Operads and Hecke operators on Modular Forms

by

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\[ \mathcal{M}_{k_1}(\Gamma) \times \cdots \times \mathcal{M}_{k_n}(\Gamma) \to \mathcal{M}_{k_1+\cdots+k_n}(\Gamma), \]

and we show these operators form an algebraic structure called operad. Then we define a Galois action on this operad which is compatible with the Galois action on modular forms. By taking the Galois orbit, we find a suboperad which acts on the integral modular forms.
Chapter 1

Introduction

1.1 Notations and Conventions

Throughout this thesis, we will denote

$k$: a commutative ring with identity.

$P, Q, R, O, \delta$: an operad in some coefficient $k$.

$A$: a subring of $\mathbb{C}$.

$G$: the Galois group of algebraic closure of $\mathbb{Q}$ over $\mathbb{Q}$.

$G^{ab}$: the Galois group of maximal abelian extension of $\mathbb{Q}$ over $\mathbb{Q}$.

$M(\Gamma)(M_k(\Gamma))$: modular form(over $\mathbb{C}$) of $\Gamma$(of weight $k$).

$M(\Gamma; A)(M_k(\Gamma; A))$: modular form over $A$ of $\Gamma$(of weight $k$).

$H$: the upper half plane.

$H(\Gamma)(H_k(\Gamma))$: Hecke operators over $\mathbb{C}$(in weight $k$).

$H(\Gamma; A)(H_k(\Gamma; A))$: Hecke operators over $A$(in weight $k$).

$(\Gamma, S)$: a Hecke pair.

$H(\Gamma, S)$: abstract Hecke ring of a Hecke pair.

$H^C$: abstract Hecke ring over $\mathbb{C}$.

$H_A$: a subring of Hecke ring in coefficient $A$. 
δ(Γ, S): abstract Z-operad of a Hecke pair.
δ^C: abstract operad over C.
δ(Γ): operad operators over C.
δ_A: a suboperad of abstract operad over A.
δ(Γ; A): the image of δ_A in End_M(Γ).
L^n(Γ, S): free Z-module generated by Γ^n g, g ∈ S^n.
D^n(Γ, S): right Γ-invariant elements in L^n(Γ, S).
α, β, γ: elements in some L^n(Γ, S).
Σ_n: symmetric group of n elements.
ζ_M: M^{th} primitive root.

1.2 Summary of the thesis

Let M_k(Γ) be the collection of modular forms over C of weight k with respect to a congruence subgroup Γ. Consider one arbitrary double coset ΓgΓ, where g is an element in GL^+_2(Q). Suppose it is decomposed into disjoint union of right cosets Γg_i. It is well-known that this double coset acts on M_k(Γ) by summing over all results acted by g_i. See for example [DS05](p. 163-168). Those operators are known as Hecke operators. Hecke operators have many applications in number theory. For example, if we consider the eigenforms for some class of Hecke operators, their L-functions have Euler product. Also if f is a normalized eigenform, there exists a cuspidal automorphic representation whose L-function correspondees to the L-function for f. Now we try to extend these linear maps to multilinear maps M_{k_1}(Γ) × ··· × M_{k_n}(Γ) → M_{k_1+···+k_n}(Γ).

More precisely, we consider the new double coset Γ^n gΔΓ, where g = (g^1, ···, g^n) and each g^i ∈ GL^+_2(Q), where Δ(Γ) = {(x, ···, x)|x ∈ Γ} is the diagonal subgroup. Similar to the Hecke-algebra case, we consider its decompston into left cosets Γ^n g_a, where g_a = (g^1_a, ···, g^n_a). We define for this double coset a multilin-
ear map from $\mathcal{M}_{k_1}(\Gamma) \times \cdots \times \mathcal{M}_{k_n}(\Gamma)$ to $\mathcal{M}_{k_1+\cdots+k_n}(\Gamma)$ as follows

$$\Gamma^n g \Delta \Gamma(f_1, \cdots, f_n) := \sum_a f_1 \begin{bmatrix} g^1_{a_{k_1}} & \cdots & f_n \end{bmatrix} \begin{bmatrix} g^n_{a_{k_n}} \end{bmatrix}.$$ (1.1)

Due to the product structure on $\oplus_k \mathcal{M}_k(\Gamma)$, this gives a multi-linear map and clearly we get the Hecke operators as a special case when $n = 1$.

In the formulation of Hecke operators, as motivated by actions of double cosets, we first define some abstract ring, called abstract Hecke ring, then make it act on modular forms. See for example [AZ95](Ch. 3, 4). As an analogue, we find the corresponding abstract object with respect to our multi-linear operators, and we find an operadic structure on this object, i.e. the analogue of abstract Hecke ring is an operad, and if we call it analogously the abstract operad, then this operad acts on modular forms. Now our abstract Hecke ring and Hecke operators can be viewed(almost) as a special part of the abstract operad and operadic operators, and general modular forms can be regarded as an operad algebra in our operad language.

Historically, operad was first defined by J. P. May in his book [May72] and studied as a tool in homotopy theory. Then it has been used extensively in homological algebra, category theory, algebraic geometry and mathematical physics. See [Mar06] for more detailed information.

As is also well-known, we usually have a $\mathbb{Z}$-structure over modular forms, i.e. if we consider all modular forms such that the coefficients of their $q$-expansion are all integers, then after base change, they reproduce all the modular forms over $\mathbb{C}$. We can show that in nice cases the abstract Hecke ring or its proper subring acts on those modular forms with integral coefficients. We can do this by computing the coefficients by brute force or using geometric methods. See [Ser73] and [Hid93].

So the analogous question is if the operad action preserves the $\mathbb{Z}$-structure of
modular forms. We consider the simplest case when \( \Gamma = SL_2(\mathbb{Z}) \). Our idea is that we first define a Galois action on the abstract operad, which is compatible with the Galois action on modular forms (acting on the coefficients of their \( q \)-expansions). Then in order to get a \( \mathbb{Z} \)-coefficient we shall take the Galois orbit of this action. In particular, as previously stated, we know the Galois action on the abstract Hecke ring part is trivial. We expect the same thing happens for the abstract operad, but strangely enough, we find an example which contradicts our intuition. The example shows the Galois action is non-trivial, hence seemingly we have an abstract operad acting on integral modular forms, getting a modular form which is non-integral, or even non-rational.

1.3 Outline of the thesis

In chapter 2, we review some basics on operad theory, i.e. the two equivalent definitions of operads, the correspondence between two definitions, the key motivating example of operads, the definition of operad algebras, the equivalent definition of operad algebras.

In chapter 3, we review some basics on modular forms and Hecke algebras. We recall the definition of modular forms (over \( \mathbb{C} \)) with respect to a congruence subgroup, its \( q \)-expansion, modular form in coefficient \( A \). We recall some theorems of \( \mathbb{Z} \)-structures of modular forms. Then we define the abstract Hecke ring with respect to a Hecke pair, consider the special case when restricted to congruence subgroups. Also we define the double coset operators, and verify by computing the coefficients that the abstract Hecke ring acts on integral modular form in the special case when the congruence subgroup is \( SL_2(\mathbb{Z}) \).

In chapter 4, we define the operad structure on \( \mathbb{Z} \)-linear combinations of double cosets \( \Gamma^n g \Delta \Gamma \), where \( \Gamma = SL_2(\mathbb{Z}) \), verify with detailed calculation that it satisfies
the axioms of operads. Then we define the operadic actions on modular forms, verify they give the modular forms an operad algebra structure. Finally we define the Galois action on the abstract operad and verify it is compatible with the action on modular forms. We prove some properties of this action. By restricting to some smaller set and taking Galois-action orbits, we show some suboperad of the abstract operad acts on integral modular forms. In the end we give an example showing this Galois action is nontrivial and argue how this example indicates there are integral modular forms which do not preserve integral structure under the action of some element in the abstract operad.
Chapter 2

Basic Operad Theory

In this chapter, we review some basic things of operad theory.

2.1 Definition of Operads

Let $k$ be a commutative ring with identity. Let $\Sigma_n$ be the symmetric group of $n$ elements. $k[\Sigma_n]$ is the corresponding group ring. We recall the definition of operads due to May [May72] [Mar06].

**Definition 2.1.1.** An operad (in the sense of May) in the category of $k$-modules is a collection $\mathcal{P} = \{\mathcal{P}(n)\}_{n \geq 0}$ of right $k[\Sigma_n]$-modules, and two $k$-linear maps called operadic compositions

\[ \gamma : \mathcal{P}(n) \otimes \mathcal{P}(k_1) \otimes \cdots \otimes \mathcal{P}(k_n) \to \mathcal{P}(k_1 + \cdots + k_n) \]

\[ \theta \otimes \theta_1 \otimes \cdots \otimes \theta_n \mapsto \theta \circ (\theta_1, \cdots, \theta_n) \]

\[ \eta : k \to \mathcal{P}(1) \text{ satisfying,} \]
1. Associativity. Suppose

\[ m_{1,1} + \cdots + m_{1,k_1} = r_1 \]
\[ m_{2,1} + \cdots + m_{2,k_2} = r_2 \]
\[ \vdots \]
\[ m_{n,1} + \cdots + m_{n,k_n} = r_n, \]

where all \( m_{i,j} \) are non-negative integers, then the following diagram commutes:

\[ \left( \mathcal{P}(n) \otimes \bigotimes_{i=1}^{n} \mathcal{P}(k_i) \right) \otimes \bigotimes_{i,j} \mathcal{P}(m_{i,j}) \xrightarrow{\gamma \otimes \text{id}} \mathcal{P}(n) \otimes \bigotimes_{i=1}^{n} \left( \mathcal{P}(k_i) \otimes \bigotimes_{j=1}^{k_i} \mathcal{P}(m_{i,j}) \right) \]

\[ \xrightarrow{\text{id} \otimes \gamma \otimes n} \mathcal{P}(n) \otimes \bigotimes_{i,j} \mathcal{P}(m_{i,j}) \]

2. Equivariance. Suppose \( \sigma \in \Sigma_n \), \( \tau_1 \in \Sigma_{k_1}, \cdots, \tau_n \in \Sigma_{k_n} \), and define \( \tau := \tau_1 \otimes \cdots \otimes \tau_n \), \( \tilde{\tau} := \tau_1 \oplus \cdots \oplus \tau_n \), namely \( \tilde{\tau} \in \Sigma_{k_1 + \cdots + k_n} \) denotes the permutation that permutes each block of \((1, \cdots, k_1), \cdots, (k_1 + \cdots + k_{n-1} + 1, \cdots, k_1 + \cdots + k_n)\) as \( \tau_i \), then the following two diagrams commute:

\[ \mathcal{P}(n) \otimes \mathcal{P}(k_1) \otimes \cdots \otimes \mathcal{P}(k_n) \xrightarrow{\sigma \otimes \sigma^{-1}} \mathcal{P}(n) \otimes \mathcal{P}(k_{\sigma(1)}) \otimes \cdots \otimes \mathcal{P}(k_{\sigma(n)}) \]

\[ \xrightarrow{\gamma} \mathcal{P}(k_1 + \cdots + k_n) \xrightarrow{\sigma(k_{\sigma(1)}, \cdots, k_{\sigma(n)})} \mathcal{P}(k_{\sigma(1)} + \cdots + k_{\sigma(n)}) \]

\[ \mathcal{P}(n) \otimes \mathcal{P}(k_1) \otimes \cdots \otimes \mathcal{P}(k_n) \xrightarrow{\text{id} \otimes \tau} \mathcal{P}(n) \otimes \mathcal{P}(k_1) \otimes \cdots \otimes \mathcal{P}(k_n) \]

\[ \xrightarrow{\gamma} \mathcal{P}(k_1 + \cdots + k_n) \xrightarrow{\tilde{\tau}} \mathcal{P}(k_1 + \cdots + k_n), \]

where \( \sigma(k_1, \cdots, k_n) \in \Sigma_{k_1 + \cdots + k_n} \) denotes the permutation that permute the \( n \) blocks of \((1, \cdots, k_1), \cdots, (k_1 + \cdots + k_{n-1} + 1, \cdots, k_1 + \cdots + k_n)\) as \( \sigma \in \Sigma_n \).

3. Unitality. The following two diagrams commute for each \( n \):

\[ \gamma \]
\[ \mathcal{P}(k_1 + \cdots + k_n) \]
We can even extend our definition to any symmetric monoidal category. See [LV09]. But here we will just use the definition of operads in the category of $k$-modules.

**Example 2.1.2.** Consider $\mathcal{E}nd_V = \{\mathcal{E}nd_V(n)\}_{n \geq 0}$, where $V$ is a module over $k$, and $\mathcal{E}nd_V(n)$ is the set of all module morphisms $V^\otimes n \to V$. We define $V^0 = k$ and consider the image of 1 in the following. Then it is an operad with the following operadic compositions:

$\forall f \in \mathcal{E}nd_V(n)$, $g_1 \in \mathcal{E}nd_V(k_1)$, $\cdots$, $g_n \in \mathcal{E}nd_V(k_n)$, and $\forall \lambda \in k$, define

$$\gamma(f, g_1, \cdots, g_n) := f \circ (g_1, \cdots, g_n)$$

$$\eta(\lambda) := \lambda \text{Id} \in \mathcal{E}nd_V(1),$$

and the right $k[\Sigma_n]$-module structure:

$\forall \sigma \in \Sigma_n$ and $f \in \mathcal{E}nd_V(n)$, define $f^\sigma(v_1 \otimes \cdots \otimes v_n) := f(v_{\sigma^{-1}(1)} \otimes \cdots \otimes v_{\sigma^{-1}(n)})$.

Now it follows from the definition that $\mathcal{E}nd_V$ is a operad. In fact all of our axioms in definition 2.1.1 come from this motivating example.

There is an equivalent definition of operads given in Markl’s [Mar06].

**Definition 2.1.3.** An operad (in the sense of Markl) in the category of $k$-modules is a collection $\delta = \{\delta(n)\}_{n \geq 0}$ of right $k[\Sigma_n]$-modules, and $k$-linear maps called operadic compositions

$$\circ_i : \delta(m) \otimes \delta(n) \to \delta(m+n-1),$$

for $i \leq m$ satisfying,

1. **Associativity.** Suppose $f \in \delta(m)$, $g \in \delta(n)$, $h \in \delta(l)$, then

$$\begin{cases}
(f \circ_j g) \circ_i h = (f \circ_i h) \circ_{j+i-1} g & 1 \leq i < j \leq m \\
(f \circ_j g) \circ_i h = f \circ_j (g \circ_{i-j+1} h) & j \leq i < j + n
\end{cases}(2.1)$$

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2. Equivariance. Suppose $\tau \in \Sigma_m$, $\sigma \in \Sigma_n$, define $\tau \circ_i \sigma \in \Sigma_{m+n-1}$ to be the permutation such that the block $(i, i+1, \cdots, i+n-1)$ permuted by $\sigma$, then all $m$ blocks permuted by $\tau$. Then $\forall f \in \delta(m), g \in \delta(n)$, we require

$$f^\tau \circ_i g^\sigma = (f \circ_{\tau(i)} g)^{\tau \circ_i \sigma}.$$ (2.2)

3. Unitality. $\exists$ a unit $e \in \delta(1)$ such that $f \circ_1 e = f$ and $e \circ_1 g = g$ for any $f \in \delta(m), g \in \delta(n)$.

**Example 2.1.4.** In the last example 2.1.2, we see $\mathcal{E}nd_V$ is an operad in the definition due to May. It is also an operad in the definition given in Markl’s [Mar06] with the following operadic compositions:

Suppose $f \in \delta(m), g \in \delta(n)$, define

$$f \circ_1 g(v_1 \otimes \cdots \otimes v_{m+n-1})$$

$$:= f(v_1 \otimes \cdots \otimes v_{i-1} \otimes g(v_i, \cdots, v_{i+n-1}) \otimes v_{i+n} \otimes \cdots \otimes v_{m+n-1}).$$ (2.3)

also $e$ is the identity in $\mathcal{E}nd_V(1) = \text{End}(V, V)$ and we define the same $k[\Sigma_n]$-module structure on $\mathcal{E}nd_V(n)$ as before. We can easily check three requirements in two definitions correspond to each other respectively in this example.

**Definition 2.1.5.** Suppose $\mathcal{P} = \{\mathcal{P}(n)\}$ and $\mathcal{Q} = \{\mathcal{Q}(n)\}$ are two operads in the category of $k$-modules. A morphism $F$ between $\mathcal{P}$ and $\mathcal{Q}$ is a collection of $k[\Sigma_n]$-morphisms $F(n) : \mathcal{P}(n) \rightarrow \mathcal{Q}(n)$ compatible with the operadic structure. A suboperad $\mathcal{R} = \{\mathcal{R}(n)\}$ of $\mathcal{P}$ consists of $\mathcal{R}(n) \subset \mathcal{P}(n)$ as a $k[\Sigma_n]$-submodule and they form an operad with the inherited operadic structures from $\mathcal{P}$.

**Proposition 2.1.6** (Equivalence of two definitions of Operads). The category of operads defined in 2.1.1 is equivalent to that defined in 2.1.3.

**Proof.** We sketch the proof. Given an operad $\delta = \{\delta(n)\}$ in Markl’s definition, let $\mathcal{P} = \{\mathcal{P}(n) = \delta(n)\}$ and construct the operad compositions

$$\gamma : \mathcal{P}(n) \otimes \mathcal{P}(k_1) \otimes \cdots \otimes \mathcal{P}(k_n) \rightarrow \mathcal{P}(k_1 + \cdots + k_n)$$

$$\theta \otimes \theta_1 \otimes \cdots \otimes \theta_n \mapsto (\cdots ((\theta \circ_n \theta_{n-1}) \circ_{n-1} \theta_{n-1}) \cdots) \circ_1 \theta_1$$
and \( \eta(\lambda) := \lambda e \) for \( \lambda \in k \).

Conversely, suppose given \( \mathcal{P} = \{\mathcal{P}(n)\} \) an operad in May’s definition, let \( \delta = \{\delta(n) = \mathcal{P}(n)\} \) and construct the operad compositions as follows.

Given \( f \in \delta(m) \), \( g \in \delta(n) \), define

\[
f \circ_i g := \gamma(f, e, \ldots, e, g, e \ldots, e),
\]

where

\[
\gamma : \mathcal{P}(m) \otimes \mathcal{P}(1) \otimes \cdots \otimes \mathcal{P}(1) \otimes \mathcal{P}(n) \otimes \mathcal{P}(1) \otimes \cdots \otimes \mathcal{P}(1) \longrightarrow \mathcal{P}(n + m - 1),
\]

and \( e \) is just \( \eta(1) \). We can easily see the correspondences above are inverse to each other and the axioms in two different definitions are correspondent to each other through these correspondences by direct check. These correspondences give an equivalence between two categories created using different definitions. From now on, we will say operads without specifying operads in which category. \( \square \)

### 2.2 Definition of Operad Algebras

**Definition 2.2.1 (Operad Algebras).** Let \( V \) be a \( k \)-module and \( \mathcal{E}nd_V \) be the operad defined in the example 2.1.2. \( \delta \) is another operad in the category of \( k \)-modules. We call \( \rho \) a \( \delta \)-operad algebra if \( \rho : \delta \rightarrow \mathcal{E}nd_V \) is a morphism between two operads.

**Proposition 2.2.2.** Given \( \rho : \delta \rightarrow \mathcal{E}nd_V \) a \( \delta \)-operad algebra is equivalent to given \( V \) a \( k \)-module and given \( k \)-linear maps (here we use definition in 2.1.3)

\[
a : \delta(n) \otimes V \otimes \cdots \otimes V \longrightarrow V
\]

\[
\theta_n \otimes v_1 \otimes \cdots \otimes v_n \longrightarrow \theta_n(v_1, \cdots, v_n) \text{ satisfying,}
\]

1. **Associativity.** The following diagram commute:
\[
\delta(m) \otimes \left( V^{\otimes (m-1)} \otimes (\delta(n) \otimes V^{\otimes n}) \right) \xrightarrow{\cong} (\delta(m) \otimes \delta(n)) \otimes (V^{\otimes (m-1)} \otimes V^{\otimes n})
\]

where \(1 \leq i \leq m\) and \(\circ_i : V^{\otimes m-1} \otimes V^{\otimes l} \to V^{\otimes (m+l-1)}\) is the insertion to the \(i^\text{th}\) position, i.e.

\[
\circ_i \left( (v_1 \otimes \cdots \otimes v_{m-1}) \otimes w \right) := v_1 \otimes \cdots \otimes v_{i-1} \otimes w \otimes v_i \otimes \cdots \otimes v_{m-1}.
\]

2. Equivariance. The following diagram commute:

\[
\begin{array}{ccc}
\delta(n) \otimes V^{\otimes n} & \overset{\alpha \otimes \sigma^{-1}}{\longrightarrow} & \delta(n) \otimes V^{\otimes n} \\
\downarrow \alpha & & \downarrow \alpha \\
V & \overset{\alpha}{\longrightarrow} & V,
\end{array}
\]

where \(\sigma^{-1}(v_1 \otimes \cdots \otimes v_n) := v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(n)}\).

3. Unitality. \(e \in \delta(1)\) satisfying: \(\alpha(e,v) = v, \forall v \in V\).

Proof. In fact we are just rewriting the maps \(\delta(n) \to \text{End}_V(n)\) into the maps \(\delta(n) \otimes V^{\otimes n} \to V\), and write conditions for the corresponding preserving structures accordingly. It’s easy to check that all the properties here and in definition 2.2.1 (associativity and so on) are exactly the same.

Remark 2.2.3. Since two definitions of operads are equivalent, we can also write down the equivalent definition of an operad algebra in the sense of May as the above proposition 2.2.2 in the sense of Markl. This is done in [Mar06].
Chapter 3

Basics on Modular Forms and Hecke Algebras

In this chapter, we review some basic things of modular forms, abstract Hecke rings and Hecke operators.

3.1 Basics on Modular Forms

Let $N$ be a positive integer, define

$$\Gamma(N) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) \mid \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \pmod{N} \right\}, \quad (3.1)$$

$$\Gamma_0(N) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) \mid \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \pmod{N} \right\}, \quad (3.2)$$

$$\Gamma_1(N) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) \mid \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \pmod{N} \right\}. \quad (3.3)$$
Obviously, $\Gamma_0(N) \supset \Gamma_1(N) \supset \Gamma(N)$. For any $\Gamma \subset SL_2(\mathbb{Z})$ a subgroup, we call it a congruence subgroup if it contains $\Gamma(N)$ for some $N$. Since $\Gamma(N)$ is of finite index in $SL_2(\mathbb{Z})$, any congruence subgroup has finite index in $SL_2(\mathbb{Z})$.

We fix a congruence subgroup $\Gamma$. Let $H := \{ \tau \in \mathbb{C} | \text{Im}(\tau) > 0 \}$ be the upper half plane. Given $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2^+(\mathbb{Q})$, define

$$\gamma \cdot \tau := \frac{a\tau + b}{c\tau + d}$$

$$j(\gamma, \tau) := c\tau + d$$

(3.4)

(3.5)

Now we define a right action of $GL_2^+(\mathbb{Q})$ on the space of holomorphic functions from $H$ to $\mathbb{C}$. Given a holomorphic function $f : H \to \mathbb{C}$, and an integer $k$, define

$$f | [\gamma]_k(\tau) := f(\gamma \cdot \tau)j(\gamma, \tau)^{-k} \text{det}(\gamma)^{k-1}.$$  

(3.6)

**Definition 3.1.1 (Modular Forms).** For a holomorphic function $f : H \to \mathbb{C}$, it is called a modular form of $\Gamma$ of weight $k$ if it satisfies

1. $f | [\gamma]_k = f$ for any $\gamma \in \Gamma$.

2. Given any $\alpha \in SL_2(\mathbb{Z})$, $f | [\alpha]_k$ is $\alpha^{-1}\Gamma\alpha$-invariant, which consists some $\Gamma(N)$, hence the group $\left\{ \begin{pmatrix} 1 & Nl \\ 0 & 1 \end{pmatrix} \mid l \in \mathbb{Z} \right\}$. Let $q := e(\tau) = \exp(2\pi i\tau)$, then $f | [\alpha]_k$ has a $q$-expansion (since it is invariant under translation by $N$)

$$f | [\alpha]_k := \sum_{i \in \mathbb{Z}} a_i q^{i/N}.$$  

(3.7)

We require $a_i = 0$ when $i < 0$, for each $\alpha \in SL_2(\mathbb{Z})$.

If $f$ is a modular form and in condition 2 we have $a_0 = 0$ for each $\alpha \in SL_2(\mathbb{Z})$, then we call it a cusp form. We denote $M_k(\Gamma)$ (resp. $S_k(\Gamma)$) to be the $\mathbb{C}$-vector space of modular forms (resp. cusp forms) of $\Gamma$ of weight $k$. 

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It can be checked that if $f \in \mathcal{M}_{k_1}(\Gamma), g \in \mathcal{M}_{k_2}(\Gamma)$, then $f \cdot g \in \mathcal{M}_{k_1+k_2}(\Gamma)$. This gives $\mathcal{M}(\Gamma) := \bigoplus_{k>0} \mathcal{M}_k(\Gamma)$ the product structure.

**Definition 3.1.2.** Let $\Gamma$ to be $\Gamma_0(N)$ or $\Gamma_1(N)$. Since $\Gamma$ contains

$$
\begin{pmatrix}
1 & 1 \\
0 & 1
\end{pmatrix}
$$

then for any modular form $f$ of $\Gamma$, its $q$-expansion can be regarded as a formal power series of $q$, so we can regard $\mathcal{M}(\Gamma) \subset \mathbb{C}[[q]]$ as a subring. Let $A$ be a subring of $\mathbb{C}$, define

$$
\mathcal{M}_k(\Gamma; A) := \mathcal{M}_k(\Gamma) \cap A[[q]], \quad (3.8)
$$

$$
\mathcal{M}(\Gamma; A) := \bigoplus_{k>0} \mathcal{M}_k(\Gamma; A). \quad (3.9)
$$

We call $f \in \mathcal{M}_k(\Gamma; A)$ a modular form of $\Gamma$ of weight $k$ in coefficient $A$. Since $\mathcal{M}(\Gamma)$ has a product structure, $\mathcal{M}(\Gamma; A)$ also has a product structure.

**Theorem 3.1.3.** $\Gamma = SL_2(\mathbb{Z}), k > 0$, then for any subring $A$ of $\mathbb{C}$ we have:

$$
\mathcal{M}_k(\Gamma; A) = \mathcal{M}_k(\Gamma; \mathbb{Z}) \otimes_{\mathbb{Z}} A. \quad (3.10)
$$

*Proof.* Basically that’s because we have Eisenstein series $E_4$, $E_6$ as generators of $\mathcal{M}_k(\Gamma)$ and their $q$-expansion coefficients are integral (up to a constant). See [Ser73](Ch. VII). □

**Theorem 3.1.4.** If $k > 6$ and $\Gamma = \Gamma_0(p^r)$, where $p$ a prime, $r \geq 2$ when $p = 2$, nonnegative otherwise, then for any subring $A$ of $\mathbb{C}$ we have:

$$
\mathcal{M}_k(\Gamma; A) = \mathcal{M}_k(\Gamma; \mathbb{Z}) \otimes_{\mathbb{Z}} A. \quad (3.11)
$$

*Proof.* See [Hid93](p. 150). □
3.2 Basics of abstract Hecke rings and Hecke operators

**Definition 3.2.1.** Let $G$ be a group, $\Gamma_1, \Gamma_2$ are two subgroups of $G$, we say $\Gamma_1, \Gamma_2$ are commensurable if the indexes $[\Gamma_1 : \Gamma_1 \cap \Gamma_2]$ and $[\Gamma_2 : \Gamma_1 \cap \Gamma_2]$ are both finite. Denote $\Gamma_1 \sim \Gamma_2$. It’s easy to check that commensurable condition is an equivalence relation.

**Lemma 3.2.2.** Let $G$ be a group, $\Gamma$ is a subgroup of $G$, then the set

$$\tilde{\Gamma} := \{ g \in G \mid g \Gamma g^{-1} \sim \Gamma \} \quad (3.12)$$

is a group containing $\Gamma$. It is called the commensurator of $\Gamma$ in $G$. $g \in \tilde{\Gamma}$ is called a $\Gamma$-rational element of $G$.

**Proof.** $a, b \in \tilde{\Gamma} \Rightarrow \Gamma \sim b \Gamma b^{-1} \Rightarrow b^{-1} \Gamma b \sim \Gamma \Rightarrow ab^{-1} \Gamma b a^{-1} \sim \Gamma \Rightarrow ab^{-1} \in \tilde{\Gamma}$, hence $\tilde{\Gamma}$ is a group. $\square$

**Definition 3.2.3** (Hecke pair). Let $\Gamma \subset G$ be a subgroup, $S \subset G$ contains 1 and is multiplicatively closed. If we have $\Gamma \subset S \subset \tilde{\Gamma}$, we call $(\Gamma, S)$ a Hecke pair.

Define $\mathcal{L} = \mathcal{L}(\Gamma, S) := \oplus_{g \in S} \mathbb{Z} \Gamma g$, the free $\mathbb{Z}$-module generated by right cosets in $S$. Consider the right $S$-action on $\mathcal{L}$: $\Gamma g \mapsto \Gamma gs$, define $\mathcal{D} = \mathcal{D}(\Gamma, S)$ to be the $\Gamma$-invariant element in $\mathcal{L}$.

**Lemma 3.2.4.** For $g$ in $S$, $\Gamma \backslash \Gamma g \Gamma$ is finite.

**Proof.** See [DS05](Ch. 5) or as a special case($n = 1$) of lemma 4.1.1. $\square$

By this lemma, given $g$ in $S$, $\Gamma g \Gamma = \bigsqcup_{g_i \in C} \Gamma g_i$, where $C \subset S$ is finite. Define $[g] = [g]_r := \sum_{g_i \in C} \Gamma g_i$.

**Lemma 3.2.5.** $\{ [g] \mid g \in S \}$ forms a $\mathbb{Z}$-basis of $\mathcal{D}$. 

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Proof. See [AZ95](Ch. 3) or as a special case \((n = 1)\) of lemma 4.1.2. □

Now we define an operation on \(\mathcal{D}\). We define it on basis then extend it \(\mathbb{Z}\)-linearly. Suppose we have \(g, h \in S\), and assume \([g] = \sum_i \Gamma g_i\), \([h] = \sum_j \Gamma h_j\). Define

\[
[g] \cdot [h] := \sum_{i,j} \Gamma g_i h_j. \tag{3.13}
\]

**Lemma 3.2.6.** *This operation is well-defined and it gives a ring structure on \(\mathcal{D}\).*

We call \(\mathcal{H} = \mathcal{H}(\Gamma, S) = \mathcal{D}\) to be the abstract Hecke ring of the Hecke pair \((\Gamma, S)\).

Proof. See [AZ95](Ch. 3) or as a special case \((n = 1)\) of proposition 4.1.4. □

Now we restrict to the case when \(\Gamma\) is a congruence group, \(G = GL_2^+(\mathbb{Q})\).

**Lemma 3.2.7.** *The commensurator \(\tilde{\Gamma}\) of \(\Gamma\) is \(G\).*

Proof. Suppose \(\Gamma \supset \Gamma(N)\). It is enough to show for any \(g \in G\), \(g\Gamma g^{-1}\) contains some \(\Gamma(M)\). By multiplication of scalar matrices, we can assume \(g \in M_2^+(\mathbb{Z})\).

\(D := \det(g)\), take \(M = DN\). Then \(\forall \gamma \in \Gamma(M)\)

\[
g^{-1} \gamma g = g^{-1} \left( I + \begin{pmatrix} aM & bM \\ cM & dM \end{pmatrix} \right) g = I + D^{-1} g^* \begin{pmatrix} aDN & bDN \\ cDN & dDN \end{pmatrix} g
\]

\[
= I + g^* \begin{pmatrix} aN & bN \\ cN & dN \end{pmatrix} g \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{(mod } N)\). \tag{3.14}
\]

So \(g^{-1} \Gamma(M)g \subset \Gamma(N) \subset \Gamma\), i.e. \(g\Gamma g^{-1} \supset \Gamma(M)\). □

Now we take this Hecke pair \((\Gamma, G)\). \(\mathcal{H} = \mathcal{H}(\Gamma, G)\) is the corresponding abstract Hecke ring. Define \(\mathcal{H}^\mathbb{C} = \mathcal{H}(\Gamma, G)^\mathbb{C} := \mathcal{H}(\Gamma, G) \otimes_{\mathbb{Z}} \mathbb{C}\) to be the Hecke ring over \(\mathbb{C}\).

We define a right action of \(\mathcal{H}^\mathbb{C}\) on \(\mathcal{M}_k(\Gamma)\). Given \([g] = \sum_i \Gamma g_i\) a basis element, \(f \in \mathcal{M}_k(\Gamma)\), define

\[
f \parallel [g]_k := \sum_i f \parallel [g_i]_k. \tag{3.15}
\]
Proposition 3.2.8. If we extend the operations defined above \( \mathbb{C} \)-linearly to \( \mathcal{H}^C \), then it gives a representation of the \( \mathbb{C} \)-algebra \( \mathcal{H}^C \) on the \( \mathbb{C} \)-vector space \( \mathcal{M}_k(\Gamma) \) (also \( \mathcal{M}(\Gamma) \)). Denote \( \mathcal{H}_k(\Gamma) \) to be the image of \( \mathcal{H}^C \) in \( \text{End}(\mathcal{M}_k(\Gamma)) \) (\( \mathcal{H}(\Gamma) \) in \( \text{End}(\mathcal{M}(\Gamma)) \) respectively). We call these operators in \( \mathcal{H}_k(\Gamma) \) Hecke operators over \( \mathbb{C} \).

Proof. See [DS05](Ch. 5) or proposition 4.2.3 as a special case.

Suppose \( \Gamma = SL_2(\mathbb{Z}) \), take \( S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \). It is well-known that they generate \( \Gamma \), see e.g. [Ser73](Ch. VII).

Lemma 3.2.9. \( \Gamma = SL_2(\mathbb{Z}) \). For any \( g \in M_2^+(\mathbb{Z}) \),

1. \( \exists! \tilde{g} \in \Gamma g \Gamma \) of the form \( \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} \) where \( 0 < a \mid d \).

2. \( \exists! \tilde{g} \in \Gamma g \Gamma \) of the form \( \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \) where \( a, d > 0, 0 \leq b < d \).

Proof. Assume \( g = \begin{pmatrix} x & y \\ z & u \end{pmatrix} \),

1. First use row and column operations of \( S \) and \( T \) to make the number \( a = \gcd(x, y, z, u) \) appear, then make it diagonal with first entry \( a \), which is of the form we want. The uniqueness follows.

2. We can use row operations of \( S \) and \( T \) to make \( g \) upper triangular, then use column operations of \( T \) to adjust \( b \) to make it into the form we require. For uniqueness, assume there is \( \gamma \in \Gamma, \gamma g_1 = g_2 \), where \( g_1, g_2 \) are two different matrices of the given form, then \( \gamma \) is upper triangular, so \( \gamma = \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \), hence \( \gamma = Id \).
Now let $\Gamma = SL_2(\mathbb{Z})$. Define $\mathcal{H}_\mathbb{Z}$ to be $\mathbb{Z}$-submodule of $\mathcal{H}$ generated by all possible $[\alpha]$, $\alpha \in M_2^+(\mathbb{Z})$. For any subring $A$ of $\mathbb{C}$, define $\mathcal{H}_A := \mathcal{H}_\mathbb{Z} \otimes_\mathbb{Z} A$.

**Proposition 3.2.10.** $\mathcal{H}_A$ acts on $\mathcal{M}_k(\Gamma; A)$. Denote $\mathcal{H}_k(\Gamma; A)$ to be its image in $End(\mathcal{M}_k(\Gamma))$ ($\mathcal{H}(\Gamma; A)$ in $End(\mathcal{M}(\Gamma))$ respectively), then $\mathcal{H}_k(\Gamma; A) \otimes_A \mathbb{C} = \mathcal{H}_k(\Gamma)$. We call these operators in $\mathcal{H}_k(\Gamma; A)$ Hecke operators over $A$.

**Proof.** We need only to show for $\mathcal{H}_\mathbb{Z}$, then by base change argument, $\mathcal{H}_A := \mathcal{H}_\mathbb{Z} \otimes_\mathbb{Z} A$ satisfies the condition. Given any $[g] \in \mathcal{H}$, suppose $g = \lambda \alpha$ for some $\lambda \in \mathbb{Q}$ and $\alpha \in M_2^+(\mathbb{Z})$. For any $f$ in $\mathcal{M}_k(\Gamma)$, $f \parallel [g]_k = f \parallel [\lambda \alpha]_k = \lambda^{k-2} \cdot f \parallel [\alpha]_k$.

Hence the images of $[g]$ and $[\alpha]$ in $\mathcal{H}_k(\Gamma)$ differ by a scalar in $\mathbb{Q}$. Thus all possible $[\alpha]$ also generate $\mathcal{H}_k(\Gamma)$ over $\mathbb{C}$, so we have $\mathcal{H}_k(\Gamma; \mathbb{Z}) \otimes_\mathbb{Z} \mathbb{C} = \mathcal{H}_k(\Gamma)$.

Now we verify for any $\alpha \in M_2^+(\mathbb{Z})$, $[\alpha]$ acts on $\mathcal{M}_k(\Gamma; \mathbb{Z})$. Suppose $f \in \mathcal{M}_k(\Gamma; \mathbb{Z})$, then $f = \sum_{n \geq 0} a_n q^n$, $a_n \in \mathbb{Z}$. Denote $C$ to be the set of all representatives that appear in $[\alpha]$. By lemma 3.2.9, we can uniquely make all elements $g \in C$ of the form $g = \begin{pmatrix} a & b \\ 0 & d \end{pmatrix}$ where $a, d > 0$, $0 \leq b < d$. Denote $l = \text{gcd}(a, b, d)$, $a = a_1 l$, $d = d_1 l$, $l_1 = \text{gcd}(a_1, d_1)$, define $C_g \subset C$ consisting of elements in $C$ with the same $a, d$ of $g$, then by lemma 3.2.9

$$C_g = \left\{ \begin{pmatrix} a & \lambda l \\ 0 & d \end{pmatrix} \mid 0 \leq \lambda < d, \text{gcd}(\lambda, l_1) = 1 \right\}.$$  \hspace{1cm} (3.16)

Because they are all in the double coset of $\begin{pmatrix} l & 0 \\ 0 & ad/l \end{pmatrix}$. Thus, we have

$$f \parallel [\alpha]_k (\tau) = \sum_{C_g} \sum_{\theta \in C_g} f \parallel [\theta]_k (\tau)$$

$$= \sum_{C_g} \sum_{\theta \in C_g} a_n e(n \tau) \parallel [\theta]_k \cdot (ad)^{k-1} \cdot d^{-k}$$

$$= \sum_{n \geq 0, C_g} a_n d^{k-1} l^{-1} \sum_{\lambda} e(n \cdot a \tau + \lambda l / d)$$

$$= \sum_{n \geq 0, C_g} a_n q^{an/d} d^{k-1} l^{-1} \sum_{\lambda} e(nl \cdot \lambda / d).$$  \hspace{1cm} (3.17)
We need a lemma.

**Lemma 3.2.11.** Let \( m, d \) be two positive integers and \( m \mid d \), \( k \) is an integer.
then \( \sum_{0 \leq \lambda < d, \gcd(\lambda, m) = 1} e(k\lambda/d) \) is an integer and divisible by \( d/m \).

**Proof.** Let \( p_1, \cdots, p_r \) be all prime factors of \( m \). Denote \( X_i := \{ \lambda \mid 0 \leq \lambda < d, \gcd(\lambda, p_i) = 1 \} \), \( X_\emptyset := \cup X_i \), \( X_{i_1 \cdots i_s} := X_{i_1} \cap \cdots \cap X_{i_s} = \{ \lambda \mid 0 \leq \lambda < d, \gcd(\lambda, p_{i_1} \cdots p_{i_s}) = 1 \} \). Define \( S(X) := \sum_{\lambda \in X} e(k\lambda/d) \). Then in fact
\[
\sum_{0 \leq \lambda < d, \gcd(\lambda, m) = 1} e(k\lambda/d) = S(X_{1 \cdots r}). \tag{3.18}
\]
We prove by induction on \( r \). Denote \( \zeta = e(k/d) \). \( r = 0 \),
\[
S(X_\emptyset) = 1 + \zeta + \cdots + \zeta^{d-1} = \begin{cases} 
1 - \zeta^d & \text{if } \zeta \neq 1, \\
1 - \zeta & \text{if } \zeta = 1.
\end{cases} \tag{3.19}
\]
which is divisible by \( d \), so is \( d/m \).
For \( r > 0 \), denote \( p = p_1 \cdots p_r \), then \( S(X_\emptyset) = (1 + \zeta + \cdots + \zeta^{d-1}) - (1 + \zeta^p + \cdots + \zeta^{d-p}) \) is divisible by \( d/p \), hence also, \( d/m \). By inclusion-exclusion principle, we have
\[
\sum_{s=0}^r (-1)^s \sum_{i_1 \cdots i_s} S(X_{i_1 \cdots i_s}) = 0 \tag{3.20}
\]
\[
\Rightarrow S(X_{1 \cdots r}) = (-1)^r S(X_\emptyset) + \sum_{s=1}^{r-1} (-1)^{r-s} \sum_{i_1 \cdots i_s} S(X_{i_1 \cdots i_s}). \tag{3.21}
\]
By induction on \( p_{i_1} \cdots p_{i_s} \), \( S(X_{i_1 \cdots i_s}) \) is divisible by \( d/p_{i_1} \cdots p_{i_s} \), hence also \( d/m \).
So \( S(X_{1 \cdots r}) \) is also divisible by \( d/m \).

Using this lemma, we see \( \sum_\lambda e(n\lambda / \lambda/d) \) is divisible by \( d/l_1 \). In [Ser73] (Ch. VII) we know \( k \geq 2 \) if \( M_k(\Gamma) \) is non-zero. Hence in the equation \( 3.17 \), \( l_1 \mid a \) is cancelled out by \( a^{k-1} \). So we get all integral coefficients in equation \( 3.17 \) in the \( q \)-expansion of \( f \ | \alpha \rangle_k \). Therefore it lies inside \( \mathbb{Z}[q] \), but also in \( M_k(\Gamma) \), thus in \( M_k(\Gamma; \mathbb{Z}) \). \qed
Remark 3.2.12. 1. In fact the previous proposition \(3.2.10\) can be viewed as a special part in the later more general theorem \(4.3.16\).

2. Notice in our definition \(\mathcal{H}_C^C\) and \(\mathcal{H}_C\) are different, but their images inside \(\text{End}(\mathcal{M}_k(\Gamma))\) are the same.
Chapter 4

Main Results

In this chapter, we first construct an abstract operad for each Hecke pair, which generalizes abstract Hecke ring. We also find that, this operad acts on the modular forms over \( \mathbb{C} \) which generalizes the usual Hecke operators. Then we study a Galois action on this operad and by taking Galois orbits we find a suboperad acting on the modular forms over \( \mathbb{Z} \), hence on arbitrary modular forms over \( A \), where \( A \) is a subring of \( \mathbb{C} \).

4.1 Abstract Operads of Hecke Pairs

Let \( G \) be a group, \((\Gamma, S)\) a Hecke pair in \( G \). \( \mathcal{L} = \mathcal{L}(\Gamma, S) \), the free \( \mathbb{Z} \)-module generated by right cosets of \( \Gamma \) in \( S \). Define \( \mathcal{L}^n = \mathcal{L}^n(\Gamma, S) := \bigotimes_{\mathbb{Z}}^n \mathcal{L}(\Gamma, S) \) (we define \( \mathcal{L}^0 = 0 \)), and let \( S \) act on \( \mathcal{L}^n \) diagonally. Define \( \mathcal{D}^n = \mathcal{D}^n(\Gamma, S) \) to be the \( \Gamma \)-invariant element in \( \mathcal{L}^n \). Let \( \Delta : S \to S^n \) to be the diagonal map, i.e. \( \Delta(g) = (g, \cdots, g) \).

Lemma 4.1.1. For \( g = (g^1, \cdots, g^n) \in S^n \), \( \Gamma^n \backslash \Gamma^n g \Delta \Gamma \) is finite, i.e. if we consider the right \( \Gamma \)-action on \( \Gamma^n \backslash S^n \), each orbit is finite.
Proof. $\Gamma^n \setminus \Gamma^n g \Delta g^{-1} \cong \Gamma^n \cap g \Delta g^{-1} \setminus g \Delta g^{-1}$

$\cong g^{-1} \Gamma^n g \cap \Delta \setminus \Delta g = \Gamma \cap i(g_i)^{-1} \Delta g^{-1}$, hence

$|\Gamma^n \setminus \Gamma^n g \Delta | = |\Gamma \setminus \Gamma \cap i(g_i)^{-1} \Delta g^{-1}|$

$\leq \prod_i |\Gamma \cap i(g_i)^{-1} \Delta g^{-1}| < \infty$. \hfill \Box

By the previous lemma 4.1.1, given $g$ in $S^n$, we have the decomposition of the double coset $\Gamma^n g \Delta = \bigcup_{g_i \in C} \Gamma^n g_i$, where $C \subset S^n$ is finite. Define $[g] = [g]r := \sum_{g_i \in C} \Gamma^n g_i \in \mathcal{L}^n$.

Lemma 4.1.2. $\{[g] \mid g \in S^n\}$ forms a $\mathbb{Z}$-basis of $\mathcal{D}^n$.

Proof. In fact, it follows from the general result that if $G$ acts on $X$ with finite orbit, then if we take the sum of elements in the orbit, they form a $\mathbb{Z}$-basis for all $G$-fixed elements in the free $\mathbb{Z}$-module generated by elements in $X$. \hfill \Box

Theorem 4.1.3. $\delta = \{\delta(n)\}_{n \geq 0}$ where $\delta(n) = \mathcal{D}^n(\Gamma, S)$ has a natural operadic structure in the category of $\mathbb{Z}$-modules. We call it the abstract $\mathbb{Z}$-operad of the Hecke pair $(\Gamma, S)$.

Proof. We construct the operadic structure and verify it is an operad in the sense of Markl. To define $\circ_i$, we define it on the $\mathbb{Z}$-basis $\{[g]\}$, then we extend it $\mathbb{Z}$-linearly to $\delta(n) = \mathcal{D}^n(\Gamma, S)$. Suppose $\alpha \in \delta(m)$, $\beta \in \delta(n)$, assume

$$\begin{cases} [\alpha] = \sum_{\alpha_a \in C_\alpha} \Gamma^m \alpha_a, & C_\alpha \subset S^n \text{ is finite.} \\ [\beta] = \sum_{\beta_b \in C_\beta} \Gamma^n \beta_b, & C_\beta \subset S^n \text{ is finite.} \end{cases}$$

where $\alpha_a = (\alpha_a^1, \cdots, \alpha_a^m)$, $\beta_b = (\beta_b^1, \cdots, \beta_b^n)$. Define

$$[\alpha] \circ_i [\beta] := \sum_{a, b} \Gamma^{m+n-1}(\alpha_a^1, \cdots, \alpha_a^{i-1}, \beta_b^1 \alpha_a^i, \cdots, \beta_b \alpha_a^i, \alpha_a^{i+1}, \cdots, \alpha_a^m). \quad (4.1)$$

It is well-defined and it is in $\mathcal{D}^n(\Gamma, S)$.

Well-definedness:
Independent of Choices of $\alpha_a$ and $\beta_b$. Suppose $\tilde{\alpha}_a = (\gamma_a^1\alpha_a^1, \ldots, \gamma_a^m\alpha_a^m)$ and $\tilde{\beta}_b = (\gamma_b^1\beta_b^1, \ldots, \gamma_b^m\beta_b^m)$, $\gamma_a, \gamma_b \in \Gamma$, then
\[ [\beta] = \sum_b \Gamma^n \beta_b = \sum_b \Gamma^n \beta_b \Delta(\gamma_b^i), \]
therefore
\[ \sum_{a,b} \Gamma^{m+n-1}(\alpha_a^1, \ldots, \alpha_a^{i-1}, \beta_b^1, \alpha_a^i, \ldots, \beta_b^m, \alpha_a^{i+1}, \ldots, \alpha_a^m) \]
\[ = \sum_{a,b} \Gamma^{m+n-1}(\gamma_a^1, \ldots, \gamma_a^{i-1}, \beta_b^1, \gamma_a^i, \ldots, \beta_b^m, \gamma_a^{i+1}, \ldots, \gamma_a^m) \]
\[ = \sum_{a,b} \Gamma^{m+n-1}(\alpha_a^1, \ldots, \alpha_a^{i-1}, \beta_b^1, \alpha_a^i, \ldots, \beta_b^m, \alpha_a^{i+1}, \ldots, \alpha_a^m), \]
\[ = \sum_{a,b} \Gamma^{m+n-1}(\alpha_a^1, \ldots, \alpha_a^{i-1}, \beta_b^1, \alpha_a^i, \ldots, \beta_b^m, \alpha_a^{i+1}, \ldots, \alpha_a^m). \]
\[ (4.2) \]

In $\mathcal{D}^n(\Gamma, S)$:

Given $\gamma \in \Gamma$, $[a] = \sum_a \Gamma^m \alpha_a = \sum_a \Gamma^m \alpha_a \Delta(\gamma)$, by well-definedness, we have
\[ (\alpha_a, \beta) \]}
\[ = \sum_{a,b} \Gamma^{m+n-1}(\alpha_a^1\gamma, \ldots, \alpha_a^{i-1}\gamma, \beta_b^1\alpha_a^i\gamma, \ldots, \beta_b^m\alpha_a^{i+1}\gamma, \ldots, \alpha_a^m\gamma) \]
\[ = \sum_{a,b} \Gamma^{m+n-1}(\alpha_a^1, \ldots, \alpha_a^{i-1}, \beta_b^1\alpha_a^i, \ldots, \beta_b^m\alpha_a^{i+1}, \ldots, \alpha_a^m) \]
\[ = [a] \alpha_i [\beta]. \]
\[ (4.3) \]

So it is in $\mathcal{D}^n(\Gamma, S)$.

$k[\Sigma_n]$-structure. Suppose $\sigma \in \Sigma_n$, $g \in S^n$ define the right action on $\mathcal{L}^n$ by
\[ (\Gamma^n(g^1, \ldots, g^n))^{\sigma} := \Gamma^n(g^{\sigma(1)}, \ldots, g^{\sigma(n)}). \]
\[ (4.4) \]

This action commutes with the right $S$-action, hence it is a right action on $\mathcal{D}^n(\Gamma, S)$.

To check the axioms of operads, we need only to check it on basis $[g]$. Suppose $f = [a] \in \delta(m)$, $g = [\beta] \in \delta(n)$, $h = [\gamma] \in \delta(l)$, assume
\[ [\alpha] = \sum_{a \in C_\alpha} \Gamma^m \alpha_a, \quad C_\alpha \subset S^m \text{ is finite.} \]
\[ [\beta] = \sum_{b \in C_\beta} \Gamma^n \beta_b, \quad C_\beta \subset S^n \text{ is finite.} \]
\[ [\gamma] = \sum_{c \in C_\gamma} \Gamma^n \gamma_c, \quad C_\gamma \subset S^l \text{ is finite.} \]
We check the axioms in 2.1.3.

1. Associativity.

Given $1 \leq i < j \leq m$,

\[
(f \circ_j g) \circ_i h
= \left( \sum_{a,b} \Gamma^{m+n-1}(\alpha_a^1, \ldots, \alpha_a^{i-1}, \beta_b^1 \alpha_a^j, \ldots, \beta_b^m \alpha_a^{j+1}, \ldots, \alpha_a^m) \right) \circ_i [\gamma]
= \sum_{a,b,c} \Gamma^{m+n+1-2}(\alpha_a^1, \ldots, \alpha_a^{i-1}, \gamma_c^1 \alpha_a^i, \ldots, \gamma_c^l \alpha_a^i, \alpha_a^{i+1}, \ldots, \alpha_a^{j-1}, \beta_b^1 \alpha_a^j, \ldots, \beta_b^m \alpha_a^{j+1}, \ldots, \alpha_a^m).
\]

(4.5)

\[
(f \circ_i h) \circ_{j+l-1} g
= \left( \sum_{a,c} \Gamma^{m+n-1}(\alpha_a^1, \ldots, \alpha_a^{i-1}, \gamma_c^1 \alpha_a^i, \ldots, \gamma_c^l \alpha_a^i, \alpha_a^{i+1}, \ldots, \alpha_a^m) \right) \circ_{j+l-1} [\beta]
= \sum_{a,b,c} \Gamma^{m+n+1-2}(\alpha_a^1, \ldots, \alpha_a^{i-1}, \gamma_c^1 \alpha_a^i, \ldots, \gamma_c^l \alpha_a^i, \alpha_a^{i+1}, \ldots, \alpha_a^{j-1}, \beta_b^1 \alpha_a^j, \ldots, \beta_b^m \alpha_a^{j+1}, \ldots, \alpha_a^m).
\]

(4.6)

Hence $(f \circ_j g) \circ_i h = (f \circ_i h) \circ_{j+l-1} g$.

Given $j \leq i < j + n$,

\[
(f \circ_j g) \circ_i h
= \left( \sum_{a,b} \Gamma^{m+n-1}(\alpha_a^1, \ldots, \alpha_a^{i-1}, \beta_b^1 \alpha_a^j, \ldots, \beta_b^m \alpha_a^{j+1}, \ldots, \alpha_a^m) \right) \circ_i [\gamma]
= \sum_{a,b,c} \Gamma^{m+n+1-2}(\alpha_a^1, \ldots, \alpha_a^{i-1}, \beta_b^1 \alpha_a^j, \ldots, \beta_b^j \alpha_a^j, \gamma_c^1 \beta_b^{i-j+1} \alpha_a^j, \ldots, \gamma_c^l \beta_b^{i-j+1} \alpha_a^j, \alpha_a^{i+1}, \ldots, \alpha_a^{j+1}, \beta_b^1 \alpha_a^{j+2}, \ldots, \beta_b^m \alpha_a^{j+1}, \ldots, \alpha_a^m).
\]

(4.7)

\[
f \circ_j (g \circ_{i-j+1} h)
= f \circ_j \left( \sum_{b,c} \Gamma^{m+n-1}(\beta_b^1, \ldots, \beta_b^j \gamma_c^{i-j+1}, \ldots, \gamma_c^l \beta_b^{i-j+1}, \beta_b^{i-j+2}, \ldots, \beta_b^m) \right)
= \sum_{a,b,c} \Gamma^{m+n+1-2}(\alpha_a^1, \ldots, \alpha_a^{i-1}, \beta_b^1 \alpha_a^j, \ldots, \beta_b^j \alpha_a^{j+1}, \gamma_c^1 \beta_b^{i-j+1} \alpha_a^{j+1}, \ldots, \gamma_c^l \beta_b^{i-j+1} \alpha_a^{j+1}, \beta_b^1 \alpha_a^{j+2}, \ldots, \beta_b^m \alpha_a^{j+1}, \ldots, \alpha_a^m).
\]

(4.8)

Hence $(f \circ_j g) \circ_i h = f \circ_j (g \circ_{i-j+1} j)$. 

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2. Equivariance. Suppose \( \tau \in \Sigma_m, \sigma \in \Sigma_n, \)

\[
f^\tau \circ_i g^\sigma
\]

\[
= (\sum_a \Gamma^m(\alpha_a^1, \ldots, \alpha_a^m))^\tau \circ_i (\sum_b \Gamma^n(\beta_b^1, \ldots, \beta_b^n))^\sigma
\]

\[
= (\sum_a \Gamma^m(\alpha_a^{(1)}, \ldots, \alpha_a^{(m)})) \circ_i (\sum_b \Gamma^n(\beta_b^{\sigma(1)}, \ldots, \beta_b^{\sigma(n)}))
\]

\[
= \sum_{a,b} \Gamma^{m+n-1}(\alpha_a^{\tau(1)}, \ldots, \alpha_a^{\tau(i-1)}, \beta_b^{\sigma(1)} \alpha_a^{\tau(i)}, \ldots, \beta_b^{\sigma(n)} \alpha_a^{\tau(i)}, \ldots, \alpha_a^{\tau(m)})
\]

(4.9)

\[
(f \circ_{\tau(i)} g)^{\tau \circ_{\tau(i)} \sigma}
\]

\[
= \left(\left(\sum_a \Gamma^m(\alpha_a^1, \ldots, \alpha_a^m)\right) \circ_{\tau(i)} \left(\sum_b \Gamma^n(\beta_b^1, \ldots, \beta_b^n)\right)\right)^{\tau \circ_{\tau(i)} \sigma}
\]

\[
= \left(\sum_{a,b} \Gamma^{m+n-1}(\alpha_a^{i=1}, \ldots, \alpha_a^{i-1}, \beta_b^{\sigma(1)} \alpha_a^{i}, \ldots, \beta_b^{\sigma(n)} \alpha_a^{i}, \ldots, \alpha_a^{i})\right)^{\tau \circ_{\tau(i)} \sigma}
\]

\[
= \sum_{a,b} \Gamma^{m+n-1}(\alpha_a^{\tau(1)}, \ldots, \alpha_a^{\tau(i-1)}, \beta_b^{\sigma(1)} \alpha_a^{\tau(i)}, \ldots, \beta_b^{\sigma(n)} \alpha_a^{\tau(i)}, \ldots, \alpha_a^{\tau(m)})
\]

(4.10)

Hence \( f^\tau \circ_i g^\sigma = (f \circ_{\tau(i)} g)^{\tau \circ_{\tau(i)} \sigma}. \)

3. Unitality. Take \( e = [1] = \Gamma \cdot \Gamma \in D(1), \) then by definition we have \([\alpha] \circ_i e = [\alpha]\)

and \( e \circ_1 [\beta] = [\beta]. \) Thus \( e \) is the required unit.

In conclusion we have checked that \( \delta \) is a \( \mathbb{Z} \)-operad. \(\)

**Proposition 4.1.4.** \( \delta(1) \) with the operadic composition \( \circ_1 : \delta(1) \otimes \delta(1) \to \delta(1) \)

is isomorphic to the opposite ring \( \mathcal{H}^{\text{op}} \) associated to the abstract Hecke ring \( \mathcal{H} = \mathcal{H}(\Gamma, S) \) of the Hecke pair \((\Gamma, S), \) where given \( x, y \) in \( \mathcal{H}^{\text{op}} = \mathcal{H}, \) \( x *_{\text{op}} y = yx. \)

**Proof.** This follows almost directly from the definition. Given \([\alpha],[\beta]\) in \( \delta(1) = \mathcal{H}, \) assume \([\alpha] = \sum_a \Gamma \alpha_a, [\beta] = \sum_b \Gamma \beta_b. \)

\([\alpha] \cdot [\beta] := \sum_{a,b} \Gamma \alpha_a \beta_b \) in the Hecke ring.

\([\alpha] \circ_1 [\beta] := \sum_{a,b} \Gamma \beta_b \alpha_a \) in the operad.

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Hence the result follows.

4.2 \(\mathbb{C}\)-Operad acting on Modular Forms

Let \(G = GL_2^+(\mathbb{Q})\) and \(\Gamma \subset SL_2(\mathbb{Z})\) be a congruence subgroup. \(\delta\) is the abstract \(\mathbb{Z}\)-operad of this Hecke pair \((\Gamma, G)\) defined in theorem 4.1.3. Define \(\delta^C = \{\delta(n)^C\} := \{\delta(n) \otimes_\mathbb{Z} \mathbb{C}\}\) to be the complex abstract operad of \(\delta\). Suppose \(\alpha \in G^m, f_1 \in \mathcal{M}_{k_1}(\Gamma), \ldots, f_m \in \mathcal{M}_{k_m}(\Gamma)\). Assume \([\alpha] = \sum_a \Gamma^m a\alpha_a\). We define the operation

\[
[\alpha] \circ (f_1, \ldots, f_m) := \sum_a f_1 \mid [\alpha_a^1]_{k_1} \cdots f_n \mid [\alpha_a^m]_{k_m},
\]

we call the end result \(s\).

**Lemma 4.2.1.** \(s\) is well-defined and in \(\mathcal{M}_{k_1 + \ldots + k_m}(\Gamma)\).

**Proof.** Well-definedness:

We show \(s\) is independent of choice of \(\alpha_a\). Suppose \(\tilde{\alpha}_a = (\gamma_a^1 a_1, \ldots, \gamma_a^m a_m)\), \(\gamma_a^i \in \Gamma\), then we get

\[
\sum_a f_1 \mid [\gamma_a^1 a_1]_{k_1} \cdots f_m \mid [\gamma_a^m a_m]_{k_m} = \sum_a f_1 \mid [\gamma_a^1]_{k_1} [\alpha_a^1]_{k_1} \cdots f_m \mid [\gamma_a^m]_{k_m} [\alpha_a^m]_{k_m} = \sum_a f_1 \mid [\alpha_a^1]_{k_1} \cdots f_n \mid [\alpha_a^m]_{k_m} = s.
\] (4.12)

Now we verify \(s\) satisfies the conditions of modular forms. Obviously \(s\) is a holomorphic function of the upper half plane \(\mathcal{H}\).

1. \(\Gamma\)-invariant. Take any \(\gamma \in \Gamma\), \((\alpha) = (\alpha \gamma) = \sum_a \Gamma^m a\alpha_a \Delta(\gamma)\).

\[
s \mid [\gamma]_{k_1 + \ldots + k_m} = \sum_a f_1 \mid [\alpha_a^1]_{k_1} [\gamma_a^1]_{k_1} \cdots f_m \mid [\alpha_a^m]_{k_m} [\gamma_a^m]_{k_m} = \sum_a f_1 \mid [\alpha_a^1]_{k_1} \cdots f_m \mid [\alpha_a^m]_{k_m} = s.
\] (4.13)
2. Holomorphic at all cusps. Take any \( \theta \in SL_2(\mathbb{Z}) \),
\[
s [[\theta]_{k_1 + \cdots + k_m}] = \sum_a f_1 \big| \left[ \alpha^1_a \theta \right]_{k_1} \cdots f_m \big| \left[ \alpha^m_a \theta \right]_{k_m}.
\] (4.14)

Since each \( f_j \in M_{k_j}(\Gamma) \), by definition, the \( q \)-expansion of each \( f_j \big| \left[ \alpha^j_a \theta \right]_{k_j} \)
has no principal part, so is their product. Hence \( s \) is holomorphic at all cusps.

In conclusion, \( s \) is a well-defined modular form. \[\square\]

Now we extend this operation \( \mathbb{C} \)-linearly to \( \delta^C \) on \( M(\Gamma) \). Hence we get a \( \mathbb{C} \)-linear map \( \delta^C \otimes_C M(\Gamma)^{\otimes n} \to M(\Gamma)^{\otimes n} \).

**Theorem 4.2.2** (Operad acting on Modular Forms). This map gives us a \( \delta^C \)-operad algebra. We denote \( \delta(\Gamma) \) to be the image of \( \delta^C \) in \( \text{End}_{M(\Gamma)} \), call elements in the image operad operators over \( \mathbb{C} \).

**Proof.** We need only to verify axioms on the generators. Denote \( V = M(\Gamma) \).

Suppose
\[
\begin{cases}
[\alpha] = \sum_{\alpha_a \in C_\alpha} \Gamma^m \alpha_a, & C_\alpha \subset S^m \text{ is finite.} \\
[\beta] = \sum_{\beta_b \in C_\beta} \Gamma^n \beta_b, & C_\beta \subset S^n \text{ is finite.}
\end{cases}
\]

We verify the conditions in proposition 2.2.2

1. Associativity. Given \( v_1, \ldots, v_i, \ldots, v_m, w_1 \cdots, w_n \in V \), \( v_s(s \neq i) \) of weight \( k_s \), \( w_t \) of weight \( l_t \). Define \( v_i := [\beta] \circ (w_1, \cdots, w_n) \), which is of weight \( l_1 + \cdots + l_n \), hence
\[
v_i \big| \left[ \alpha^i_a \right]_{l_1 + \cdots + l_n} = \left( \sum_b w_1 \big| \left[ \beta^1_b \right]_{l_1} \cdots w_n \big| \left[ \beta^n_b \right]_{l_n} \right) \big| \left[ \alpha^i_a \right]_{l_1 + \cdots + l_n}
\]
\[
= \sum_b w_1 \big| \left[ \beta^1_b \alpha^i_a \right]_{l_1} \cdots w_n \big| \left[ \beta^n_b \alpha^i_a \right]_{l_n}
\]
\[
= \sum_b \sum_i \left[ \beta^i_b \alpha^i_a \right]_{l_1} \cdots w_n \big| \left[ \beta^n_b \alpha^i_a \right]_{l_n}.
\] (4.15)
We put in \([\alpha], [\beta], v_1 \otimes \cdots \otimes v_m, w_1 \otimes \cdots w_n\) in the diagram of proposition 2.2.2. On the left-hand side of the commutative diagram in the associativity part we get

\[
[\alpha] \circ (v_1, \cdots, v_{i-1}, [\beta] \circ (w_1, \cdots, w_n), v_{i+1}, \cdots, v_m)
= [\alpha] \circ (v_1, \cdots, v_{i-1}, v_i, v_{i+1}, \cdots, v_m)
\]

\[
= \sum_a (v_1 \mid [\alpha_a^1]_{k_1} \cdots v_{i-1} \mid [\alpha_a^{i-1}]_{k_{i-1}} v_i \mid [\alpha_a^i]_{l_1+\cdots+l_n} v_{i+1} \mid [\alpha_a^{i+1}]_{k_{i+1}} \cdots v_m \mid [\alpha_a^m]_{k_m})
\]

\[
= \sum_a (v_1 \mid [\alpha_a^1]_{k_1} \cdots v_{i-1} \mid [\alpha_a^{i-1}]_{k_{i-1}} (\sum_b w_1 \mid [\beta_b^1 \alpha_a^i]_{l_1} \cdots w_n \mid [\beta_b^n \alpha_a^i]_{l_n}) v_{i+1} \mid [\alpha_a^{i+1}]_{k_{i+1}} \cdots v_m \mid [\alpha_a^m]_{k_m})
\]

\[
= \sum_{a,b} (v_1 \mid [\alpha_a^1]_{k_1} \cdots v_{i-1} \mid [\alpha_a^{i-1}]_{k_{i-1}} w_1 \mid [\beta_b^1 \alpha_a^i]_{l_1} \cdots w_n \mid [\beta_b^n \alpha_a^i]_{l_n} v_{i+1} \mid [\alpha_a^{i+1}]_{k_{i+1}} \cdots v_m \mid [\alpha_a^m]_{k_m}) \tag{4.16}
\]

\[
[\alpha] \circ_i [\beta] = \sum_{a,b} \Gamma^{m+n-1}(\alpha_a^{i-1}, \cdots, \alpha_a^1, \beta_b^1 \alpha_a^i, \cdots, \beta_b^n \alpha_a^i, \alpha_a^{i+1}, \cdots, \alpha_a^m) \tag{4.17}
\]

On the right-hand side, we get

\[
([\alpha] \circ_i [\beta]) \circ (v_1, \cdots, v_{i-1}, w_1, \cdots, w_n, v_{i+1}, \cdots, v_m)
\]

\[
= \sum_{a,b} (v_1 \mid [\alpha_a^1]_{k_1} \cdots v_{i-1} \mid [\alpha_a^{i-1}]_{k_{i-1}} w_1 \mid [\beta_b^1 \alpha_a^i]_{l_1} \cdots w_n \mid [\beta_b^n \alpha_a^i]_{l_n} v_{i+1} \mid [\alpha_a^{i+1}]_{k_{i+1}} \cdots v_m \mid [\alpha_a^m]_{k_m}) \tag{4.18}
\]

So the diagram commutes.

2. Equivariance. Given \(\tau \in \Sigma_m, v_1, \cdots, v_m \in V, v_s \) of weight \(k_s\).

\[
[\alpha]^{\tau} \circ (\tau^{-1}(v_1, \cdots, v_m))
= (\sum_a \Gamma^m \alpha_a^\tau) \circ (v_{\tau(1)}, \cdots, v_{\tau(m)})
\]

\[
= \sum_a v_{\tau(1)} \mid [\alpha_a^{\tau(1)}]_{k_{\tau(1)}} \cdots v_{\tau(m)} \mid [\alpha_a^{\tau(m)}]_{k_{\tau(m)}}
\]

\[
= \sum_a v_1 \mid [\alpha_a^1]_{k_1} \cdots v_m \mid [\alpha_a^m]_{k_m}
= [\alpha] \circ (v_1, \cdots, v_m). \tag{4.19}
\]
So the diagram commutes in proposition 2.2.2.

3. Unitality. $e = [1] = \Gamma \cdot \Gamma$, so $e \circ f = f$ for $f \in M_k(\Gamma)$. Hence $e(v) = v$ for any $v \in V$, the equation holds in proposition 2.2.2.

In conclusion, this map gives a $\delta^C$-operad algebra. 

**Proposition 4.2.3.** $\delta(1)^C$-action on $M_k(\Gamma)$ is exactly the left action of the opposite ring associated to the abstract complex Hecke ring $H^C$. So their images $\delta(1)_k(\Gamma)$ and $H_k(\Gamma)$ in $\text{End}(M_k(\Gamma))$ are the same.

**Proof.** Given $[\alpha] \in \delta(1)$ and $f \in M_k(\Gamma)$. We see directly from the definition that $[\alpha] \circ f = f \| [\alpha]_k$. Hence when extended $\mathbb{C}$-linearly to $\delta(1)^C$ and $H^C$ on $M(\Gamma)$, since these two rings are opposite to each other by proposition 4.1.4, the right action of $H^C$ is exactly opposite to the left action of $\delta(1)^C$. The result follows. 

**Remark 4.2.4.** The awkward ‘opposite’ situations happen because operads were first introduced by J. P. May in algebraic topology, which was independent of Hecke algebras. Operads are defined such that they act on the left but Hecke algebras naturally act on the right of $M_k(\Gamma)$. If we want to precisely get Hecke algebras as a special part inside some operad, we have to modify the axioms of operads.

### 4.3 $\mathbb{Z}$-Operad acting on Modular Forms

Let $G = GL_2^+(\mathbb{Q})$, $\Gamma = SL_2(\mathbb{Z})$. $\delta$ is the abstract operad defined in theorem 4.1.3. Define $\delta_Z(m)$(we require $\delta_Z(0) = 0$) to be the $\mathbb{Z}$-submodule of $\delta(m)$ generated by all possible $[\alpha]$, where $\alpha = (\alpha^1, \cdots, \alpha^m)$, each $\alpha^k \in M_2^+(\mathbb{Z})$. Clearly by definiton of operadic compositions of $\delta$, $\delta_Z = \{\delta_Z(n)\}$ forms a $\mathbb{Z}$-suboperad of $\delta$. Given any
subring $A$ of $\mathbb{C}$, define $\delta_A = \delta_Z \otimes Z A$, which is an $A$-operad. Denote $V = \mathcal{M}(\Gamma)$. Denote $\delta(\Gamma; A)$ to be the image of $\delta_A$ in $\text{End}_V$. It is a natural $A$-operad, but it does not necessarily act on $\mathcal{M}(\Gamma; A)$.

**Lemma 4.3.1.** $\delta(\Gamma; A) \otimes_A \mathbb{C} = \delta(\Gamma)$.

**Proof.** By base change argument, we need only to show it for $\mathbb{Z}$. Given any $[g] \in \delta(m)$, suppose $g = \lambda \alpha$ for some $\lambda \in \mathbb{Q}$ and $\alpha \in M_2^+(\mathbb{Z})^m$. Suppose $f_1, \cdots, f_m$ $\mathbb{C}$-modular forms, $f_j$ of weight $k_j$,

$$[g] \circ (f_1, \cdots, f_m) = ([\lambda I] \circ_1 [\alpha]) \circ (f_1, \cdots, f_m)$$

$$= ([\alpha] \circ (f_1, \cdots, f_m)) ([\lambda I]_{k_1+\cdots+k_m})$$

$$= (\lambda^{2k_1+\cdots+k_m})^{1-\lambda^{-k_1-\cdots-k_m}} ([\alpha] \circ (f_1, \cdots, f_m))$$

$$= \lambda^{k_1+\cdots+k_m-2} ([\alpha] \circ (f_1, \cdots, f_m)) . \quad (4.20)$$

By checking on the generators, we conclude that the images of $[g]$ and $[\alpha]$ in $\delta(\Gamma)$ differ by a scalar in $\mathbb{Q}$. Thus all possible $[\alpha]$ also generate $\delta(\Gamma)$ over $\mathbb{C}$, so we have $\delta(\Gamma; \mathbb{Z}) \otimes Z \mathbb{C} = \delta(\Gamma)$.

Let $\overline{\mathbb{Q}}$ be the algebraic closure of $\mathbb{Q}$, $\mathbb{Q}^{ab}$ be the abelian closure of $\mathbb{Q}$. Denote $G = G_\mathbb{Q} = Gal(\overline{\mathbb{Q}}/\mathbb{Q})$, $G^{ab} = G^{ab}_\mathbb{Q} = Gal(\mathbb{Q}^{ab}/\mathbb{Q})$. We now define an action of $G$ on $Q = \delta_Z$ which factors through $G^{ab}$. For $[\alpha] \in \mathcal{Q}(m)$, suppose $\alpha = (\alpha^1, \cdots, \alpha^m)$, each $\alpha^k \in M_2^+(\mathbb{Z})$. By lemma 3.2.9, we can use left $\Gamma^m$-action to find a representative $\alpha$ such that each $\alpha^i = \begin{pmatrix} a_i & b_i \\ 0 & d_i \end{pmatrix}$, $a_i, d_i > 0$, $0 \leq b_i < d_i$.

Denote $A = a_1 \cdots a_n$, $D = d_1 \cdots d_n$, $M = AD$, $\zeta_k = k^{th}$ primitive root. For any $\sigma \in G$, $\sigma(\zeta_M) = \zeta_M^N$ for some integer $N$ such that $gcd(N, M) = 1$. Define

$$[\alpha]_\sigma = [\alpha^\sigma], \text{ where } \alpha^\sigma_i := \begin{pmatrix} a_i & Nb_i \\ 0 & d_i \end{pmatrix} . \quad (4.21)$$

Different choices of $N$ differ by a multiple of $M$, hence a multiple of each $d_i$. Use
left-$\begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}$ action on each $\alpha^i_\sigma$, we see this operation is independent of choice of $N$.

**Theorem 4.3.2.** This operation is independent of choice of $\alpha$, hence it is well-defined.

**Proof.** Suppose $\tilde{\alpha} = g\alpha \Delta x$ is another representative we pick. Assume $\tilde{\alpha}^i = \begin{pmatrix} \tilde{a}_i & \tilde{b}_i \\ 0 & \tilde{d}_i \end{pmatrix}$, then

$$\begin{pmatrix} a_i & b_i \\ 0 & d_i \end{pmatrix} \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix} \begin{pmatrix} \tilde{a}_i & \tilde{b}_i \\ 0 & \tilde{d}_i \end{pmatrix}^{-1} = (g^i)^{-1} \in \Gamma.$$  

**Lemma 4.3.3.** For integers $x_{11}, x_{21}, M, N$ satisfying $gcd(x_{11}, x_{21}) = 1$, $gcd(M, N) = 1$, we can find $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$ such that

$$\begin{cases} c \equiv -x_{21} \pmod{N} \\ d \equiv x_{11} \pmod{N} \\ a \equiv d \equiv 1 \pmod{M} \\ b \equiv c \equiv 0 \pmod{M}. \end{cases}$$  

(4.22)

**Proof.** By Chinese Remainder Theorem, $gcd(M, N) = 1$ so we can first find $c, d$ satisfying the congruence relation involving $c$ and $d$. Modify $c$ by $c + MN \prod_{p \mid d, p \mid c} p$ to make it coprime with $d$. Check: $q$ prime,

$$\begin{cases} q \mid d, q \mid c \Rightarrow q \nmid MN, q \nmid \prod_{p \mid d, p \mid c} p \Rightarrow q \nmid c + MN \prod_{p \mid d, p \mid c} p. \\ q \mid d, q \mid c \Rightarrow q \mid MN \prod_{p \mid d, p \mid c} p \Rightarrow q \nmid c + MN \prod_{p \mid d, p \mid c} p. \end{cases}$$  

(4.23)

Since $c, d$ are coprime, find $a, b$ such that $ad - bc = 1$. Modify $a$ by $a - bc$ and $b$ by $b - bd$, we see $a \equiv 1 \pmod{M}, b \equiv 0 \pmod{M}$. □

Apply the lemma, pick such $\gamma$, then

$$x := \gamma x = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix} = \begin{pmatrix} * & * \\ cx_{11} + dx_{21} & * \end{pmatrix} \in \Gamma_0(N).$$  

(4.24)
\begin{align*}
\begin{pmatrix} a_i & b_i \\ 0 & d_i \end{pmatrix} x \begin{pmatrix} \tilde{a}_i & \tilde{b}_i \\ 0 & \tilde{d}_i \end{pmatrix}^{-1} &= \begin{pmatrix} a_i & b_i \\ 0 & d_i \end{pmatrix} \gamma \begin{pmatrix} a_i & b_i \\ 0 & d_i \end{pmatrix}^{-1} \begin{pmatrix} a_i & b_i \\ 0 & d_i \end{pmatrix} x \begin{pmatrix} \tilde{a}_i & \tilde{b}_i \\ 0 & \tilde{d}_i \end{pmatrix}^{-1} \\
(4.25)
\end{align*}

\begin{align*}
\begin{pmatrix} a_i & b_i \\ 0 & d_i \end{pmatrix} \gamma \begin{pmatrix} a_i & b_i \\ 0 & d_i \end{pmatrix}^{-1} = I + \begin{pmatrix} a_i & b_i \\ 0 & d_i \end{pmatrix} \begin{pmatrix} uM & vM \\ yM & zM \end{pmatrix} (a_id_i)^{-1} \begin{pmatrix} d_i & -b_i \\ 0 & a_i \end{pmatrix} \\
&\in M_2(\mathbb{Z}), \text{ hence in } SL_2(\mathbb{Z}) = \Gamma.
\end{align*}

\begin{align*}
\begin{pmatrix} a_i & b_i \\ 0 & d_i \end{pmatrix} x \begin{pmatrix} \tilde{a}_i & \tilde{b}_i \\ 0 & \tilde{d}_i \end{pmatrix}^{-1} \in \Gamma \text{ by the assumption.}
\end{align*}

So \( \begin{pmatrix} a_i & b_i \\ 0 & d_i \end{pmatrix} x \begin{pmatrix} \tilde{a}_i & \tilde{b}_i \\ 0 & \tilde{d}_i \end{pmatrix}^{-1} \in \Gamma \) for each \( i \). Define

\begin{align*}
\tilde{x} := \begin{pmatrix} 1 & 0 \\ 0 & N^{-1} \end{pmatrix} x \begin{pmatrix} 1 & 0 \\ 0 & N \end{pmatrix} = \begin{pmatrix} \tilde{x}_{11} & N\tilde{x}_{12} \\ N^{-1} \tilde{x}_{11} & \tilde{x}_{11} \end{pmatrix} \in \Gamma \text{ since } \tilde{x} \in \Gamma_0(N).
\end{align*}

\begin{align*}
\begin{pmatrix} \alpha^i_\sigma \tilde{x} & (\alpha^i_\sigma)^{-1} \\
\end{pmatrix}
&= \begin{pmatrix} a_i & Nb_i \\ 0 & d_i \end{pmatrix} x \begin{pmatrix} \tilde{a}_i & N\tilde{b}_i \\ 0 & \tilde{d}_i \end{pmatrix}^{-1} \\
&= \begin{pmatrix} 1 & 0 \\ 0 & N^{-1} \end{pmatrix} \begin{pmatrix} a_i & b_i \\ 0 & d_i \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & N \end{pmatrix} \tilde{x} \begin{pmatrix} 1 & 0 \\ 0 & N^{-1} \end{pmatrix} \begin{pmatrix} \tilde{a}_i & \tilde{b}_i \\ 0 & \tilde{d}_i \end{pmatrix}^{-1} \begin{pmatrix} 1 & 0 \\ 0 & N \end{pmatrix} \\
&= \begin{pmatrix} 1 & 0 \\ 0 & N^{-1} \end{pmatrix} \begin{pmatrix} a_i & b_i \\ 0 & d_i \end{pmatrix} x \begin{pmatrix} \tilde{a}_i & \tilde{b}_i \\ 0 & \tilde{d}_i \end{pmatrix}^{-1} \begin{pmatrix} 1 & 0 \\ 0 & N \end{pmatrix} \quad (4.26)
\end{align*}

\begin{align*}
\det \begin{pmatrix} \tilde{a}_i & \tilde{b}_i \\ 0 & \tilde{d}_i \end{pmatrix} = a_id_i \text{ is coprime to } N, \text{ hence it has an inverse when passing to } GL_2(\mathbb{Z}/N\mathbb{Z}). \text{ So the integral matrix}
\begin{pmatrix} a_i & b_i \\ 0 & d_i \end{pmatrix} x \begin{pmatrix} \tilde{a}_i & \tilde{b}_i \\ 0 & \tilde{d}_i \end{pmatrix}^{-1} \equiv \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \equiv \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \pmod{N} . \quad (4.27)
\end{align*}
Therefore conjugated by \( \begin{pmatrix} 1 & 0 \\ 0 & N^{-1} \end{pmatrix} \), it is also integral. Also it is of determinant 1, hence in \( SL_2(\mathbb{Z}) = \Gamma \). Therefore \( \alpha^i \bar{x} (\bar{\alpha}^i)^{-1} \in \Gamma \) for each \( i \), i.e. \( \bar{\alpha} \in \Gamma^m \alpha \Delta \bar{x} \subseteq \Gamma^m \alpha \Delta \Gamma \), namely \([\alpha] = [\bar{\alpha}]\).

**Example 4.3.4.** Take \( p \) a prime, let

\[
X = \left\{ M_0 = \begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix}, \ldots, M_{p-1} = \begin{pmatrix} 1 & p-1 \\ 0 & p \end{pmatrix}, M_\infty = \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix} \right\}. \tag{4.28}
\]

be the set of all representatives of right cosets in \( \Gamma M_0 \Gamma \). Take \( \alpha = (M_x_1, \ldots, M_x_n) \), \( \sigma(\zeta_p) = \zeta_p^N \). Assume \( ap + bN = 1 \), \( S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \). According to last theorem 4.3.2, we have \([\alpha]\sigma = [\alpha \Delta S]_\sigma\). Find \( y_i \) such that \( p \mid x_iy_i + 1 \) (\( y_i = \infty \) if \( x_i = 0 \), \( y_i = 0 \) if \( x_i = \infty \)).

\[
\begin{pmatrix} 1 & x_i \\ 0 & p \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 1 & y_i \\ 0 & p \end{pmatrix}^{-1}
= \begin{pmatrix} -x_i & 1 \\ -p & 0 \end{pmatrix} \begin{pmatrix} 1 & -y_ip^{-1} \\ 0 & p^{-1} \end{pmatrix} = \begin{pmatrix} -x_i & \frac{x_iy_i + 1}{p} \\ -p & y_i \end{pmatrix} \in \Gamma. \tag{4.29}
\]

Hence \( \Gamma \cdot M_{x_i} S = \Gamma \cdot M_{y_i} \), for \( x_i \neq 0, \infty \). The same is true for \( x_i = 0, \infty \).

\[
\begin{pmatrix} 1 & Nx_i \\ 0 & p \end{pmatrix} \begin{pmatrix} ap & N(1+ap) \\ -b & ap \end{pmatrix} \begin{pmatrix} 1 & Ky_i \\ 0 & p \end{pmatrix}^{-1}
= \begin{pmatrix} ap - bNx_i & Nap(1 + x_i) + N \\ -bp & ap^2 \end{pmatrix} \begin{pmatrix} 1 & -y_iNp^{-1} \\ 0 & p^{-1} \end{pmatrix}
= \begin{pmatrix} ap - bNx_i & Na(1 + x_i - y_i - x_iy_i) + \frac{N(1 + x_iy_i)}{p} \\ -bp & ap + by_iN \end{pmatrix} \in \Gamma. \tag{4.30}
\]

\[
\begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix} \begin{pmatrix} ap & N(1+ap) \\ -b & ap \end{pmatrix} \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}^{-1}
= \begin{pmatrix} ap & N(1+ap) \\ -bp & ap^2 \end{pmatrix} \begin{pmatrix} p^{-1} & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a & N(1 + ap) \\ -b & ap^2 \end{pmatrix} \in \Gamma. \tag{4.31}
\]

33
\[
\begin{pmatrix}
p & 0 \\
0 & 1
\end{pmatrix}
\begin{pmatrix}
ap & N(1 + ap) \\
-b & ap
\end{pmatrix}
\begin{pmatrix}
1 & 0 \\
0 & p
\end{pmatrix}^{-1}
= \begin{pmatrix}
ap^2 & Np(1 + ap) \\
-b & ap
\end{pmatrix}
\begin{pmatrix}
1 & 0 \\
0 & p^{-1}
\end{pmatrix}
= \begin{pmatrix}
ap^2 & N(1 + ap) \\
-b & a
\end{pmatrix} \in \Gamma. \tag{4.32}
\]

Denote \( U = \begin{pmatrix}
ap & N(1 + ap) \\
-b & ap
\end{pmatrix} \), so \( \Gamma \cdot M_{Nz_i}U = \Gamma \cdot M_{Ny_i} \), i.e. \( \Gamma^m\alpha_{\sigma}\Delta U = \Gamma^m(\alpha\Delta S)_{\sigma} \). \([\alpha_{\sigma}] = [(\alpha\Delta S)_{\sigma}] \) as in the last theorem 4.3.2.

**Proposition 4.3.5.** When extended \( \mathbb{Z} \)-linearly, this operation gives an action of \( \mathbb{G} \) on \( \mathcal{Q} \), factoring through \( \mathbb{G}^{ab} \).

**Proof.** Take \( \sigma, \tau \in \mathbb{G} \), take \([\alpha]\) as in the previous argument. Suppose \( \sigma(\zeta_M) = \zeta_{M_i}^N, \tau(\zeta_M) = \zeta_{M_i}^N \), then \( \sigma\tau(\zeta_M) = N_1N_2(\zeta_M) \), thus

\[
[\alpha]_{\sigma\tau}^i = \begin{pmatrix}
a_i & N_1N_2b_i \\
0 & d_i
\end{pmatrix}.
\]

\[
[\alpha]_{\sigma}^i = \begin{pmatrix}
a_i & N_1b_i \\
0 & d_i
\end{pmatrix}, \quad \text{so} \quad ([\alpha]_{\sigma})^i = \begin{pmatrix}
a_i & N_2N_1b_i \\
0 & d_i
\end{pmatrix}.
\]

Hence \( [\alpha]_{\sigma\tau} = ([\alpha]_{\sigma})^i \). It is an action of \( \mathbb{G} \) which depends on its value in \( Gal(\mathbb{Q}(\zeta_M)/\mathbb{Q}) \) only, so this action factors through \( \mathbb{G}^{ab} \). (So it acts commutatively, we do not specify it as a left or right action.) \( \square \)

**Proposition 4.3.6.** Suppose \([\alpha] = \sum_a \Gamma^m\alpha_a\), each \( \alpha_a^i \) is upper-triangular. Adopt the previous notations, we have

\[
[\alpha]_{\sigma} = \sum_a \Gamma^m\alpha_{a,\sigma}. \tag{4.33}
\]

**Proof.** Denote \( C_\alpha := \{ \alpha_a | \alpha_a \text{ are all upper triangular representatives of right cosets in double coset of } \alpha \} \). Since \([\alpha]_{\sigma} = [\alpha_{\sigma}]\), it is enough to show \( \Gamma^mC_{\alpha_{\sigma}} = \{ \Gamma^m\alpha_{a,\sigma} \} := \Gamma^mC_{\alpha}^\sigma \) as sets of right cosets. By theorem 4.3.2 we can use representative \( \alpha_a \), i.e. \( [\alpha_{\sigma}] = [\alpha_{a,\sigma}] \), hence \( \Gamma^mC_{\alpha_{\sigma}} \supset \Gamma^mC_{\alpha}^\sigma \). Thus,
\[ \Gamma^m C_\alpha = \Gamma^m (C_{(\alpha^\sigma)} \sigma^{-1}) \supset \Gamma^m C_{\alpha^\sigma}^{-1} \supset \Gamma^m C_{\alpha^\sigma \sigma^{-1}} = \Gamma^m C_\alpha. \]

\[ \Gamma^m C_\alpha \supset \Gamma^m C_{\alpha^\sigma} \supset \Gamma^m C_{\alpha^\sigma \sigma^{-1}}. \] Hence they are equal. \qed

**Remark 4.3.7.** By lemma 3.2.9, we can extend the action to \( L_\mathbb{Z} = \{ L_\mathbb{Z}(n) \} \) where \( L_\mathbb{Z}(m) \) is the free \( \mathbb{Z} \)-module generated by all possible \( \Gamma^m \alpha \), where \( \alpha = (\alpha^1, \cdots, \alpha^m) \), each \( \alpha^k \in M_2^+(\mathbb{Z}) \). But this action does not commute with the right \( \Gamma \)-action. Hence it is not clear that it is an action on \( \mathcal{O} \). So we directly define the action on \( \mathcal{O} \).

**Example 4.3.8.** \( p > 3 \) is an odd prime. We adopt notation as in example 4.3.4.

\[ T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \ \alpha = (M_0, M_1), \ \sigma(\zeta_p) = \zeta_p^2, \ 	ext{then} \]

\[ \alpha^\sigma \Delta T = (M_1, M_3), \ (\alpha \Delta T)_\sigma = (M_2, M_4), \ 	ext{but } \Gamma^2(M_1, M_3) \neq \Gamma^2(M_2, M_4). \]

Hence right \( \Gamma \)-action and Galois action does not commute. This does not contradict to our theorem 4.3.2 since \( [(M_1, M_3)] = [(M_2, M_4)] \).

**Example 4.3.9.** Let \( \beta = (\beta^1, \cdots, \beta^n) \), where \( \beta^j = \begin{pmatrix} a_j & 0 \\ c_j & d_j \end{pmatrix}, \ \sigma(\zeta_M) = \zeta_M^N. \)

We compute the coefficients for \( [\beta]_\sigma \). \( l_j = \gcd(a_j, c_j) \), \( a_j = \overline{a}_jl_j \), \( c_j = \overline{c}_jl_j \). Find \( \overline{b}_j, \overline{d}_j \) such that \( \overline{a}_jd_j - \overline{b}_j\overline{c}_j = 1 \).

\[ \begin{pmatrix} \overline{d}_j & -\overline{b}_j \\ -\overline{c}_j & \overline{a}_j \end{pmatrix} \begin{pmatrix} a_j & 0 \\ c_j & d_j \end{pmatrix} = \begin{pmatrix} l_j & -\overline{b}_jd_j \\ 0 & \overline{a}_jd_j \end{pmatrix}. \] (4.34)

If \( c_j \) non-zero, choose any \( \overline{c}_j^* \) such that it is inverse of \( \overline{c}_j \) (mod \( \overline{a}_j \)), then \( -\overline{b}_j \equiv \overline{c}_j^* \) (mod \( \overline{a}_j \)). So we can modify the result in 4.34 by left multipliciton of some \( \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \) to be \( \kappa^j := \begin{pmatrix} l_j & \overline{c}_j^*d_j \\ 0 & \overline{a}_jd_j \end{pmatrix} \) and as a right coset, it is independent of the choice of \( \overline{c}_j^* \) for the same reason. Therefore

\[ \Gamma^n \beta = \Gamma^n \kappa = \Gamma^n (\kappa^j)_j. \] (4.35)
Choose $N^*$ such that $NN^* \equiv 1 \pmod{M}$, replace $c_j$ by $N^*c_j$ in the above argument, then $\tilde{c}_j^*$ changes to $N\tilde{c}_j^*$. We find

$$[\beta]_\sigma = \begin{pmatrix} l_j & N\tilde{c}_j d_j \\ 0 & \tilde{a}_j d_j \end{pmatrix}_j = \begin{pmatrix} a_j & 0 \\ (N^*c_j \ d_j) \end{pmatrix}_j. \quad (4.36)$$

This formula is also true when $c_j = 0$, hence it is true for all $\beta$.

**Proposition 4.3.10.** $G$ acts on $Q = \delta_\mathbb{Z}$ as operad automorphisms.

**Proof.** Take $\sigma \in G$, take $[\alpha] \in Q(m)$, $[\beta] \in Q(n)$. Assume

$$[\alpha] = \sum_a \Gamma^m a \quad \text{where } \alpha_a^s = \begin{pmatrix} A_{a,s} & B_{a,s} \\ 0 & D_{a,s} \end{pmatrix},$$

$$[\beta] = \sum_b \Gamma^n b \quad \text{where } \beta_b^t = \begin{pmatrix} A_{b,t} & B_{b,t} \\ 0 & D_{b,t} \end{pmatrix}.$$

Let $M = \prod A_{*,*}D_{*,*}$ and assume $\sigma(\zeta_M) = \zeta_M^N$. We prove that $\sigma$ preserves the structure of operad by checking on the generators.

1. Associativity. For $1 \leq i \leq m$,

$$[\alpha] \circ_i [\beta] = \sum_{a,b} \Gamma^{m+n-1}(\alpha_1^a, \ldots, \alpha_i^a, \beta_1^b, \alpha_1^b, \ldots, \beta_n^i, \alpha_i^b, \ldots, \alpha_m^b). \quad (4.37)$$

By the proposition 4.3.6, we see

$$([\alpha] \circ_i [\beta])_\sigma = \sum_{a,b} \Gamma^{m+n-1}(\alpha_1^a, \ldots, \alpha_i^a, (\beta_1^b, \alpha_i^b)\sigma, \ldots, \beta_n^i, \alpha_i^b, \ldots, \alpha_m^b). \quad (4.38)$$

$$[\alpha]_\sigma \circ_i [\beta]_\sigma = \sum_{a,b} \Gamma^{m+n-1}(\alpha_1^a, \alpha_i^a, \beta_1^b, \alpha_1^b, \alpha_i^b, \ldots, \beta_n^i, \alpha_i^b, \ldots, \alpha_m^b). \quad (4.39)$$

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But in general given \( M_1 = \begin{pmatrix} a_1 & b_1 \\ 0 & d_1 \end{pmatrix}, \ M_2 = \begin{pmatrix} a_2 & b_2 \\ 0 & d_2 \end{pmatrix}, \)

\[
(M_1 M_2)_\sigma = \begin{pmatrix} a_1 & b_1 \\ 0 & d_1 \end{pmatrix} \begin{pmatrix} a_2 & b_2 \\ 0 & d_2 \end{pmatrix}_\sigma
= \begin{pmatrix} a_1 a_2 & a_1 b_2 + b_1 d_2 \\ 0 & d_1 d_2 \end{pmatrix} = \begin{pmatrix} 0 & N(a_1 b_2 + b_1 d_2) \\ 0 & d_1 d_2 \end{pmatrix}. \quad (4.40)
\]

\[
M_{1,\sigma} M_{2,\sigma} = \begin{pmatrix} a_1 & Nb_1 \\ 0 & d_1 \end{pmatrix} \begin{pmatrix} a_2 & Nb_2 \\ 0 & d_2 \end{pmatrix}
= \begin{pmatrix} a_1 a_2 & a_1 Nb_2 + Nb_1 d_2 \\ 0 & d_1 d_2 \end{pmatrix} = \begin{pmatrix} 0 & N(a_1 b_2 + b_1 d_2) \\ 0 & d_1 d_2 \end{pmatrix}. \quad (4.41)
\]

So \( \sigma \) commutes with the matrix multiplication in the equation 4.38, therefore,

\[
\left( [\alpha] \circ_{i} [\beta] \right)_\sigma = [\alpha]_\sigma \circ_{i} [\beta]_\sigma. \quad (4.42)
\]

2. Equivariance. Given \( \tau \in \Sigma_m \), we have

\[
\left( [\alpha]^\tau \right)_\sigma = \sum_a \Gamma_m^{\alpha} \left( \alpha_1^{(1)}, \ldots, \alpha_m^{(m)} \right)_\sigma = \sum_a \Gamma_m^{\alpha_{a,\sigma}} \left( \alpha_1^{(1)}, \ldots, \alpha_m^{(m)} \right). \quad (4.43)
\]

\[
\left( [\alpha]_{\sigma}^\tau \right) = \sum_a \Gamma_m^{\alpha_{a,\sigma}} \left( \alpha_1^{(1)}, \ldots, \alpha_m^{(m)} \right) = \sum_a \Gamma_m^{\alpha_{a,\sigma}} \left( \alpha_1^{(1)}, \ldots, \alpha_m^{(m)} \right). \quad (4.44)
\]

Therefore, \( \left( [\alpha]^\tau \right)_\sigma = \left( [\alpha]_{\sigma}^\tau \right) \), i.e. \( \sigma \) preserves equivariance.

3. Unitality. By definiton, \( \sigma \) preserves \( e = \Gamma \Gamma \).

**Proposition 4.3.11.** Given \( \sigma \in \mathbb{G} \), \( \sigma^2 \) acts on \( \mathcal{Q} \) as identity. Hence this action factor through \( \mathbb{G}^{ab}/(\mathbb{G}^{ab})^2 \).

**Proof.** We adopt the notion used as previous.

\[
[\alpha] = [\Delta S]^{-1} \alpha \Delta S
= \left[ \left( \begin{array}{cc} 0 & -1 \\ 1 & 0 \end{array} \right) \begin{pmatrix} a_i & b_i \\ 0 & d_i \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right]_{i=1}^m
= \left[ \begin{array}{c} d_i \\ -b_i \end{array} \right]_{i=1}^m. \quad (4.45)
\]
By the computation in example 4.3.9, we have
\[
[\alpha]_\sigma = \left[ \begin{array}{cc}
  d_i & 0 \\
- b_i & a_i \\
\end{array} \right]_{i=1}^{m}
\]
\[
= \left[ \begin{array}{cc}
  d_i & 0 \\
- N^* b_i & a_i \\
\end{array} \right]_{i=1}^{m}
\]
\[
= \left[ \begin{array}{cc}
  a_i & N^* b_i \\
0 & d_i \\
\end{array} \right]_{i=1}^{m}.
\] (4.46)

replace \(b_i\) by \(Nb_i\), we have \([\alpha]_{\sigma^2} = [\alpha]\), i.e. \(\sigma^2\) acts trivially.

\[\square\]

**Corollary 4.3.12.** The number of elements in \(G\)-orbit of any element \(x \in Q(n)\) is a power of 2.

**Proof.** Suppose the action on \(G\)-orbit of \(x\) factors through some Galois group \(G = Gal(Q(\zeta_M)/Q)\), which is finite abelian. So by proposition 4.3.11 it factors through \(G/G^2\) whose number of elements is a power of 2. So the number of elements in \(G\)-orbit of \(x\) is a power of 2.

\[\square\]

**Corollary 4.3.13.** Given \([\alpha]\) such that \(D = \prod d_i\) is a power of \(p\), \(p\) prime. Then the orbit of \([\alpha]\) has 1, 2, or 4 elements if \(p = 2\), it has 1 or 2 elements for \(p\) an odd prime.

**Proof.** The action factors through \(G = Gal(Q(\zeta_D)/Q)\), and \(G = (Z/p^mZ)^*\). The result follows from the fact that \(G\) is acyclic of even number order when \(p\) is an odd prime and \(G = Z/2^{m-2}Z \times Z/2Z\) when \(p = 2\).

\[\square\]

On \(V_\mathbb{Q} := M(\Gamma; \mathbb{Q})\) by theorem 3.1.3 \(M(\Gamma; \mathbb{Q}) = M(\Gamma; Z) \otimes Z \mathbb{Q}\), so we have a natural Galois action on the \(\mathbb{Q}\)-part, namely acting on the coefficient in the \(q\)-expansion. We can extend this action to \(\text{End}_{V_\mathbb{Q}}\) by
\[
\sigma T(v_1 \otimes \cdots \otimes v_m) := \sigma T(\sigma^{-1}v_1 \otimes \cdots \otimes \sigma^{-1}v_m).
\] (4.47)

We can easily check by definition this \(\sigma\) acts as an operad automorphism.
Proposition 4.3.14. The Galois action on $Q$ is compatible with the action on $V_Q$, namely the following diagram commutes:

\[
\begin{array}{ccc}
Q & \longrightarrow & \text{End}_{V_Q} \\
\downarrow^{\sigma} & & \downarrow^{\sigma} \\
Q & \longrightarrow & \text{End}_{V_Q}.
\end{array}
\]

where $\sigma$ acts as operad automorphism.

Proof. Enough to show the following diagram commute:

\[
\begin{array}{ccc}
Q(m) \otimes Z V_Q^{\otimes m} & \longrightarrow & V_Q \\
\downarrow^{\sigma \otimes \sigma^{\otimes m}} & & \downarrow^{\sigma} \\
Q(m) \otimes Z V_Q^{\otimes m} & \longrightarrow & V_Q.
\end{array}
\]

Take $[\alpha] = \sum_a \Gamma^m \alpha_a$, where $\alpha^i_a = \begin{pmatrix} A_{a,i} & B_{a,i} \\ 0 & D_{a,i} \end{pmatrix}$, $M = \prod A_{*,*} D_{*,*}$, $\sigma$ in $G$, $\sigma(\zeta_M) = \zeta_M^N$. $f_1, \cdots, f_m$ in $\mathcal{M}(\Gamma)$.

\[f_i(\tau) = a^0_i + a^1_i q + \cdots + a^j_i q^j + \cdots\text{ and } f_i \text{ of weight } k_i.\]

\[
[\alpha] \circ (f_1, \cdots, f_m) = \sum_a \prod_i \left( \sum_{j \geq 0} a^j_i q^j \right) [\alpha^i_a]_{k_i}
\]

\[= \sum_a \prod_i \sum_{j \geq 0} a^j_a (A_{a,i} D_{a,i})^{k_i-1} D_{a,i}^{-k_i} e \left( j \frac{A_{a,i} \tau + B_{a,i}}{D_{a,i}} \right)
\]

\[= \sum_a \sum_{j_1 A_{a,1} + \cdots + j_m A_{a,m} = n} a^1_{j_1} \cdots a^m_{j_m} e \left( \frac{j_1 B_{a,1}}{D_{a,1}} + \cdots + \frac{j_m B_{a,m}}{D_{a,m}} \right) q^n
\]

\[\times \prod_{i=1}^m A_{a,i}^{k_i-1} D_{a,i}^{-1}. \] (4.48)
Substitute in $[\alpha]_{\sigma} = \sum_a \Gamma^m \alpha_{a,\sigma}$, where $\alpha_{a,\sigma}^i = \begin{pmatrix} A_{a,i} & NB_{a,i} \\ 0 & D_{a,i} \end{pmatrix}$, we have

$$[\alpha]_{\sigma} \circ (\sigma f_1, \cdots, \sigma f_m) = \sum_a \sum_{j_1, \cdots, j_m} \sigma(a_{j_1}^1) \cdots \sigma(a_{j_m}^m)e(j_1NB_{a,1}D_{a,1}^{-1} + \cdots + j_mNB_{a,m}D_{a,m}^{-1})q^n$$

$$\times \prod_{i=1}^m A_{a,i}^{k_i-1}D_{a,i}^{-1}.$$  \hspace{1cm} (4.49)

Since $D_{a,i}$ is a divisor of $M$, $\sigma(e(k/D_{a,i})) = e(Nk/D_{a,i})$. Thus,

$$\sigma([\alpha] \circ (f_1, \cdots, f_m)) = [\alpha]_{\sigma} \circ (\sigma f_1, \cdots, \sigma f_m).$$  \hspace{1cm} (4.50)

By checking on the basis, we see the diagram commutes. \hfill \Box

We see in the equation 4.48 we have all algebraic integer coefficients except the part involving $D_{a,i}$. Define $P$ such that $P(1) = Q(1)$, for $m > 1$ and all possible $[\alpha]$ in $Q(m)$, we take $\prod_a D_{a,i} \cdot [\alpha]$. $P$ is the suboperad of $Q$ generated by all $\prod_a D_{a,i} \cdot [\alpha]$.

**Lemma 4.3.15.** The Galois action on $Q(1)$ is trivial.

**Proof.** By lemma 3.2.9, given $[\alpha]$ in $Q(1)$, we can take representative $\alpha$ which is diagonal. From the definition of Galois action and theorem 4.3.2, we see the action is trivial. \hfill \Box

**Theorem 4.3.16.** Galois action of $Q$ descends to $P$. If we take $O = \{O(n)\}$ to be all the $G$-action invariant element in $P$, it is a suboperad and it acts on $M(\Gamma; \mathbb{Z})$. Hence by base change $O_A = O \otimes A$ acts on $M(\Gamma; A)$ for any subring $A$ of $\mathbb{C}$.

**Proof.** We see from the definition that the elements in Galois group map generators of $P$ to generators of $P$. Therefore $P$ is $G$-action closed. Take all the sums over elements inside one $G$-orbit, they form a $\mathbb{Z}$-basis for $P$. Since $G$ acts
on $Q$ as automorphisms by proposition $4.3.10$, hence descends to automorphisms in $P$, so $O$ is a suboperad. $O(1) = P(1) = Q(1)$ by lemma $4.3.15$, so it acts on $M(\Gamma; \mathbb{Z})$ by proposition $3.2.10$ as Hecke operators, and from equation $4.48$ we see $O$ acting on $f_1, \cdots, f_m \in M(\Gamma; \mathbb{Z})$ gives some function $f$ with algebraic integer coefficients in $q$-expansion. By compatibility property $4.3.14$, $f$ is $G$-invariant, hence its coefficients are indeed integers, i.e. $f \in M(\Gamma; \mathbb{Z})$. So the results hold for arbitrary subring $A$ of $\mathbb{C}$. $\square$

**Example 4.3.17.** One might expect that in the equation $4.48$, for fixed $j_1, \cdots, j_m$, the total sum will give you some integer or at least some rational number as in the Hecke operator case. Therefore, one might expect the Galois action to be trivial as in the Hecke case. However in the operad case some strange things can happen. The following example shows that the Galois action is not trivial in general.

Denote $M_j = \begin{pmatrix} 1 & j \\ 0 & 3 \end{pmatrix}$ for $j = 0, 1, 2$, $M_\infty = \begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix}$, $X = \{ M_0, M_1, M_2, M_\infty \}$. We find all $[\alpha] \in Q(3)$, $\alpha^i \in X$.

$$
\begin{align*}
[(M_0, M_0, M_0)] &= [(M_0, M_0, M_0)] = [(M_1, M_1, M_1)] \\
&= [(M_2, M_2, M_2)] = [(M_\infty, M_\infty, M_\infty)].
\end{align*}
$$

$$
\begin{align*}
[(M_0, M_0, M_1)] &= [(M_0, M_0, M_1)] = [(M_1, M_1, M_2)] = [(M_2, M_2, M_0)] \\
&= [(M_\infty, M_\infty, M_2)] = [(M_\infty, M_\infty, M_0)] = [(M_\infty, M_\infty, M_1)] \\
&= [(M_2, M_2, M_1)] = [(M_0, M_0, M_2)] = [(M_1, M_1, M_0)] \\
&= [(M_1, M_1, M_\infty)] = [(M_2, M_2, M_\infty)] = [(M_0, M_0, M_\infty)].
\end{align*}
$$

$$
\begin{align*}
[(M_0, M_1, M_0)] &= [(M_0, M_1, M_0)] = [(M_1, M_2, M_1)] = [(M_2, M_0, M_2)] \\
&= [(M_\infty, M_2, M_\infty)] = [(M_\infty, M_0, M_\infty)] = [(M_\infty, M_1, M_\infty)] \\
&= [(M_2, M_1, M_2)] = [(M_0, M_2, M_0)] = [(M_1, M_0, M_1)] \\
&= [(M_1, M_\infty, M_1)] = [(M_2, M_\infty, M_2)] = [(M_0, M_\infty, M_0)].
\end{align*}
$$

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\[
[(M_1, M_0, M_0)] = [(M_1, M_0, M_0)] = [(M_2, M_1, M_1)] = [(M_0, M_2, M_2)] \\
= [(M_2, M_\infty, M_\infty)] = [(M_0, M_\infty, M_\infty)] = [(M_1, M_\infty, M_\infty)] \\
= [(M_1, M_2, M_2)] = [(M_2, M_0, M_0)] = [(M_0, M_1, M_1)] \\
= [(M_\infty, M_1, M_1)] = [(M_\infty, M_2, M_2)] = [(M_\infty, M_0, M_0)].
\]

\[
[(M_0, M_1, M_2)] = [(M_0, M_1, M_2)] = [(M_1, M_2, M_0)] = [(M_2, M_0, M_1)] \\
= [(M_\infty, M_2, M_1)] = [(M_\infty, M_0, M_2)] = [(M_\infty, M_1, M_0)] \\
= [(M_2, M_\infty, M_1)] = [(M_0, M_\infty, M_2)] = [(M_1, M_\infty, M_0)] \\
= [(M_1, M_\infty, M_2)] = [(M_2, M_\infty, M_0)] = [(M_0, M_\infty, M_1)].
\]

\[
[(M_0, M_2, M_1)] = [(M_0, M_2, M_1)] = [(M_1, M_0, M_2)] = [(M_2, M_1, M_0)] \\
= [(M_\infty, M_1, M_2)] = [(M_\infty, M_2, M_0)] = [(M_\infty, M_0, M_1)] \\
= [(M_2, M_\infty, M_1)] = [(M_0, M_\infty, M_2)] = [(M_1, M_\infty, M_0)] \\
= [(M_1, M_2, M_\infty)] = [(M_2, M_0, M_\infty)] = [(M_0, M_1, M_\infty)].
\]

\(G\)-action factors through \(\text{Gal}(\mathbb{Q}(\zeta_3)/\mathbb{Q}) = \{1, \sigma\}\) where \(\sigma\) is the complex conjugation. \(G\) acts trivially on \([(M_0, M_0, M_0)], [(M_0, M_0, M_1)], [(M_0, M_1, M_0)], [(M_1, M_0, M_0)],\) but \([(M_0, M_1, M_2)] \not\rightarrow [(M_0, M_2, M_1)].\) The number of elements inside one orbit is either 1 or 2, which agrees with our result 4.3.13.

In fact the list above also tells us all the information of the right cosets decomposition in each double cosets. Suppose given \(f, g, h\) three modular forms with \(\mathbb{Z}\)-coefficients. Assume that

\[
\begin{align*}
f &= a_0 + a_1 q + \cdots + a_n q^n + \cdots \\
g &= b_0 + b_1 q + \cdots + b_n q^n + \cdots \\
h &= c_0 + c_1 q + \cdots + c_n q^n + \cdots
\end{align*}
\]

Take \([\alpha] = [(M_0, M_1, M_2)].\) In equation 4.48 if we fix \(j_1 = 10, j_2 = 2, j_3 = 1\) and consider the sum over all \([\alpha_a]\), we find that the coefficient of \(q^{31}\) can be only
contributed by elements \((M_\infty, M_2, M_1)\), \((M_\infty, M_0, M_2)\) and \((M_\infty, M_1, M_0)\). The only different and non-rational parts \(e^{\left(\frac{j_1 B_{a,1}}{D_{a,1}} + \cdots + \frac{j_m B_{a,m}}{D_{a,m}}\right)}\) sum up together gives

\[
e(10 \cdot 0 + 2 \cdot \frac{2}{3} + 1 \cdot \frac{1}{3}) + e(10 \cdot 0 + 2 \cdot \frac{0}{3} + 1 \cdot \frac{2}{3}) + e(10 \cdot 0 + 2 \cdot \frac{1}{3} + 1 \cdot \frac{0}{3}) = 3e^{2/3} = 3e^{\frac{4\pi i}{3}}. \tag{4.51}
\]

which is not a rational number. Hence we do not expect \([\alpha]\) acting on general \(f, g, h\) modular forms over \(\mathbb{Z}\) gives a modular form over \(\mathbb{Z}\).
Bibliography


