f_1 \cdot f_2 \cdots f_p) \leq \sum_{i=1}^{p} T(r, f_i) . \]

Replacing \( f \) by \( f - a \) in the characteristic function \( T(r, f) \), we obtain

\[ T(r, f - a) = T(r, \frac{1}{f - a}) + \log |f(0) - a| \]

We simply state, without proof, the First and Second Fundamental Theorem of Nevanlinna Theory:

**Theorem 1.2 (First Fundamental Theorem)** Let \( f(z) \neq 0 \) be meromorphic in \( |z| < R \leq \infty \). If \( a \) is an arbitrary complex number and \( 0 < r < R \), then

\[ T(r, \frac{1}{f - a}) = T(r, f) - \log |f(0) - a| + \epsilon(r, a) , \]

where \( \epsilon(r, a) \leq \log^+ |a| + \log 2 \).

The function \( N(r, \frac{1}{f - a}) \) is really a weighted measure of the density of the \( a \)-points of \( f(z) \) in \( |z| < r \), while \( m(r, \frac{1}{f - a}) \) is in some sense an inverse measure of the mean deviation of \( f(z) \) from \( a \) on \( |z| = r \). The theorem means that the sum of these two quantities remains the same up to a bounded term, i.e. for sufficiently large \( r \)

\[ T(r, \frac{1}{f - a}) = T(r, f) + O(1) . \]
Theorem 1.3 (Second Fundamental Theorem) Suppose that $f(z)$ is a non-constant meromorphic function in $|z| < R$. If $a_i (i = 1, 2, \ldots, q)$ are $q(\geq 2)$ finite, distinct complex numbers such that $|a_i - a_j| \geq \delta$ for $1 \leq i, j \leq q$. If $f(0) \neq 0, \infty$ and $f'(0) \neq 0$, then we have for every $r \in (0, R)$

$$m(r, f) + \sum_{i=1}^{q} m\left(r, \frac{1}{f - a_i}\right) \leq 2T(r, f) - N_1(r, f) + S(r, f),$$

where $N_1(r) = N\left(r, \frac{1}{f}\right) + 2N(r, f) - N(r, f')$ and

$$S(r, f) = m\left(r, \frac{f'}{f}\right) + m\left(r, \sum_{i=1}^{q} \frac{f'}{f - a_i}\right) + q \log^+ \frac{3q}{\delta} + \log 2 + \log \frac{1}{|f'(0)|}.$$

We introduce a number of definitions which are as follow:

Let $\{a_n\}$ be a sequence of non-zero complex numbers whose moduli tending to infinity. Suppose $\{a_n\}$ are the zeros of the function $f$, then the exponent of convergence of zeros of $f$ is defined as:

$$\lambda(f) = \inf\{q : \sum_{n=1}^{\infty} \frac{1}{|a_n|^q} < \infty, \ q \text{ is a positive real number}\}$$

The order $\rho(f)$ and lower order $\mu(f)$ of an entire function $f$ are defined by:

$$\rho(f) = \lim_{r \to \infty} \log \log M(r, f) / \log r \quad \text{and} \quad \mu(f) = \lim_{r \to \infty} \log \log M(r, f) / \log r$$

where

$$M(r, f) = \max_{|z|=r} |f(z)| \quad \text{and} \quad L(r, f) = \min_{|z|=r} |f(z)|.$$
For a set $E \subseteq [1, \infty)$, we define the linear measure of $E$ by

$$m(E) = \int_0^\infty \chi_E(t) dt$$

where $\chi_E(t)$ is the characteristic function of the set $E$, and we also define the logarithmic measure of $E$ by

$$m_l(E) = \int_0^\infty \frac{\chi_E(t)}{t} dt.$$

The upper density and upper logarithmic density of $E$ are defined by

$$\overline{dens} E = \lim_{r \to \infty} \frac{m(E \cap [1, r])}{r - 1} \quad \text{and} \quad \overline{logdens} E = \lim_{r \to \infty} \frac{m_l(E \cap [1, r])}{\log r}$$

The lower density and lower logarithmic density, $\underline{dens} E$ and $\underline{logdens} E$, are defined similarly with $\lim \sup$ replaced by $\lim \inf$. It is easy to verify that

$$0 \leq \underline{dens} E \leq \underline{logdens} E \leq \overline{logdens} E \leq \overline{dens} E \leq 1$$

for any $E \subseteq [1, \infty)$. 

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Chapter 2

Lemmas and Theorems

We will list a set of lemmas and theorems which will be used in our proofs in Chapter 3.

Theorems A, B, C and D will be used to prove Theorems 3.1, Propositions 3.1 to 3.3 and Theorem 3.2.

Theorem A is similar to Theorem 7 of [7, P.419].

**Theorem A.** Let \( \{\phi_k\} \) and \( \{\theta_k\} \) be two finite collections of real numbers that satisfy \( \phi_1 < \theta_1 < \phi_2 < \theta_2 < \cdots < \phi_n < \theta_n < \phi_{n+1} \) where \( \phi_{n+1} = \phi_1 + 2\pi \), and set

\[
\mu = \max_{1 \leq k \leq n} (\phi_{k+1} - \theta_k).
\]

Suppose that \( A(z) \) and \( B(z) \) are entire functions such that for some constants \( \alpha \geq 0 \),

\[
|A(z)| = O(|z|^\alpha)
\]
as \( z \to \infty \) in \( \phi_k \leq \text{arg} \, z \leq \theta_k \) for \( k = 1,2,\cdots,n \), and where \( B(z) \) is transcendental with \( \rho(B) < \pi/\mu \). Then every solution \( f \neq 0 \) of the differential
\begin{equation}
    f''' + A(z)f' + B(z)f = 0
\end{equation}

has infinite order.

**Proof** Suppose that \( f \not\equiv 0 \) is a solution of the above differential equation and assume that the order of \( f \) is finite. By Lemma 1(i) of [7], there exists a set \( E \subset [0, 2\pi) \) that has linear measure zero, such that for any \( \psi \in [0, 2\pi) - E \), there is a constant \( R = R(\psi) > 0 \) so that for all \( z \) satisfying \( \arg z = \psi \) and \( |z| \geq R \), we have the inequalities:

\[
    |\frac{f''(z)}{f(z)}| \leq |z|^\rho - 1 + \epsilon \quad \text{and} \quad |\frac{f'''(z)}{f(z)}| \leq |z|^{3(\rho - 1 + \epsilon)}
\]  
(2.1)

where \( \epsilon > 0 \) is a small number.

Rewrite the differential equation as follows:

\[
    B(z) = -\left( \frac{f'''(z)}{f(z)} + A(z)\frac{f''(z)}{f(z)} \right)
\]  
(2.2)

By (2.1), (2.2) and our assumption, we will have:

\[
    |B(z)| \leq |\frac{f'''}{f}| + |A(z)||\frac{f''}{f}| = O(|z|^\rho + 3(\rho - 1 + \epsilon))
\]  
(2.3)

Now, replace (10.1) and (10.2) of [7, P.426] by (2.1) and (2.3) respectively. Then apply the same argument as in [7, P.426] and we have proved the theorem.

**Theorem B.** [12, P.84-98] All non-trivial solutions \( f \) of

\[
    f'' + e^{P(z)}f = 0
\]

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where \( P(z) \) is a non-constant polynomial, satisfy \( \lambda(f) = \infty \).

Theorems C and D will be used to prove Theorem 3.2.

Let \( f \) be an entire function whose Taylor expansion is

\[
f = \sum_{n=0}^{\infty} a_n z^n
\]

Clearly, the power series \( \sum_{n=0}^{\infty} |a_n| r^n \) converges for every \( r > 0 \). Then, for a given \( r > 0 \)

\[
\lim_{n \to \infty} |a_n| r^n = 0
\]

and the maximum term

\[
\mu(r) = \mu(r, f) = \max_{n \geq 0} |a_n| r^n
\]

is well-defined. We define the central index \( v(r) = v(r, f) \) as the greatest exponent \( m \) such that \( |a_m| r^m = \mu(r, f) \).

**Theorem C.** [12, P.51] If \( f \) is an entire function of order \( \sigma \), then

\[
\sigma = \lim_{r \to \infty} \frac{\log^+ v(r, f)}{\log r} = \lim_{r \to \infty} \frac{\log^+ \log^+ \mu(r, f)}{\log r}.
\]

Theorem D below is a simplified version in a paper by Ishizaki [11, Theorem 1.1].

**Theorem D.** Suppose that the equation

\[
w'' + g_1(z) w' + g_0(z) w = 0
\]

where \( g_1(z) \) and \( g_0(z) \) are meromorphic in the complex plane, and it has a fundamental set of linearly independent meromorphic solutions \( \{w_1, w_2, w_3\} \). If
\[ \rho(w_1) > \rho(w_2), \] then we have

\[ \rho(w_1) = \rho(w_3) > \rho(w_2). \]

Lemma 2.1 will be used to prove Theorem 3.3.

For a non-constant meromorphic function \( f \), if it satisfies the condition

\[ \lim_{r \to \infty} \frac{\log T(r, f)}{r} = 0 \]

then \( f \) is said to be subnormal [8].

**Lemma 2.1** [8, Lemma 4.1] Consider an \( n^{\text{th}} \) order linear differential equation of the form

\[ P_0(e^z, e^{-z})f^{(n)} + P_1(e^z, e^{-z})f^{(n-1)} + \cdots + P_n(e^z, e^{-z})f = P_{n+1}(e^z, e^{-z}) \]

where each \( P_j(z, w) \) is a polynomial in \( z \) and \( w \) with \( P_0(z, w) \neq 0 \). Suppose that \( f = \phi(z) \) is an entire subnormal solution of the above equation. If \( \phi \) is periodic with period \( 2\pi i \), then

\[ \phi(z) = S_1(e^z) + S_2(e^{-z}) \]

where \( S_1(z) \) and \( S_2(z) \) are polynomials in \( z \).

Theorems E, F and G, and Lemmas 2.2 to 2.7 will be used to prove Theorem 3.4.

Theorem E is a simplified version of [2, Theorem 4]
Theorem E. For an entire function $g$, define
\[ E(\alpha, \sigma) = \{ r : L(r, g) > M(r, g) \cos \pi \alpha > r^\sigma \} \]

$0 \leq \lambda < \frac{1}{2}$ and $\lambda < \rho$, then if $\lambda \leq \sigma < \min(\rho, \frac{1}{2})$ and $\sigma < \alpha < \frac{1}{2}$, we have
\[ \overline{\log dens} E(\alpha, \sigma) \geq 1 - \frac{\sigma}{\alpha} \]

Theorem F. (The classical "cos $\pi \rho$ -theorem") [3, P.40-43] Let $f(z)$ be an entire function of order $\rho < 1$, then
\[ \lim_{|z|=r \to \infty} \frac{\log L(r, f)}{\log M(r, f)} \geq \cos \rho \pi. \]

Theorem G is a portion of Theorem 8.1 of [5], when it specialized to the case $\rho = \frac{1}{2}$, it becomes:

Theorem G. Suppose $g$ is an entire function of order $\frac{1}{2}$ and satisfies the condition
\[ \log L(r, g) = o(\log M(r, g)) \quad as \quad r \to \infty \]
Then there exists a set $G$ of logarithmic density 1, a set $H$ of density 0, a real-valued function $\tau = \tau(r)$, and a positive function $\mathcal{L}(r)$ varying slowly in the sense that
\[ \lim_{r \in G} \frac{\mathcal{L}(sr)}{\mathcal{L}(r)} = 1 \]
for all $\sigma > 0$, such that for $r \in G - H$, we have
\[ \log |g(re^{i\tau})| = (\cos(\tau/2) + o(1))r^{1/2}\mathcal{L}(r) \quad as \quad r \to \infty \]
uniformly for $r \in [-\pi, \pi]$.

We let $\Omega_r = \{ z : |z| = r \}$ in Lemma 2.2:

**Lemma 2.2** [10, P.696 - 697] For an entire function $f$ with zeros $\{a_\mu\}$. We choose $r$ so that there is no zero in the circles $\Omega_{r/e}$ and $\Omega_r$. Consider the differentiated Poisson-Jensen formula for $|z| = r < R$:

$$
z f'(z) = f(z) = \frac{1}{2\pi} \int_0^{2\pi} \log |f(Re^{i\theta})| \left( \frac{2zRe^{i\theta}}{(Re^{i\theta} - z)^2} \right) d\beta + \sum_{|a_\mu| < r/e} \left( \frac{z}{z - a_\mu} + \frac{\bar{a}_\mu z}{R^2 - \bar{a}_\mu z} \right) + \sum_{r/e < |a_\mu| < r} \left( \frac{z}{z - a_\mu} + \frac{\bar{a}_\mu z}{R^2 - \bar{a}_\mu z} \right) + \sum_{r < |a_\mu| < R} \left( \frac{z}{z - a_\mu} + \frac{\bar{a}_\mu z}{R^2 - \bar{a}_\mu z} \right) = f_1(z) + f_2(z) + f_3(z) + f_4(z)
$$

For $z = re^{i\theta}$, we have

(i) If $|a_\mu| < r$, then

$$
\text{Re}\left( \frac{re^{i\theta}}{re^{i\theta} - a_\mu} + \frac{\bar{a}_\mu re^{i\theta}}{R^2 - \bar{a}_\mu re^{i\theta}} \right) > 0
$$

and

$$
\frac{1}{2\pi} \int_0^{2\pi} \text{Re}\left( \frac{re^{i\theta}}{re^{i\theta} - a_\mu} + \frac{\bar{a}_\mu re^{i\theta}}{R^2 - \bar{a}_\mu re^{i\theta}} \right) d\theta = 1
$$
(ii) If \( |a_\mu| < r/e \), then

\[
\left| \frac{re^{i\theta}}{re^{i\theta} - a_\mu} + \frac{\bar{a}_\mu re^{i\theta}}{R^2 - \bar{a}_\mu re^{i\theta}} \right| \leq \frac{e + 1}{e - 1} < 4
\]

(iii) For \( r < |a_\mu| < R \), we have

\[
Re\left( \frac{re^{i\theta}}{re^{i\theta} - a_\mu} + \frac{\bar{a}_\mu re^{i\theta}}{R^2 - \bar{a}_\mu re^{i\theta}} \right) < \frac{1}{2} + \frac{r}{R - r}
\]

(iv) When \( R = 3r \)

\[
\frac{1}{2\pi} \int_0^{2\pi} \left| Re \frac{re^{i\theta} f'(re^{i\theta})}{f(re^{i\theta})} \right| d\theta \leq \frac{3}{2} T(R, f) + n(R, 0, f)
\]

\[\leq \frac{5}{2} T(R, f)\]

We remark that our estimate in (iv) is different but more accurate than the estimate in [10, (3.8) in P.697].

**Proof** We only prove (ii) and (iv).

Proof of (ii):

\[
\left| \frac{re^{i\theta}}{re^{i\theta} - a_\mu} + \frac{\bar{a}_\mu re^{i\theta}}{R^2 - \bar{a}_\mu re^{i\theta}} \right| \leq \left| \frac{re^{i\theta}}{re^{i\theta} - a_\mu} \right| + \left| \frac{\bar{a}_\mu re^{i\theta}}{R^2 - \bar{a}_\mu re^{i\theta}} \right|
\]

\[\leq \left| \frac{e}{e - a_\mu} \right| + \left| \frac{1}{R^2 - \bar{a}_\mu re^{i\theta} - 1} \right|
\]

\[\leq \frac{e}{e - 1} + \frac{1}{e - 1} = \frac{e + 1}{e - 1}
\]
Proof of (iv):

\[
\frac{1}{2\pi} \int_0^{2\pi} |Re\ e^{i\theta} f'(re^{i\theta})| d\theta \leq \frac{1}{2\pi} \int_0^{2\pi} \int_0^{2\pi} \log |f(Re^{i\beta})| \frac{2z Re^{i\beta}}{(Re^{i\beta} - z)^2} d\beta d\theta
\]

\[+\frac{1}{2\pi} \int_0^{2\pi} |Re \sum_{|a_\mu|<r} \left( \frac{z}{z - a_\mu} + \frac{\bar{a}_\mu z}{R^2 - \bar{a}_\mu z} \right)| d\theta \]

\[+\frac{1}{2\pi} \int_0^{2\pi} |Re \sum_{r<|a_\mu|<R} \left( \frac{z}{z - a_\mu} + \frac{\bar{a}_\mu z}{R^2 - \bar{a}_\mu z} \right)| d\theta \]

\[= g_1 + g_2 + g_3 \]

Let us first consider \( g_1 \). Since

\[\left| \frac{2z Re^{i\beta}}{(Re^{i\beta} - z)^2} \right| \leq \frac{2Rr}{(R - r)^2} = \frac{3}{2} \]

Therefore \( g_1 = \frac{3}{2} T(R, f) \).

For \( g_2 \), since from (i)

\[Re\left( \frac{re^{i\theta}}{re^{i\theta} - a_\mu} + \frac{\bar{a}_\mu re^{i\theta}}{R^2 - \bar{a}_\mu re^{i\theta}} \right) > 0 \]

So

\[g_2 = \frac{1}{2\pi} \int_0^{2\pi} Re \sum_{|a_\mu|<r} \left( \frac{re^{i\theta}}{re^{i\theta} - a_\mu} + \frac{\bar{a}_\mu re^{i\theta}}{R^2 - \bar{a}_\mu re^{i\theta}} \right) d\theta \]

\[= n(r, 0, f) \]

For \( g_3 \), since

\[g_3 \leq \frac{1}{2\pi} \int_0^{2\pi} \left| \sum_{r<|a_\mu|<R} \left( \frac{1}{2} + \frac{r}{R - r} \right) \right| d\theta \]
\[ = n(R, 0, f) - n(r, 0, f) \]

Therefore
\[
\frac{1}{2\pi} \int_0^{2\pi} |\text{Re} \, r e^{i\theta} f'(re^{i\theta})| \frac{1}{f(re^{i\theta})} \, d\theta \leq \frac{3}{2} T(R, f) + n(R, 0, f)
\]
\[
\leq \frac{5}{2} T(R, f).
\]

**Lemma 2.3** [10, P.697-698] For non-constant entire function \( f \), let \( A(r, f) \) be the average number of solutions of \( f(z) = a \) in \( |z| \leq r \) as \( a \) varies over the Riemann sphere and \( \varphi(r) \) be the mean covering number of the unit circle under the map \( f \) restricted to \( \{ z : |z| \leq r \} \), then there exists a set \( E_1 \) with \( m_1(E_1) < \infty \) such that
\[
\lim_{r \to 1} \frac{\varphi(r)}{A(r, f)} = 1
\]

and
\[
A(r, f) \leq \frac{1 + o(1)}{2\pi} \int_0^{2\pi} (\text{Re} \, r e^{i\theta} \frac{f'(re^{i\theta})}{f(re^{i\theta})})^+ \, d\theta, \ r \notin E_1.
\]

where
\[
\frac{1}{2\pi} \int_0^{2\pi} (\text{Re} \, r e^{i\theta} \frac{f'(re^{i\theta})}{f(re^{i\theta})})^+ \, d\theta = \frac{1}{2\pi} \int_{B_r} \text{Re} \, r e^{i\theta} \frac{f'(re^{i\theta})}{f(re^{i\theta})} \, d\theta
\]

and
\[
B_r = \{ \theta \in [0, 2\pi] : |f(re^{i\theta})| > 1 \}.
\]
Lemma 2.4 [10, Lemma 2] Suppose $f$ is an entire function of order $\lambda < \infty$. Let $n(r) = n(r, \frac{1}{f})$ and suppose $n(1) \geq 1$. For $K > 1$, let

$$E_2(K) = \{r > e : n(r) > Kn(r/e)\}$$

then

$$\frac{\log dens E_2(K)}{\log K} \leq \frac{4\lambda}{\log K}.$$ 

Lemma 2.5 is very similar to Lemma 4 of [10].

Lemma 2.5 Suppose $f$ is an entire function of order $\lambda < \lambda' < \infty$. If $\rho_1 > 0$, there exists a set $E(\rho_1) \subset [1, \infty)$ satisfying

$$m(E(\rho_1) \cap [r/e, er]) < 2r^{\lambda'}e^{-r^{\rho_1}} \text{ for } r > r_0(\lambda')$$

and such that if $|z| = r \notin E(\rho_1)$, then

$$\left| \frac{f'''(z)}{f(z)} \right| < r^{3\lambda'}e^{3(9r)^{\rho_1}} \text{ for } r > r_0(\lambda')$$

In particular, $E(\rho_1)$ has logarithmic density 0.

Proof We apply the same argument in P.700 of [10] to $f'$ and $f''$ then we will obtain the above result.

Lemma 2.6 [10, Lemma 5] Let $f$ be an entire function of finite order $\lambda > 0$. If $\epsilon > 0$, there exists $c(\epsilon) > 0$ and a set $E_3(\epsilon) \subset [1, \infty)$ with lower logarithmic
density at least $2c(\epsilon)$ such that for all $r \in E_3(\epsilon)$, there exists $h = h_r > 0$ such that if $R' = re^h$ then

$$T_0(R', f) < h(e + \epsilon)A(r, f)$$

and for all $K > K_0(\epsilon)$

$$T_0(R', f) < h^2 K \lambda(e + \epsilon)A(r, f).$$

**Lemma 2.7** [10, Lemma 6] Suppose $g$ is an entire function of order $\rho \in (0, \infty)$ and suppose $0 < \rho_2 < \rho_3 < \rho < \rho_4 < \infty$. Suppose $\log L(s, g) > s^{\rho_2}$ where $L(s, g) = \min_{|z|=s} |g(z)|$. For $s < r < 2s$, let

$$C_r = \{\theta \in [0, 2\pi] : \log |g(re^{i\theta})| < r^{\rho_2}\}$$

Then for sufficiently large $s$ we have

$$m(C_r) < s^{\rho_4 - \rho_3 - 1}(r - s).$$
Chapter 3

Theorems and Proofs

For the differential equation

\[ f'' + A(z)f' + B(z)f = 0 \]

where \( A(z) \) and \( B(z) \) are both entire functions of finite order and \( A(z) \) is transcendental. It is well known that all solutions of the above equation are entire functions, and that there exists a solution that has infinite order.

In 1980, M. Ozawa studied the equation

\[ w'' + e^{-z}w' + (az + b)w = 0 \] (3.1)

where \( a \) and \( b \) are constants. If we let \( a = 0 \) and \( b = -1 \), then it is not difficult to see that \( w = e^z + 1 \) is a solution and, in particular, it is a finite order solution. He then asked: Under what conditions will we have a solution of infinite order. He proved that if \( a \neq 0 \), then every solution of (3.1) is of infinite order [14, Theorem 6].
In 1986, G. Gundersen[6] consider the equation

$$f'' + e^{-z} f' + P(z)f = 0 \tag{3.2}$$

where $P(z)$ is an entire function. He proved that if $P(z)$ is either (i) a transcendental entire function with $\text{order}(P) \neq 1$, or (ii) a polynomial of odd degree, then all non-trivial solutions of (3.2) has infinite order. In the same year, J. Langley[13, Theorem 2] proved that if $P(z)$ is a non-constant polynomial, then all non-trivial solutions of (3.2) must have infinite order.

In Theorem 3.1, Propositions 3.1 to 3.3, Theorem 3.2 and Theorem 3.3, I will consider the third order differential equation

$$f''' + e^{-z} f' + B(z)f = 0 \tag{3.3}$$

where $B(z)$ is an entire function.

**Theorem 3.1** If $B(z)$ is a transcendental entire function with $\rho(B) \neq 1$, then all non-trivial solutions $f$ of (3.3) has infinite order.

**Proof** We first show that $\rho(B) < 1$. Assume that $f \neq 0$ is a solution of (3.3). If we rewrite (3.3) as follows:

$$B(z) = -\left( \frac{f'''}{f} + e^{-z} \frac{f'}{f} \right) \tag{3.4}$$

By Nevanlinna's fundamental estimate of the logarithmic derivative, we obtain

$$m(r, B) \leq m(r, e^{-z}) + O(\log r)$$

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as \( r \to \infty \). If \( \rho(B) > 1 \), then we will get a contradiction and so we must have \( \rho(B) \leq 1 \). However, by our assumption, \( \rho(B) \neq 1 \), therefore \( \rho(B) < 1 \).

Since \( e^{-z} \) is bounded in \( Re(z) \geq 0 \). If we let \( \phi_1 = -\frac{\pi}{2}, \theta_1 = \frac{\pi}{2} \) and \( \phi_2 = \phi_1 + 2\pi \), then by Theorem A, every solution \( f \neq 0 \) of (3.3) has infinite order and we have proved the theorem.

We remark that Theorem 3.1 also holds for \( f^{(k)} + e^{-z}f' + B(z)f = 0 \) where \( k \geq 4 \)

In Theorem 3.1, we have solved the case for \( \rho(B) \neq 1 \).

In Propositions 3.1 - 3.3, I will consider two classes of transcendental entire functions of \( B(z) \) which have order one.

**Proposition 3.1** If \( B(z) = -\frac{1}{2}e^{-z} \), then for all non-trivial solutions \( f \) of (3.3), we have \( \lambda(f) = \infty \).

**Proof** The equation (3.3) will become

\[
f''' + e^{-z}f' - \frac{1}{2}e^{-z}f = 0 \tag{3.5}\]

Let \( u \) be a solution of the second order differential equation

\[
u'' + \frac{1}{2}e^{-z}u = 0 \tag{3.6}\]

We will show that \( f = u^2 \) is a solution of (3.5) by substitution. Since \( f' = 2u \cdot u', f'' = 2(u')^2 + 2u \cdot u'' \) and \( f''' = 6u' \cdot u'' + 2u \cdot u''' \), substitute into (3.5) will give
\[ f''' + e^{-z}f' - \frac{1}{2}e^{-z}f \]
\[ = 6u' \cdot u'' + 2u \cdot u''' + 2u \cdot u'e^{-z} - \frac{1}{2}e^{-z}u^2 \]
\[ = 2u(u'' + \frac{1}{4}e^{-z}u' - \frac{1}{4}e^{-z}u) + 6u'(u'' + \frac{1}{4}e^{-z}u) \]
\[ = 0 \]

Since
\[ u'' + \frac{1}{4}e^{-z}u' - \frac{1}{4}e^{-z}u = (u'' + \frac{1}{4}e^{-z}u)' = 0 \]
so we have proved that \( f = u^2 \) is a solution of (3.5). By Theorem B, we have \( \lambda(u) = \infty \), and the theorem is proved.

**Proposition 3.2** If \( B(z) = -(c + 1)e^{-z} - c^3 \) where \( c \) is a constant, then all non-trivial solutions \( f \) of (3.3) has infinite order. In particular, when \( c = 0 \) we have \( \lambda(f) = \infty \).

**Proof** Consider the differential equation
\[ u'' + cu' + (e^{-z} + c^2)u = 0 \]  
(3.7)

We will eliminate the first order term by making the transformation \( u = y \cdot \exp(-\frac{cz}{2}) \), then

\[ u' = y' \cdot \exp(-\frac{cz}{2}) - \frac{c}{2}y \cdot \exp(-\frac{cz}{2}) \]

and

\[ u'' = y'' \cdot \exp(-\frac{cz}{2}) - cy' \cdot \exp(-\frac{cz}{2}) + \left(\frac{c}{2}\right)^2 y \cdot \exp(-\frac{cz}{2}) \]
Substitute the above equations into (3.7), we get

\[ u'' + cu' + (e^{-z} + c^2)u \]
\[ = [y'' + (\frac{c}{2})^2 y + (e^{-z} + c^2)y - (\frac{c^2}{2})y] \cdot exp(-\frac{cz}{2}) \]
\[ = [y'' + (\frac{3}{4}c^2 + e^{-z})y] \cdot exp(-\frac{cz}{2}) = 0 \quad (\ast) \]

It is an elementary fact in the studies of second order differential equations that all the non-trivial solutions \( y \) of equation (\ast) will have infinite order.

We differentiate (3.7) once and get

\[ u''' + cu'' + (e^{-z} + c^2)u' - e^{-z}u = 0 \quad (3.8) \]

We then subtract (3.8) by \( c \) times the equation (3.7) and we will get

\[ u''' + e^{-z}u' - [(c + 1)e^{-z} + c^3]u = 0 \quad (3.9) \]

which is the equation we are considering. Since \( \rho(y) = \infty \), so \( \rho(u) = \infty \) too and the theorem is almost proved.

Now, if \( c = 0 \), we first remark that (3.9) is actually equal to (3.8) and (3.7) will become

\[ u'' + e^{-z}u = 0 \]

By Theorem B, \( \lambda(u) = \infty \) and the theorem is proved.

**Proposition 3.3** If \( B(z) = -(1 + az) + 3a^2z - (az)^3 \) where \( a \) is a non-zero constant, then all non-trivial solutions \( f \) of (3.3) has infinite order.
**Proof** The proof is very similar to that of Proposition 3.2. We consider the differential equation

$$u'' + azu' + (a^2z^2 + e^{-z} - a)u = 0$$  \hspace{1cm} (3.10)

We then transform it by \( u = y \cdot \exp(-a\frac{z^2}{4}) \), so

$$u' = y' \cdot \exp(-a\frac{z^2}{4}) - \frac{az}{2}y \cdot \exp(-a\frac{z^2}{4})$$

and

$$u'' = y'' \cdot \exp(-a\frac{z^2}{4}) - azy' \cdot \exp(-a\frac{z^2}{4}) + \left((\frac{az}{2})^2 - \frac{a}{2}\right)y \cdot \exp(-a\frac{z^2}{4})$$

Substitute the above equations into (3.10), we get

$$u'' + azu' + (a^2z^2 + e^{-z} - a)u$$

$$= [y'' + (e^{-z} - \frac{3a}{2} + \frac{3}{4}a^2z^2)y] \cdot \exp(-a\frac{z^2}{4})$$

$$= 0$$

Again, we have \( \rho(y) = \infty \).

We differentiate (3.10) once and get

$$u''' + azu'' + (a^2z^2 + e^{-z})u' + (2a^2z - e^{-z})u = 0$$  \hspace{1cm} (3.11)

Subtract (3.11) by \( az \) times the equation (3.10), we get

$$u''' + e^{-z}u' + [3a^2z - (1 + az)e^{-z} - (az)^3]u = 0$$
By the same reason as stated in Proposition 3.2, we have \( \rho(u) = \infty \) and the theorem is proved.

Theorem 3.1, Propositions 3.1 to 3.3 deal with the case when \( B(z) \) is a transcendental entire function. We will consider the case when \( B(z) \) is a non-constant polynomial and we have the following theorem:

**Theorem 3.2** If \( B(z) \) is a non-constant polynomial of degree \( b \) and \( \{f_1, f_2, f_3\} \) be a fundamental set of linearly independent solutions of (3.3) with \( \rho(f_1) \leq \rho(f_2) \leq \rho(f_3) \), then either

\[
\rho(f_1) = \rho(f_2) = \rho(f_3) \geq \frac{b+3}{3}
\]

or

\[
\rho(f_3) = \rho(f_2) > \rho(f_1) \quad \text{and} \quad \rho(f_2) = \rho(f_3) \geq \frac{b+3}{3}
\]

**Proof** Let \( v(r) \) be the central index of the non-trivial solutions \( f(z) \) of (3.3). By the Wiman-Valiron theory [15, Chapter 4], we have

\[
\left(\frac{v(r)}{z}\right)^3 f(z) + e^{-z}\left(\frac{v(r)}{z}\right)(1 + o(1))f(z) + B(z)(1 + o(1))f(z) = 0
\]

\[
v(r)^3 + e^{-z}z^2v(r)(1 + o(1)) + B(z)z^3(1 + o(1)) = 0
\]

The solution of the cubic equation was solved by Tartaglia and known as Cardan’s solution.

\[
v(r) = \left\{-\frac{B(z)z^3}{2}(1 + o(1)) + \sqrt{\left(\frac{(B(z)z^3)^2}{4}(1 + o(1)) + \frac{e^{-3z}z^6}{27}\right)}\right\}^{\frac{1}{3}}
\]

\[
+\left\{-\frac{B(z)z^3}{2}(1 + o(1)) - \sqrt{\left(\frac{(B(z)z^3)^2}{4}(1 + o(1)) + \frac{e^{-3z}z^6}{27}\right)}\right\}^{\frac{1}{3}}
\]
And there exists a positive constant $A$ such that

$$v(r) \geq A|B(z)z^3(1 + o(1))|^{\frac{1}{3}}$$

Since $B(z)z^3$ is asymptotic to $kz^{b+3}$ where $k$ is a non-zero constant, so by Theorem C

$$\rho(f) = \lim_{r \to \infty} \frac{\log v(r)}{\log r} \geq \frac{b + 3}{3} \tag{3.12}$$

If $\rho(f_3) > \rho(f_2)$, by Theorem D, we obtain $\rho(f_1) = \rho(f_3)$ which is impossible. So we either have

$$\rho(f_3) = \rho(f_2) > \rho(f_1)$$

or

$$\rho(f_3) = \rho(f_2) = \rho(f_1) \tag{3.13}$$

By (3.12) and (3.13), we get either

$$\rho(f_2) = \rho(f_3) \geq \frac{b + 3}{3} \text{ and } \rho(f_2) > \rho(f_1)$$

or

$$\rho(f_1) = \rho(f_2) = \rho(f_3) \geq \frac{b + 3}{3}$$

and the theorem is proved.

In Theorem 3.3, we will consider the subnormal solutions of a class of functions of $B(z)$ which has order one.
Theorem 3.3 Let \( B(z) = \sum_{k=-n}^{m} a_k e^{kz} \) where \( B(z) \neq 0 \), \( m \geq 0 \) and \( n \geq 1 \). If there exists a non-trivial subnormal solution \( f \) of (3.3) which is periodic with period \( 2\pi i \), then

\[
a_0 = -t^3, a_{-1} = s \text{ and } a_m = a_{m-1} = \cdots = a_1 = a_{-2} = \cdots = a_{-n} = 0
\]

where \( t \) and \( s \) are not both equal to zero.

Furthermore, the subnormal solutions \( f \) will be of the form

\[
f = \sum_{k=-s}^{t} b_k e^{kz}
\]

where all \( b_k \neq 0 \) and they satisfy the recurrence relation

\[
b_k (k^3 + a_0) + (k + 1 + a_{-1}) b_{k+1} = 0 \text{ for } k = -s, -s + 1, \ldots, t - 1.
\]

Proof By Lemma 2.1, the non-trivial solutions \( f \) will be of the form

\[
f = S_1(e^z) + S_2(e^{-z}) \tag{3.14}
\]

where \( S_1(z) \) and \( S_2(z) \) are polynomials in \( z \).

We can rewrite (3.14) as

\[
f = \sum_{k=-s}^{t} b_k e^{kz} \tag{3.15}
\]

where \( b_t \) and \( b_{-s} \) both are not equal to zero, either \( t \geq 0 \) and \( s \geq 1 \), or \( t > -s \geq 0 \) and \( t > 0 \), or \( 0 > t > -s \) and \( s > 0 \).

Substitute (3.15) into equation (3.3), we get

\[
\sum_{k=-s}^{t} b_k (k)^3 e^{kz} + \sum_{k=-(s+1)}^{t-1} b_{k+1}(k + 1)e^{kz} + (\sum_{j=-n}^{m} a_j e^{jz})(\sum_{k=-s}^{t} b_k e^{kz}) = 0 \tag{3.16}
\]
By equating the coefficients of $e^{kz}$ where $k = -(s + n), -(s + n - 1), \cdots, -(s - 2)$ and $k = t + 1, t + 2, \cdots, t + m$ we obtain

$$a_m = a_{m-1} = \cdots = a_1 = a_{-2} = \cdots = a_{-n} = 0$$

Therefore, equation (3.16) will become

$$\sum_{k=-s}^{t} b_k(k)^3 e^{kz} + \sum_{k=-(t+1)}^{t-1} b_{k+1}(k+1)e^{kz} + (a_0 + a_{-1}e^{-z})(\sum_{k=-s}^{t} b_k e^{kz}) = 0$$

$$\sum_{k=-s}^{t} b_k(k)^3 e^{kz} + \sum_{k=-(t+1)}^{t-1} b_{k+1}(k+1)e^{kz} + \sum_{k=-(t+1)}^{t-1} a_{-1}b_{k+1}e^{kz} + \sum_{k=-s}^{t} a_0 b_k e^{kz} = 0$$

(3.17)

By equating the coefficients of $e^{kz}$ where $k = -(s + 1), -s, \cdots, t$. We have the following equations

$$b_t(t^3 + a_0) = 0$$  (3.18)

$$b_{-s}(-s + a_{-1}) = 0$$  (3.19)

$$b_t(t^3 + a_0) + (i + 1 + a_{-1})b_{i+1} = 0 \text{ for } i = -s, -(s - 1), \cdots, t - 1$$  (3.20)

From (3.18), we either have $b_t = 0$ or $a_0 = -t^3$. If $b_t = 0$, then by (3.20), all $b_k$'s where $k = -s, -(s - 1), \cdots, t$ will be equal to zero and it is a trivial solution. So we only have $a_0 = -t^3$.

For (3.19), by the same argument as above. We can only have $a_{-1} = s$ and the theorem is proved.

We will now give some examples which will illustrate some possibilities that can occur in $B(z)$:

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Example 3.3.1. This example shows that \( a_0 = 0 \) and \( a_{-1} \neq 0 \) is possible. Let \( B(z) = e^{-z} \), then \( f = 2 + 2e^{-z} \) will be a solution of (3.3).

Example 3.3.2. Similarly, \( a_0 \neq 0 \) and \( a_{-1} = 0 \) is possible too. Let \( B(z) = -8 \) then \( f = 28e^{2z} + 8e^{z} + 1 \) will satisfy equation (3.3).

Example 3.3.3. Let \( B(z) = -1 + 2e^{-z} \), then a solution of (3.3) will be: \( f = 3e^{z} + 9 + 9e^{-z} + e^{-2z} \).

In Theorem 3.4, we will consider a different type of equation and the argument used in here follows the proof in Hellerstein, Miles and Rossi[10].

**Theorem 3.4** Consider the third order differential equation:

\[
 f''' + g f' + h f = 0 \tag{3.21}
\]

where \( g \) and \( h \) are entire functions with \( \rho(h) < \rho(g) \leq \frac{1}{2} \), then any non-constant solution of (3.21) has infinite order.

**Proof** We denote the order of \( f \) by \( \lambda \), the order of \( g \) by \( \rho \) and the order of \( h \) by \( \rho' \) throughout the proof. We also choose parameters \( \rho_j, 1 \leq j \leq 4 \), satisfying

\[
 \rho' < \rho_1 < \rho_2 < \rho_3 < \rho < \rho_4 \tag{3.22}
\]

For the entire function \( g \), we let \( L(t, g) = \min_{|z|=t} |g(z)|, \rho = \rho(g) \) and \( \mu(g) \) as the order and lower order of \( g \) respectively. If \( \mu(g) < \rho_3 < \rho(g) \leq \frac{1}{2} \), by Theorem E, there exists \( s \to \infty \) with

\[
 \log L(s, g) > s^{\rho_3} \tag{3.23}
\]
If \( \rho_3 < \mu(g) = \rho(g) < \frac{1}{2} \), again, by Theorem F we have either there exists a sequence \( s_n \to \infty \) satisfying

\[
\log L(s_n, g) > \epsilon \log M(s_n, g)
\]

for some \( \epsilon > 0 \) and hence also satisfies (3.23), or

\[
\log L(r, g) = o(\log M(r, g)) \quad \text{as} \quad r \to \infty \quad (3.24)
\]

By the last inequality in P.9, (3.24) and Theorem G, we note that \( G^* = G - H \) has logarithmic density 1. For \( \rho_3 < \frac{1}{2} \), we conclude from Theorem G that

\[
\mathcal{L}(r)r^{\frac{1}{2} - \rho_3} \to \infty \quad \text{as} \quad r \to \infty \quad (3.25)
\]

Define

\[
K_r = \{ \theta \in [0, 2\pi] : \log |g(re^{i\theta})| < r^{\rho_3} \} \quad (3.26)
\]

we conclude from Theorem G and (3.25) that

\[
m(K_r) \to 0, \quad r \in G^* \quad \text{and} \quad r \to \infty \quad (3.27)
\]

In summary, if \( \rho(g) \leq \frac{1}{2} \), then for all \( \rho_3 < \rho(g) \) we have two possibilities:

(I) There are arbitrarily large \( s \) satisfying:

\[
\log |g(se^{i\theta})| > s^{\rho_3} \quad \text{where} \quad 0 \leq \theta \leq 2\pi. \quad (3.28)
\]

(II) We have (3.27) for some set \( G^* \) of logarithmic density 1 where \( K_r \) is defined as in (3.26).

In both cases, we presume that the non-constant entire function \( f \) has finite order \( \lambda < \lambda' \). We first consider Case I. For large \( s \) satisfying (3.28), we
apply Lemma 2.5 to obtain

\[ r \in (s, s + 3s^{\lambda'} e^{-s\rho_1}) - E(\rho_1) \]  \hspace{1cm} (3.29)

Defining \( C_r \) as in Lemma 2.7, we conclude that

\[ m(C_r) < s^{\lambda' + \rho_4 - \rho_3 - 1} e^{-s\rho_1} \]  \hspace{1cm} (3.30)

From (3.21) we have

\[ \frac{f'(z)}{f(z)} = -\frac{h(z) + \frac{f'''(z)}{f''(z)}}{g(z)} \]  \hspace{1cm} (3.31)

From Lemma 2.5, (3.22), Lemma 2.7, (3.29) and (3.31) it follows that there exists an unbounded set \( H \) for which \( \theta \notin C_r \) implies

\[ \left| re^{i\theta} \frac{f'(re^{i\theta})}{f(re^{i\theta})} \right| \leq \frac{r(e^{\rho_1} + r^{3\lambda'} e^{3(\theta \rho_1)})}{e^{r\rho_2}} = o(1) \text{ as } r \to \infty, r \in H \]  \hspace{1cm} (3.32)

We now estimate

\[ \frac{1}{2\pi} \int_{C_r} (Re re^{i\theta} \frac{f'(re^{i\theta})}{f(re^{i\theta})})^+ d\theta \]

from above for \( r \in H \). By Lemma 2.2, we have

\[ \frac{1}{2\pi} \int_{C_r} |f_1(re^{i\theta})| d\theta \leq \frac{3}{2} T(3r, f, m(C_r)) = o(1) \text{ as } r \to \infty \]  \hspace{1cm} (3.33)

by letting \( R = 3r \) with (3.29) and (3.30).

If \( f \) has no zeros, then

\[ \frac{1}{2\pi} \int_{C_r} |re^{i\theta} \frac{f'(re^{i\theta})}{f(re^{i\theta})}| d\theta = \frac{1}{2\pi} \int_{C_r} |f_1(re^{i\theta})| d\theta \leq o(1) \]  \hspace{1cm} (3.34)
by (3.33), we therefore conclude from (3.32) and (3.34) that the total variation of \( \arg f(re^{i\theta}) \) on \([0, 2\pi]\) is \( o(1) \). Since \( f \) is non-constant, this is incompatible with the Casorati-Weierstrass theorem [4, P.109] and the argument principle, and this results in a contradiction.

If \( f \) does have zeros, we conclude from Lemma 2.2(ii) and (3.30) for \( r \in H \) that

\[
\frac{1}{2\pi} \int_{C_r} |\text{Re} f_2(re^{i\theta})| d\theta \leq \frac{2}{4\pi} n(r/e)m(C_r) = o(1) \quad \text{as} \quad r \to \infty
\]  

(3.35)

From Lemma 2.2(i), we have

\[
\frac{1}{2\pi} \int_{C_r} (\text{Re} f_3(re^{i\theta}))^+ d\theta \leq n(r) - n(r/e)
\]  

(3.36)

From Lemma 2.2(iii) and (3.30), we also conclude that for \( r \in H \)

\[
\frac{1}{2\pi} \int_{C_r} (\text{Re} f_4(re^{i\theta}))^+ d\theta \leq \frac{1}{2\pi} (n(3r) - n(r))m(C_r) = o(1) \quad \text{as} \quad r \to \infty
\]  

(3.37)

Combining (3.33), (3.35) to (3.37), we obtain

\[
\frac{1}{2\pi} \int_0^{2\pi} (\text{Re} \ re^{i\theta} \frac{f'(re^{i\theta})}{f(re^{i\theta})})^+ d\theta \leq n(r) - n(r/e) + o(1)
\]

\[
\leq n(r) - 1 + o(1)
\]  

(3.38)

for large \( r \) and \( r \in H \)

Since

\[
\frac{1}{2\pi} \int_0^{2\pi} \text{Re} \ re^{i\theta} \frac{f'(re^{i\theta})}{f(re^{i\theta})} d\theta = n(r)
\]  

(3.39)
by the argument principle, it contradicts with (3.38) and the theorem is proved for Case I.

We now consider Case II. For $\epsilon > 0$, by Lemma 2.3, Lemma 2.5 and Lemma 2.6, we conclude there exists a set $E \subset [1, \infty)$ such that

$$E \cap E_1 = \emptyset$$

$$E \cap E(\rho_1) = \emptyset$$

$$E \subset (G^* \cap E_3(\epsilon))$$

and

$$\log \text{dens} E > \sigma(\epsilon)$$

(3.43)

By (3.41), for large $r \in E$, we have for $\theta \not\in K_r$

$$\left| \frac{re^{i\theta}f'(re^{i\theta})}{f(re^{i\theta})} \right| \leq \frac{r(e^{r\rho_1} + r^{\lambda'e^{3(9r\epsilon)\rho_1}})}{e^{r\rho_3}} = o(1) \quad \text{as} \quad r \to \infty$$

(3.44)

We choose $K$ so large that Lemma 2.6 holds for $r \in E$ and we estimate

$$\frac{1}{2\pi} \int_{K_r} (Re \, re^{i\theta}f'(re^{i\theta}) \frac{f'(re^{i\theta})}{f(re^{i\theta})})^+ d\theta$$

(3.45)

using the differentiated Poisson-Jensen formula in Lemma 2.2 with $R = re^{h/2}$ and $R' = re^{h}$ in Lemma 2.6. We have

$$|f_1(re^{i\theta})| \leq 2T(r, f)\frac{e^{h/2}}{(e^{h/2} - 1)^2}$$

$$< \frac{9T_0(R', f)}{h^2} < 9K\lambda(e + \epsilon)A(r, f)$$
since \( r \in E_3(\varepsilon) \). Consequently

\[
\frac{1}{2\pi} \int_{K_r} |f_1(re^{i\theta})|d\theta = o(A(r,f)) \quad \text{as} \quad r \to \infty \quad r \in E
\]  

(3.46)

by (3.27) and (3.42). If \( f \) has no zeros, by Lemma 2.3, (3.44) and (3.45) it will lead to a contradiction.

If \( f \) does have zeros, we presume that \( n(1) = n(1, \frac{1}{J}) \geq 1 \). By Lemma 2.4, we may choose \( K > 1 \) so large that in addition to (3.40) to (3.43), \( E \) also satisfies

\[
E \cap E_3(K) = \emptyset
\]  

(3.47)

From Lemma 2.2(ii), we have

\[
\frac{1}{2\pi} \int_{K_r} |Re f_2(re^{i\theta})|d\theta \leq \frac{4}{2\pi} n(r/e)m(K_r) = o(n(r/e)) \quad \text{as} \quad r \to \infty \quad r \in E
\]  

(3.48)

By Lemma 2.2(i), we also have

\[
\frac{1}{2\pi} \int_{K_r} |Re f_3(re^{i\theta})|d\theta \leq n(r) - n(r/e)
\]  

(3.49)

By letting \( R = re^{h/2} \), \( R' = re^h \) and Lemma 2.6 we conclude

\[
\frac{1}{2\pi} \int_{K_r} (Re f_4(re^{i\theta}))^+ d\theta \leq \frac{1}{2\pi} (n(R) - n(r)) \left( \frac{1}{2} + \frac{1}{e^{h/2} - 1} \right)m(K_r)
\]

\[
< \frac{n(R)}{2\pi} \left( \frac{1}{2} + \frac{2}{h} \right)m(K_r) \leq \frac{N(R',1/f)}{\pi h} \left( \frac{1}{2} + \frac{2}{h} \right)m(K_r)
\]

\[
\leq \frac{[T(R',f) + O(1)]}{\pi h} \left( \frac{1}{2} + \frac{2}{h} \right)m(K_r) = o(A(r,f))
\]  

(3.50)
as \( r \to \infty \) and \( r \in E \). We conclude from (3.44), (3.46) to (3.49) that

\[
\frac{1}{2\pi} \int_0^{2\pi} \left( \text{Re} \, re^{i\theta} \frac{f'(re^{i\theta})}{f(re^{i\theta})}\right)^+ d\theta \leq n(r) - n(r/e) + o(A(r,f) + n(r/e)) + o(1)
\]

\[
\leq (1 + o(1) - 1/K)n(r) + o(A(r,f)) + o(1) \quad \text{as} \quad r \to \infty , \quad r \in E \quad (3.51)
\]

Observe that the combination of Lemma 2.3 with (3.51) is incompatible with (3.39), we therefore obtained a contradiction and the theorem is proved.
Chapter 4

Conclusions and Remarks

By looking at the results of Propositions 3.1 to 3.3, one may be tempted to conclude that for all transcendental entire functions $B(z)$ of order one, all their non-trivial solutions would have infinite order. It is false and example is not difficult to construct, the equation

$$f''' + e^{-z}f' - (1 + e^{-z})f = 0$$

has solution $f = e^z$ and it is a finite order solution.

We remark that in Proposition 3.2, when $c = -1$, $B(z) = 1$ which is a constant.

When one consider the proofs of Proposition 3.2 and Proposition 3.3, one may naively think that by changing the coefficient of $u'$ (i.e. the constant $c$ in (3.7) and $az$ in (3.10)) into a general polynomial $P(z)$, it would give another function $B(z)$ such that all non-trivial solutions would have infinite order. It seems a difficult problem even to consider a simple special case where $P(z) = az + c$ and $a$ and $c$ are non-zero constants.

For Theorem 3.2, though the author suspects that the solutions should have infinite order, he is unable to prove it.
The author regrets that he overlooked the paper of Hellerstein, Miles and Rossi in Annales Academiae Scientiarum Fennicae Series A. I. Mathematica Volume 17, 1992, P.345-365. The result that we proved in Theorem 3.4 were already contained in this paper. But the author also wants to point out that their method is different from the proof in Theorem 3.4.
Bibliography


