Spin-lowest $K$-types and Dirac cohomology

by

Chao-Ping Dong

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This is to certify that I have examined the above Ph.D thesis
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the thesis examination committee have been made.

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Spin-lowest $K$-types and Dirac cohomology

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Abstract

This thesis studies the problem that for an irreducible unitary representation, what kind of its $K$-types contributes to the Dirac cohomology. We introduce spin-norm and spin-lowest $K$-type, which offer the right framework to answer the problem. Based on our study of the spin-norm, reduction along a pencil, and by tracing certain bottom layer $K$-types, we verify that if $G$ is on the following list: real $G_2$, $F_II$, $E_{IV}$; complex $G_2$, $F_4$, $E_6$; and $X$ is any irreducible unitary $(\mathfrak{g}, K)$ module which contains some unitarily small $K$-types, then only these $K$-types can contribute to the Dirac cohomology of $X$. These results also give partial support to Conjecture 7.13 of [Salamanca-Riba and Vogan, On the classification of unitary representations of reductive Lie groups, Ann. of Math. 148 (1998), 1067–1133]. Moreover, for $G$ complex, we reveal the relation between $H_D(L_S(Z))$ and $H_D(Z)$. This result reduces the classification of unitary representations with non-zero Dirac cohomology to the classification of the spherical ones with non-zero Dirac cohomology on the Levi level.
Chapter 1

Introduction

Let $G$ be a connected semisimple Lie group with finite center. Let $K$ be the maximal compact subgroup of $G$ corresponding to a Cartan involution $\theta$. Denote by $g_0$ and $\mathfrak{k}_0$ the Lie algebras of $G$ and $K$, respectively. Let $g_0 = \mathfrak{k}_0 \oplus \mathfrak{p}_0$ be the corresponding Cartan decomposition of the Lie algebra. As usual, we drop the subscripts to denote the complexifications; thus $g = \mathfrak{k} \oplus \mathfrak{p}$ is the complexified Cartan decomposition.

Let $T_c$ be a maximal torus of $K$. Let $a_c^0 = Z_{p_0}(t_c^0)$ and let $A_c$ be the corresponding analytic subgroup. Then $\mathfrak{h}_c = \mathfrak{t}_c \oplus a_c$ is a fundamental Cartan subalgebra of $g$, and up to conjugation,

$$H_c = T_c A_c$$

is the unique $\theta$-stable maximally compact Cartan subgroup of $G$. We fix compatible choices of positive roots $\Delta^+(g, \mathfrak{h}_c)$, $\Delta^+(g, \mathfrak{t}_c)$ and $\Delta^+(\mathfrak{t}_c, \mathfrak{t}_c)$. We denote by $\rho$ the half sum of roots in $\Delta^+(g, \mathfrak{h}_c)$, by $\rho_c$ the half sum of roots in $\Delta^+(\mathfrak{t}_c, \mathfrak{t}_c)$, and by $\rho_n$ the difference $\rho - \rho_c$. Then $\rho, \rho_c, \rho_n \in (\mathfrak{t}_c)^*$. 

This thesis begins with the observation that for a cohomologically induced representation $\mathcal{L}_S(Z)$ with infinitesimal character in the weakly good range, only
its bottom layer $K$-types can contribute to its Dirac cohomology (see §2.2 and Lemma 3.1.1). We note that the bottom layer $K$-types of $L_S(Z)$ and their multiplicities can be easily traced from $Z$. This observation motivates us to explore further the problem that for an irreducible unitary representation $X$, what kind of its $K$-types can contribute to $H_D(X)$.

Based on Parthasarathy’s Dirac inequality [P] and Huang and Pandžić’s proof of Vogan conjecture [HP1], we introduce spin-norm and spin-lowest $K$-type in Definition 3.2.1. Then, for an irreducible unitary representation $X$ with infinitesimal character $\Lambda$, $H_D(X)$ is non-zero if and only if $\|X\|_{\text{spin}} = \|\Lambda\|$. Moreover, in this case, it is exactly the spin-lowest $K$-types that give rise to $H_D(X)$ (see Proposition 3.2.3). We show that the spin-norm is lower bounded by the lambda-norm introduced by Vogan [V3] in Proposition 3.3.4. As a toy application, we show that for any irreducible tempered representation, only its lambda-lowest $K$-types can contribute to the Dirac cohomology in Corollary 3.4.2. Note that an irreducible tempered representation can have more than one lambda-lowest $K$-types. Recall also that in the unitary $A_q(\lambda)$ module with $\lambda$ admissible studied in [HKP], it is the unique lambda-lowest $K$-type that gives rise to the Dirac cohomology.

At this stage, one may imagine that in general, for any irreducible unitary representation $X$ of $G$ with non-zero Dirac cohomology, at least one of its lambda-lowest $K$-types will contribute to $H_D(X)$. However, as told to us by Vogan, this is false. The spherical principal series for $SL(2n + 1, \mathbb{C})$ give counter-examples, as being immediate from Lemma 5.1.1. To be more concrete, the trivial $K$-type is the unique lambda-lowest $K$-type, while the $K$-type with highest weight $\rho_c$ is the unique spin-lowest $K$-type. Thus, spin-lowest $K$-type actually differs from lambda-lowest $K$-type.

Let us denote by $\Pi_q(G)$ the set of equivalence classes of irreducible admissible
(\mathfrak{g}, K) modules; and denote by \(\Pi_u(G)\) the unitary dual of \(G\), that is, the set of equivalence classes of irreducible unitary \((\mathfrak{g}, K)\) modules. To describe a conjecture on \(\Pi_u(G)\), Salamanca-Riba and Vogan introduce the notion of unitarily small \(K\)-type in [SV] (cf. §4.1). To be more precise, they conjecturally reduce the classification of \(\Pi_u(G)\) to that of those containing unitarily small \(K\)-types; and they conjecture that the unitarity of such a representation which is Hermitian can be tested on its unitarily small \(K\)-types.

In this thesis, we call \(X \in \Pi_u(G)\) \textit{u-small} if it contains some unitarily small \(K\)-types. The above reasons motivate us to study the following problem on examples.

**Problem A.** For any u-small \(X \in \Pi_u(G)\), can we conclude that only its unitarily small \(K\)-types can contribute to \(H_D(X)\)?

Since \(\Pi_u(G)\) remains unknown in many cases, we prefer to study the following Problem B at first. Note that by Proposition 3.2.3, if Problem B has an affirmative answer, then so does Problem A.

**Problem B.** Take any u-small \(X \in \Pi_u(G)\). Then, for any non-unitarily small \(K\)-type \(\delta'\) occurring in \(X\), can we find a unitarily small \(K\)-type \(\delta\) of \(X\) such that

\[
\|\delta'\|_{\text{spin}} > \|\delta\|_{\text{spin}}.
\] (1.1)

Now let us begin the story about Problem B. Our first finding is that if \(G\) is on the following list:

complex \(G_2\), real \(G_2\), real \(F II\), real \(E IV\),

then we have (see Example 5.1.6, 4.3.2, 4.4.2, 4.4.3)

\[
\|\delta'\|_{\text{spin}} > \|\delta\|_{\text{spin}},
\] (1.3)
where \(\delta'\) is any non-unitarily small \(K\)-type and \(\delta\) is any unitarily small \(K\)-type. In particular, Problem A answers affirmatively for any \(G\) in (1.2). Note that the unitary dual of complex \(G_2\) is obtained by Duflo [D] in 1979, and that of real \(G_2\) is obtained by Vogan [V5] in 1994. To our knowledge, besides the list (1.2), one can find no more exceptional group for which (1.3) holds.

In general, (1.3) shall fail, and the situation becomes complicated. For example, let \(G\) be the complex \(E_8\), then there are 427571 unitarily small \(K\)-types and 234568 non-unitarily small \(K\)-types whose spin-norms lie in the same interval \([2\sqrt{485}, 4\sqrt{155}]\) (see Table 1 in §7.5). It is subtle to distinguish them in any \(u\)-small \(X \in \Pi_u(G)\), and the major difficulty one encounters is the knowledge about the \(K\)-types pattern of \(X\), i.e., which \(K\)-types occur in \(X\).

For \(G\) simple, Vogan has a coarse description using the so-called pencils [V2] (see Proposition 2.5.1). Therefore, our attention is drawn to the behavior of the spin-norm along a pencil. We find on examples that the spin-norm increases strictly along a pencil once it crosses the convex hull generated by the extremal weights of all the unitarily small \(K\)-types. This is proved for all complex classical groups.

**Proposition I.** (Proposition 5.1.7) Suppose \(G\) is complex of type \(A, B, C\), or \(D\). Let \(\beta\) be the largest positive root. Then we have

\[
\|\delta_\mu\|_{\text{spin}} > \|\delta_{\mu - \beta}\|_{\text{spin}},
\]  

(1.4)

where \(\delta_\mu\) is any non-unitarily small \(K\)-type such that \(\mu - \beta\) is dominant.

The unitary dual for complex classical \(G\) is obtained by Barbasch [B] in 1989. A corollary of the above result is that to calculate the Dirac cohomology for any \(X \in \Pi_u(G)\), we need only to consider the starting \(K\)-type of a pencil provided it does not contain any unitarily small \(K\)-type; and to consider the unitarily small \(K\)-types of it otherwise.
For $G$ complex, Zhelobenko classifies all the irreducible representations of $G$ using $J(\lambda_L, \lambda_R)$, the composition factor of the principal series $X(\lambda_L, \lambda_R)$ containing the $K$-type with extremal weight $\lambda_L - \lambda_R$ (see Theorem 6.1.1). As deduced by Barbasch and Pandžić [BP, page 5] from Theorem 2.2.1, for $J(\lambda_L, \lambda_R)$ to have non-zero Dirac cohomology, one should have

$$\lambda_R = -s\lambda_L, 2\lambda_L = \tau + \rho_c,$$

where $s \in W(\mathfrak{g}_0, \mathfrak{h}_0^c)$ and $\tau$ is the highest weight of a $\widetilde{K}$-type. In particular, $2\lambda_L$ is dominant integral regular for $\Delta^+(\mathfrak{k}, \mathfrak{t}^c)$. We look at these representations more carefully via cohomological parabolic induction instead of the usual real parabolic induction.

**Proposition II.** (Proposition 6.5.1) Let $X(\lambda, \nu) \simeq J(\lambda_L, -s\lambda_L)$ (see (6.5)) be an irreducible representation of a complex group $G$, where $s \in W(\mathfrak{g}_0, \mathfrak{h}_0^c)$, $2\lambda_L$ is dominant integral regular for $\Delta^+(\mathfrak{k}, \mathfrak{t}^c)$, and $\lambda := \{\lambda_L + s\lambda_L\}$ (the unique dominant element to which $\lambda_L + s\lambda_L$ is conjugate) is dominant integral for $\Delta^+(\mathfrak{k}, \mathfrak{t}^c)$. Let $\mathfrak{q} = \mathfrak{l} + \mathfrak{u}$ be the $\theta$-stable parabolic subalgebra defined by $\lambda$. Then the set of $\theta$-stable data associated to $X(\lambda, \nu)$ can be chosen as $(\mathfrak{q}, H^c = T^c A^c, \lambda - \rho(\mathfrak{u}), 2\lambda_L - \lambda)$, and we have

$$X(\lambda, \nu) \simeq \mathcal{R}_q^S(X_L(\lambda - \rho(\mathfrak{u}), 2\lambda_L - \lambda)).$$

(1.6)

Moreover, all the bottom layer $K$-types of the RHS are contained in the $\Delta^+(\mathfrak{k}, \mathfrak{t}^c)$-dominant members of

$$\lambda + \Lambda^+_{r,L},$$

(1.7)

where $\Lambda^+_{r,L}$ is the set of all non-negative integer combinations of $\Delta^+(\mathfrak{k} \cap \mathfrak{k}, \mathfrak{t}^c)$. In particular, all of them sit on the hyperplane passing through $\lambda$ and being perpendicular to $\lambda$. Finally, only these $K$-types can contribute to $H_D(X(\lambda, \nu))$.

To handle Problem A, now the basic idea is to do reduction along a pencil (as illustrated in Proposition I) and to trace certain bottom layer $K$-types (as
described in Proposition II). In this fashion, we answer Problem A affirmatively for complex $F_4$ in Chapter 7, and for complex $E_6$ in Chapter 8. For $G$ complex, it is easy to see that the validity of [SV, Conjecture 7.13] would guarantee an affirmative answer to Problem A. Conversely, these results also give partial support to [SV, Conjecture 7.13].

Finally, for $G$ complex, we reveal the relation between $H_D(L_S(Z))$ and $H_D(Z)$. Let $E_\mu$ be the $\tilde{K}$-type with highest weight $\mu$. Let $F_\nu$ be the $\tilde{K}_L$-type with highest weight $\nu$.

**Theorem III.** Let $G$ be complex. Let $Z$ be an irreducible unitary $(\mathfrak{t}, K_L)$ module with infinitesimal character $\lambda \in i(\mathfrak{t}_0)^*$ which is dominant for $\Delta^+((\mathfrak{t} \cap \mathfrak{t}^c))$. Assume that

$$\lambda + \rho(\mathfrak{u}) \text{ is dominant integral regular for } \Delta^+(\mathfrak{t}, \mathfrak{t}^c).$$

Then $H_D(L_S(Z))$ is non-zero if and only if $H_D(Z)$ is non-zero. Actually, if

$$H_D(Z) = m2^{[l_0/2]} F_{\lambda - \rho_L},$$

where $l_0 = \dim \alpha^c$, $m$ is a non-negative integer, then

$$H_D(L_S(Z)) = m2^{[l_0/2]} E_{\lambda + \rho(\mathfrak{u}) - \rho^c}.$$

As mentioned in (1.5), (1.8) is necessary for $H_D(L_S(Z))$ to be non-zero. Moreover, it implies

$$\langle \lambda + \rho(\mathfrak{u}), \alpha \rangle > 0, \quad \forall \alpha \in \Delta(\mathfrak{u}, \mathfrak{h}^c).$$

Thus, by [V4, Theorem 1.3], $Z$ is unitary if and only if $L_S(Z)$ is unitary. Coming back to the setting of Proposition II, we have

$$X(\lambda, \nu) \text{ is unitary if and only if } X_L(\lambda - \rho(\mathfrak{u}), 2\lambda_L - \lambda) \text{ is unitary.}$$

In this case, Theorem III says that

$$H_D(X(\lambda, \nu)) \neq 0 \text{ if and only if } H_D(X_L(\lambda - \rho(\mathfrak{u}), 2\lambda_L - \lambda)) \neq 0.$$
Note that the lambda-lowest $K$-type $\lambda - \rho(u)$ of \( L(\lambda - \rho(u), 2\lambda_L - \lambda) \) is perpendicular to all roots in $\Delta(l, h^e)$. Thus up to a center character of $L$, it is a trivial $K_L$-type. Therefore, Theorem III reduces the classification of unitary representations with non-zero Dirac cohomology to the classification of the spherical ones with non-zero Dirac cohomology on the Levi level.

The thesis is organized as follows. We set the notation and collect necessary preliminaries in Chapter 2. The starting observation is made in §3.1; spin-norm and spin-lowest $K$-type are introduced in §3.2; we compare the spin-norm and the lambda-norm in §3.3; and apply it to tempered representations in §3.4. Motivation for studying Problem A is presented in §4.1; basic properties of the spin-norms for unitarily small $K$-types are studied in §4.2; some toy examples are presented in §4.3; real $F_{II}$ and $E_{IV}$ are considered in §4.4. We show Proposition I in Chapter 5, and Proposition II is proved in Chapter 6. We answer Problem A affirmatively for complex $F_4$ in Chapter 7, and for complex $E_6$ in Chapter 8. Finally, Theorem III is proved in Chapter 9.
Chapter 2

Notation and Preliminaries

2.1 Notation

Throughout this thesis, we will adopt the following notation unless otherwise specified.

Let $G$ be a connected semisimple Lie group with finite center. Let $K$ be the maximal compact subgroup of $G$ corresponding to a Cartan involution $\theta$. Denote by $\mathfrak{g}_0$ and $\mathfrak{k}_0$ the Lie algebras of $G$ and $K$, respectively. Let $\mathfrak{g}_0 = \mathfrak{k}_0 \oplus \mathfrak{p}_0$ be the corresponding Cartan decomposition of the Lie algebra. As usual, we drop the subscripts to denote the complexifications; thus $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ is the complexified Cartan decomposition. Let us fix a nondegenerate invariant symmetric bilinear form $B$ on $\mathfrak{g}$. Its restrictions to $\mathfrak{k}$, $\mathfrak{p}$, etc., will also be denoted by $B$.

Let $T^c$ be a maximal torus of $K$. Let $\mathfrak{a}_0^c = Z_{\mathfrak{p}_0}(\mathfrak{t}_0^c)$ and let $A^c$ be the corresponding analytic subgroup. Then $\mathfrak{h}^c = \mathfrak{t}^c \oplus \mathfrak{a}^c$ is a fundamental Cartan subalgebra of $\mathfrak{g}$, and up to conjugation,

$$H^c = T^c A^c$$

is the unique $\theta$-stable maximally compact Cartan subgroup of $G$. 
We will denote by $\Delta(g, h^c)$ (resp., $\Delta(g, t^c)$) the root system of $g$ with respect to $h^c$ (resp., $t^c$). The root system of $\mathfrak{k}$ with respect to $t^c$ will be denoted by $\Delta(\mathfrak{k}, t^c)$. Note that $\Delta(g, h^c)$ and $\Delta(g, t^c)$ are reduced, while $\Delta(\mathfrak{k}, t^c)$ is in general not reduced.

The Weyl groups corresponding to the above root systems will be denoted by $W(g, h^c)$, $W(g, t^c)$ and $W(\mathfrak{k}, t^c)$. We fix compatible choices of positive roots $\Delta^+(g, h^c)$, $\Delta^+(g, t^c)$ and $\Delta^+(\mathfrak{k}, t^c)$. We denote by $\rho$, $\rho_c$ the half sum of roots in $\Delta^+(g, h^c)$, $\Delta^+(g, t^c)$, and $\Delta^+(\mathfrak{k}, t^c)$, respectively. Then $\rho, \rho_c, \rho_n \in (t^c)^*$. Let $t^*_R = i(t_0^c)^*$ and $h^*_R = i(t_0^c)^* \oplus (a_0^c)^*$. Our fixed form $B$ on $g$ induces inner products on $t^*_R$ and $h^*_R$. Let $C_g(h^*_R)$ (resp., $C_g(t^*_R)$, $C_{\mathfrak{k}}(t^*_R)$) be the closed Weyl chamber corresponding to $\Delta^+(g, h^c)$ (resp., $\Delta^+(g, t^c)$, $\Delta^+(\mathfrak{k}, t^c)$). Let us define

$$W(g, t^c)^1 = \{ w \in W(g, t^c) \mid w(\Delta^+(g, t^c)) \subseteq C_{\mathfrak{k}}(t^*_R) \}.$$ 

It is clear that $W(\mathfrak{k}, t^c)$ is a subgroup of $W(g, t^c)$, and that the multiplication map induces a bijection from $W(\mathfrak{k}, t^c) \times W(g, t^c)^1$ onto $W(g, t^c)$ (cf. [K]).

Since $\Delta^+(\mathfrak{k}, t^c)$ is fixed once for all, we will freely refer to a $K$-type by its highest weight.

### 2.2 Preliminaries on Dirac cohomology

For a study of Dirac cohomology, one may refer to [HP2]. Let us fix an orthonormal basis $Z_1, \ldots, Z_n$ of $\mathfrak{p}_0$ with respect to the inner product induced by the form $B$. Let $U(g)$ be the universal enveloping algebra of $g$ and let $C(\mathfrak{p})$ be the Clifford algebra of $\mathfrak{p}$ (with respect to $B$). The Dirac operator $D \in U(g) \otimes C(\mathfrak{p})$ is defined as

$$D = \sum_{i=1}^n Z_i \otimes Z_i.$$
It is easy to check that $D$ does not depend on the choice of the orthonormal basis $Z_i$ and it is $K$-invariant for the diagonal action of $K$ given by adjoint actions on both factors. This version was introduced by Vogan [V6].

Let $\tilde{K}$ be the spin double cover of $K$. If $X$ is a $(\mathfrak{g}, K)$ module, and if $S$ denotes a spin module for $C(\mathfrak{p})$, then $X \otimes S$ is a $(\mathfrak{U}(\mathfrak{g}) \otimes C(\mathfrak{p}), \tilde{K})$ module. The action of $\mathfrak{U}(\mathfrak{g}) \otimes C(\mathfrak{p})$ is the obvious one, and $\tilde{K}$ acts on both factors, on $X$ through $K$ and on $S$ through the spin group Spin$_{p_0}$.

Now the Dirac operator acts on $X \otimes S$, and the Dirac cohomology of $X$ is the $\tilde{K}$-module

$$H_D(X) = \text{Ker} \, D/(\text{Im} \, D \cap \text{Ker} \, D).$$

One of the main result of Huang and Pandžić [HP1], conjectured by Vogan, is

**Theorem 2.2.1.** ([HP1, Th. 2.3]) Let $X$ be an irreducible $(\mathfrak{g}, K)$ module. Let $E_\gamma$ be a $\tilde{K}$-type (if there exists) contained in the Dirac cohomology. Then the infinitesimal character of $X$ is conjugate to $\gamma + \rho_c$ under $W(\mathfrak{g}, \mathfrak{h}^c)$.

Let $X$ be an admissible $(\mathfrak{g}, K)$ module. We say a $K$-type $\delta$ of $X$ **contributes** to the Dirac cohomology if a $\tilde{K}$-type of $\delta \otimes S$ occurs in $H_D(X)$.

**Proposition 2.2.2.** ([HP2, Th. 3.5.2], [HKP, Prop. 3.3]) Let $X$ be an irreducible unitary $(\mathfrak{g}, K)$ module with infinitesimal character $\Lambda$. Assume that $X \otimes S$ contains a $\tilde{K}$-type $E_\gamma$, that is, $(X \otimes S)(\gamma) \neq 0$. Then

$$||\Lambda|| \leq ||\gamma + \rho_c||, \quad (2.1)$$

and the equality holds if and only if the Dirac cohomology contains $(X \otimes S)(\gamma)$.

**Remark 2.2.3.** The inequality (2.1) appears above is Parthasarathy’s Dirac inequality [P, Lemma 2.5]. By Theorem 2.2.1, one sees easily that the equality in (2.1) holds if and only if $\Lambda$ is conjugate to $\gamma + \rho_c$ under $W(\mathfrak{g}, \mathfrak{h}^c)$. 

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2.3 Preliminaries on cohomological induction

For a study of cohomological induction, one may refer to [KV]. Recall that a
$\theta$-stable parabolic subalgebra $q = l \oplus u$ of $g$ can be defined as the sum of nonnegative eigenspaces of $\text{ad}(H)$, where $H \in i t_0^c$ is a fixed element. The Levi subalgebra $l$ of $q$ is the zero eigenspace of $\text{ad}(H)$, while the nilradical $u$ of $q$ is the sum of positive eigenspaces of $\text{ad}(H)$. Since $\theta(H) = H$, $l$, $u$ and $q$ are all $\theta$-stable. Note that $\overline{q} = l \oplus \overline{u}$ and $g = \overline{u} \oplus l \oplus u$.

Note also that $l$ is real in the sense that it is the complexification of $l_0 = l \cap g_0$.

Let $L = N_G(q)$. Then $L$ is connected since $K_L := L \cap K$ is connected, cf. [KV, Prop. 5.13].

Let us arrange the positive root systems in a compatible way, that is, $\Delta(u, t^c) \subseteq \Delta^+(l, h^c)$ and set $\Delta^+(l, h^c) = \Delta(l, h^c) \cap \Delta^+(g, h^c)$. Let $\rho^L$ (resp., $\rho^L_c$) denote the half sum of roots in $\Delta^+(l, h^c)$ (resp., $\Delta^+(l \cap t, t^c)$). Let $\rho^L_n$ denote the difference of $\rho^L - \rho^L_c$. We denote by $\rho(u)$ (resp., $\rho(u \cap p)$, $\rho(u \cap t)$) the half sum of roots in $\Delta(u, t^c)$ (resp., $\Delta(u \cap p, t^c)$, $\Delta(u \cap t, t^c)$). Then we have:

$$\rho = \rho^L + \rho(u), \rho_c = \rho^L_c + \rho(u \cap t), \rho_n = \rho^L_n + \rho(u \cap p).$$

(2.2)

Let $Z$ be an $(l, K_L)$ module, and define

$$Z^\sharp = Z \otimes_C \wedge^\text{top} u.$$

Since $u$ is an $(l, K_L)$ module, $\wedge^\text{top} u$ is a one-dimensional $(l, K_L)$ module; its unique weight relative to $h^c$ is $2 \rho(u)$. Let $\mathcal{L}_j(Z)$ and $\mathcal{R}_j(Z)$ be given by

$$\mathcal{L}_j(Z) = (\Pi_j \circ I_{q, g \cap K}^0 \circ \mathcal{F}_j^{l \cap K} \circ I_{l \cap K}^{l \cap K})(Z^\sharp)$$

(2.3)

$$\mathcal{R}_j(Z) = (\Gamma_j \circ I_{q, g \cap K}^0 \circ \mathcal{F}_j^{l \cap K} \circ I_{l \cap K}^{l \cap K})(Z^\sharp).$$

(2.4)
Here $\mathcal{F}$ is the forgetful functor, $\Pi_j$ is the $j$-th derived functor of the Bernstein functor $\Pi$, and $\Gamma_j$ is the $j$-th derived functor of the Zunckerman functor $\Gamma$. For their specific definitions, one can refer to Chapter 2 of [KV].

The functors $L_j$ and $R_j$ are the functors of cohomological induction. We call $L_j(Z)$ and $R_j(Z)$ the cohomological modules induced from $Z$. To simplify notation, let us abbreviate

$$Z^\sharp_q = \mathcal{F}_q^{\mathfrak{l}, L \cap K}(Z^{\sharp}) \quad \text{and} \quad Z^\sharp_\bar{q} = \mathcal{F}_q^{\mathfrak{l}, L \cap K}(Z^{\sharp}).$$

Note that the most interesting case happens at the degree

$$S = \dim (u \cap \mathfrak{k}).$$

Let us also introduce

$$R = \dim (u \cap \mathfrak{p}).$$

Recall that a $K$-type of $L_S(Z)$ is said to be in the bottom layer if it appears in $L^K_S(Z)$, where

$$L^K_S(Z) = ((P^{K, \mathfrak{k}}_{\mathfrak{t}, L \cap K}_s \circ P^{\mathfrak{k}, L \cap K}_{q \mathfrak{t}, L \cap K})(Z^\sharp_\bar{q})).$$

### 2.4 Preliminaries on the PRV component

Since we will have a chance to consider the tensor product of two finite dimensional irreducible representations, let us collect some fundamental results on this aspect. Let $\tau_0$ be the unique element in $W(\mathfrak{k}, \mathfrak{c})$ with the longest length. Recall that a positive root system $\Delta(\mathfrak{k}, \mathfrak{c})$ has been fixed. Then for any integrally dominant weight $\lambda \in (\mathfrak{c}^*)^*$, we have the irreducible highest weight $\mathfrak{k}$-module $V(\lambda)$. Now let us recall a theorem of Parthasarathy, Ranga Rao, and Varadarajan.

**Theorem 2.4.1.** ([PRV, Cor. 1 and Cor. 2 to Th. 2.1]) Let $\lambda_1, \lambda_2$ be two dominant integral weights. Then $V(\{\lambda_1 + \tau_0 \lambda_2\})$ occurs with multiplicity one in
$V(\lambda_1) \otimes V(\lambda_2)$. Here the braces denote the unique dominant weight to which the indicated weight is conjugate. Moreover, if $V(\lambda)$ is any irreducible component of $V(\lambda_1) \otimes V(\lambda_2)$, then $\{\lambda_1 + \tau_0 \lambda_2\}$ must be a weight of $V(\lambda)$. In particular, we have

$$\|\{\lambda_1 + \tau_0 \lambda_2\} + \rho_c\| \leq \|\lambda + \rho_c\|,$$

and the equality happens if and only if $\lambda = \{\lambda_1 + \tau_0 \lambda_2\}$.

**Remark 2.4.2.** The irreducible constituent $V(\{\lambda_1 + \tau_0 \lambda_2\})$ is called the PRV component of $V(\lambda_1) \otimes V(\lambda_2)$.

### 2.5 Preliminaries on the $K$-types pattern described by pencils

There will be occasions where we shall look at the $K$-types pattern of an irreducible representation carefully. Let us recall a result of Vogan [V2] which gives a coarse description using the so-called pencils.

**Proposition 2.5.1.** ([V2, Lemma 3.4 and Cor. 3.5]) Let $G$ be simple and suppose that $G/K$ is not Hermitian symmetric. Let $\beta$ be the highest weight of the $\mathfrak{p}$ representation of $K$. Then for any infinite dimensional $X \in \Pi_a(G)$, there is a unique set

$$\{\mu_i | i \in I\} \subseteq i(t_0^c)^*$$

of dominant weights such that all the $K$-types of $X$ are precisely

$$\{\mu_i + n\beta | i \in I, n \in \mathbb{N}\}.$$

**Remark 2.5.2.** Here the assumption that “$G/K$ is not Hermitian symmetric” is adopted for simplicity. When $G/K$ is Hermitian symmetric, one may also need $-\beta$. Following [V2], we call a set of highest weights $\{\mu + n\beta | n \in \mathbb{N}\}$ a pencil.
Chapter 3

Spin-norm and spin-lowest $K$-types

3.1 A starting observation

Lemma 3.1.1. Let $Z$ be an irreducible $(l, K_L)$ module with infinitesimal character $\lambda \in (\mathfrak{h}^c)^*$. Arrange $\lambda$ to be dominant for $\Delta^+(l, \mathfrak{h}^c)$ and suppose it satisfies

$$\Re \langle \lambda + \rho(u), \alpha \rangle \geq 0, \quad \forall \alpha \in \Delta(u, \mathfrak{h}^c),$$

(3.1)

then only the bottom layer $K$-types of $L_S(Z)$ can contribute to the Dirac cohomology.

Proof. The dominance condition (3.1) is same as that adopted in [KV, Th. 5.99 and 8.2]. In particular, all the cohomologically induced modules are zero except at the possible degree $S$, where $L_S(Z) \cong R^S(Z)$ is zero or irreducible. And if all the inequalities in (3.1) are strict, then $L_S(Z)$ is irreducible (not zero). Anyway, we can use Theorem 2.2.1 under the current setting.

On one hand, Corollary 5.25 of [KV] says that $L_S(Z)$ has infinitesimal char-
acter $\lambda + \rho(u)$. The element $H \in it L$ lies in $Z(0)$, so it acts on $Z$ by a scalar, say $c$. Then
\[
c = \chi_\lambda(H) = \lambda(H - \rho^L(H)) = \lambda(H).
\]

By Corollary 4.69 of [KV], $H_{\rho(u)}$ also defines the parabolic subalgebra $q = l \oplus u$. Therefore, up to a possible adjustment, we may and we will adopt the following identification
\[
H = H_{\rho(u)}. \tag{3.2}
\]

Let $\mu_G$ be the highest weight of any $K$-type in $L_S(Z)$, then as noted on [KV, p.369]
\[
\mu_G(H) \geq c + 2\rho(u \cap p)(H), \tag{3.3}
\]
and equality holds if and only if the $K$-type $\mu_G$ is in the bottom layer.

On the other hand, any $\tilde{K}$-highest weight in $L_S(Z) \otimes S$ is the sum of a highest weight of a $K$-type in $L_S(Z)$ and a weight of $S$. Any weight of $S$ is of the form
\[
\nu = -\rho_n + \langle \Phi \rangle,
\]
where $\Phi \subseteq \Delta^+(p, t^c)$ and $\langle \Phi \rangle$ denotes the sum of the elements in $\Phi$. Thus any highest weight $\gamma$ in $L_S(Z) \otimes S$ is
\[
\gamma = \mu_G - \rho_n + \langle \Phi \rangle.
\]

Now by Theorem 2.2.1, $\mu_G$ can contribute to Dirac cohomology only if
\[
\mu_G - \rho_n + \langle \Phi \rangle + \rho_c = w(\lambda + \rho(u))
\]
for some $w \in W(g, h^c)$ and for some $\Phi \subseteq \Delta^+(p, t^c)$. In other words, only if
\[
(\mu_G + \rho_c + \langle \Phi \rangle - \lambda - \rho(u) - \rho_n) + (\lambda + \rho(u) - w(\lambda + \rho(u)) = 0. \tag{3.4}
\]
Denote the sum of terms in the first (resp., second) parentheses by I (resp., II). Let us evaluate them at $H$. Since $\lambda$ is dominant for $\Delta^+(l, h^c)$ and satisfies (3.1),
one sees easily that

\[
\text{Re} \langle \lambda + \rho(u), \alpha \rangle \geq 0, \quad \forall \alpha \in \Delta^+(g, h^\circ).
\]

Then in view of [HKP, Lemma 2.3], the real part of II(H) is nonnegative. Since (cf.(2.2))

\[
\rho = \rho_n + \rho_c = \rho^L + \rho(u),
\]

we have

\[
\rho_c - \rho(u) = \rho^L - \rho_n.
\]

Moreover,

\[
\rho_n = \rho^L_n + \rho(u \cap p).
\]

Hence

\[
I = \mu_G - \langle \Phi \rangle + \rho^L - 2 \rho_n
\]

\[
= \mu_G - \langle \Phi \rangle + \rho^L - 2 \rho^L_n.
\]

Therefore

\[
I(H) = \mu_G(H) - c - 2 \rho(u \cap p)(H) + \langle \Phi \rangle(H) \geq 0,
\]

and in view of (3.3), the equality can hold (thus, the equation (3.4) can hold) only if \( \mu_G \) lies in the bottom layer of \( \mathcal{L}_S(Z) \) and \( \Phi \subseteq \Delta^+(l \cap p, t^c) \).

\( \square \)

**Remark 3.1.2.** As noted on [KV, p.364], one can identify the bottom layer \( K \)-types of \( \mathcal{L}_S(Z) \) as follows: let \( \mu_L \) be any \( K_L \)-type occurring in \( Z \),

(i) if \( \mu_L + 2 \rho(u \cap p) \) is dominant for \( \Delta^+(t, t^c) \), then the \( K \)-type \( \mu_G := \mathcal{L}_S^K(\mu_L) = \mu_L + 2 \rho(u \cap p) \) is in the bottom layer.

(ii) otherwise, \( \mathcal{L}_S^K(\mu_L) = 0 \).

In the first case, the multiplicity of \( \mu_G \) in \( \mathcal{L}_S(Z) \) equals to that of \( \mu_L \) in \( Z \) (cf. [KV, Cor. 5.72, Th. 5.80]), and we call \( \mu_G \) the **bottom layer \( K \)-type associated to \( \mu_L \).**
Lemma 3.1.3. In the setting of Lemma 3.1.1, let $\mu_G$ be the bottom layer $K$-type associated to $\mu_L$. Then for any subset $\Phi \subseteq \Delta^+(\mathfrak{l} \cap \mathfrak{p}, \mathfrak{t}^c)$, we have

$$\|\mu_G - \rho_n + \langle \Phi \rangle + \rho_c\| = \|\lambda + \rho(u)\| \text{ if and only if } \|\mu_L - \rho_n^L + \langle \Phi \rangle + \rho_c^L\| = \|\lambda\|. \quad (3.5)$$

Proof. It is easy to see that

$$\|\mu_G - \rho_n + \langle \Phi \rangle + \rho_c\|^2 = \|\mu_L - \rho_n^L + \langle \Phi \rangle + \rho_c^L\|^2 + \|\rho(u)\|^2 + 2c. \quad (3.6)$$

Actually, it is direct that (cf. (2.2))

$$(\mu_G - \rho_n + \langle \Phi \rangle + \rho_c) - (\mu_L - \rho_n^L + \langle \Phi \rangle + \rho_c^L) = \rho(u).$$

Now the identity (3.6) follows from (cf. the identification (3.2))

$$\langle \mu_L - \rho_n^L + \langle \Phi \rangle + \rho_c^L, \rho(u) \rangle = \mu_L(H \rho(u)) = \mu_L(H) = c.$$

Here, recall that $H \in i\mathfrak{t}_0^c$ acts on $Z$ by the scalar $c$; and $\mu_L$ is (the highest weight of) a $K_L$-type of $Z$. Therefore, we have the last equality above, i.e., $\mu_L(H) = c$.

Now, to arrive at (3.5), one just recall that

$$c = \lambda(H) = \langle \lambda, \rho(u) \rangle.$$

3.2 Definitions of spin-norm and spin-lowest $K$-type

Taking the lambda-norm defined by Vogan [V3, Def. 5.4.1] as a model, and motivated by the Dirac inequality, we introduce

Definition 3.2.1. For any $\delta \in \hat{K}$, we define its spin-norm by

$$\|\delta\|_{\text{spin}} = \min \|\gamma + \rho_c\|, \quad (3.7)$$
where the minimum is taken over all the highest weights $\gamma$ of the $\widetilde{K}$-types in $\delta \otimes S$. Now for any $X \in \Pi_a(G)$, we define

$$\|X\|_{\text{spin}} = \min_{\delta \in \widetilde{K} \text{ s.t. } X(\delta) \neq 0} \|\delta\|_{\text{spin}}.$$  \hspace{1cm} (3.8)

We call $\delta$ a **spin-lowest $K$-type** of $X$ if $X(\delta) \neq 0$ and $\|\delta\|_{\text{spin}} = \|X\|_{\text{spin}}$.

**Remark 3.2.2.** It is easy to see that $X$ has finitely many spin-lowest $K$-types. Moreover, using the $R$-group introduced by Vogan [V3, Def. 5.1.2], one sees that this definition also works for the setting that $G$ is a real reductive Lie group in Harish-Chandra’s class.

The motivation of introducing the spin-norm is that, for unitary representations, it measures whether or not a $K$-type can contribute to Dirac cohomology in a precise fashion. That is, as a direct consequence of Proposition 2.2.2, we have the following

**Proposition 3.2.3.** For $X \in \Pi_u(G)$ with infinitesimal character $\Lambda$, let $\delta$ be any $K$-type occurring in $X$. Then

1) $\|X\|_{\text{spin}} \geq \|\Lambda\|$, and the equality happens if and only if $H_D(X)$ is non-zero.

2) $\|\delta\|_{\text{spin}} \geq \|\Lambda\|$, and the equality holds if and only if $\delta$ contributes to $H_D(X)$.

3) If $H_D(X) \neq 0$, it is exactly the spin-lowest $K$-types of $X$ that give rise to $H_D(X)$.

**Lemma 3.2.4.** ([W, Lemma 9.3.2]) Let $E_\mu$ denote the highest weight representation of $\mathfrak{k}$ with highest weight $\mu$. Then

$$S = \bigoplus_{w \in \mathcal{W}(\mathfrak{g}, \mathfrak{t}^c)^1} 2^{[l_0/2]} E_{w_0 - \rho_c},$$  \hspace{1cm} (3.9)

where $l_0 = \dim \mathfrak{a}^c$ and $mE_\mu$ means $m$ copies of $E_\mu$.

Now let us describe the spin-norm more precisely, using the $\mathfrak{k}$-structure of $S$ and the PRV component.
Lemma 3.2.5. For any $K$-type $\mu \in i(\xi_0^*)$, we have
\[
\|\mu\|_{\text{spin}} = \min_{w \in W(g, t^c)^1} \|\{\mu - w\rho + \rho_c\} + \rho_c\|.
\] (3.10)
Moreover,
\[
\|\mu\|_{\text{spin}} \geq \min_{\Phi} \|\mu - \rho_n + \langle \Phi \rangle + \rho_c\|,
\] (3.11)
where $\Phi$ runs over all the subsets of $\Delta^+(p, t^c)$ such that $\mu - \rho_n + \langle \Phi \rangle$ is dominant for $\Delta^+(\mathfrak{f}, t^c)$.

Proof. Recall that $\Delta^+(\mathfrak{f}, t^c)$ is fixed once for all. Then there are possibly different choices of positive systems of $\Delta(p, t^c)$ compatible with it. Recall also that an initial one $\Delta^+(p, t^c)$ has been fixed due to the choice $\Delta^+(g, t^c)$, and we have the corresponding $W(g, t^c)^1$. Now all the possible choices of positive systems of $\Delta(p, t^c)$ compatible with $\Delta^+(\mathfrak{f}, t^c)$ can be described by the following one-one correspondence:

- a choice of a positive system of $\Delta(p, t^c)$ compatible with $\Delta^+(\mathfrak{f}, t^c)$, say, $(\Delta^+)'(p, t^c)$.
- a choice of of a positive system of $\Delta(g, t^c)$ containing $\Delta^+(\mathfrak{f}, t^c)$, say, $(\Delta^+)'(g, t^c)$.
- a choice of a dominant Weyl chamber for $\Delta(g, t^c)$.
- a choice of an element $w \in W(g, t^c)^1$.

Therefore, $(\Delta^+)'(g, t^c)$ gives $\rho' = w\rho$ and $\rho'_n = w\rho - \rho_c$. Hence the set of all the highest weights (without multiplicity) of the spin representation $S$ is
\[
\{w\rho - \rho_c \mid w \in W(g, t^c)^1\}.
\]
We note further that although $-w\rho + \rho_c$ is not necessarily the lowest weight of $E_{w\rho - \rho_c}$ (see Example 4.2.4), it must be a lowest weight of $S$, and the set of all the lowest weights (without multiplicity) of the spin representation $S$ is
\[
\{-w\rho + \rho_c \mid w \in W(g, t^c)^1\}.
\]
Therefore, (3.10) follows from Theorem 2.4.1 and Lemma 3.2.4.

For the second part, using [Hum, §24, Ex. 12, page 142], one have that any highest weight in $\mu \otimes S$ must be the sum of $\mu$ and a weight $-\rho_n + \langle \Phi \rangle$ of $S$ such that $\mu - \rho_n + \langle \Phi \rangle$ is dominant for $\Delta^+(\frak{t}, \frak{t}^c)$. Then it is obvious. \hfill \Box

### 3.3 Spin-norm and lambda-norm

In this section, we will firstly recall the lambda-norm introduced by Vogan in [V3, Def. 5.4.1], and compare it with the spin-norm.

**Lemma 3.3.1.** ([SV, Prop. 2.3 and Cor. 2.4]) Suppose $\mu \in i(\frak{t}_0)^*$ is dominant integral for $\Delta^+(\frak{t}, \frak{t}^c)$. Choose a positive root system $\Delta^+(\frak{g}, \frak{t}^c)$ making $\mu + 2\rho_c$ dominant. Write $\rho$ for the corresponding half sum of positive roots. Then

a) The weight $P(\mu + 2\rho_c - \rho)$ is dominant for $\Delta^+(\frak{g}, \frak{t}^c)$.

b) There is an orthogonal collection $\beta_i$ of positive imaginary roots such that

1. $P(\mu + 2\rho_c - \rho) = \mu + 2\rho_c - \rho + \sum_i c_i \beta_i \ (0 \leq c_i \leq 1/2)$; and
2. $\langle P(\mu + 2\rho_c - \rho), \beta_i \rangle = 0$.

Moreover, $P(\mu + 2\rho_c - \rho)$ is independent of the choice of $\Delta^+(\frak{g}, \frak{t}^c)$ making $\mu + 2\rho_c$ dominant.

**Remark 3.3.2.** The definition of $P(\cdot)$ is given in [SV, Def. 1.2]. Note that $P(\mu + 2\rho_c - \rho)$ is the unique weight associated to $\mu$ via [V3, Prop. 5.3.3], cf. [SV, Cor. 2.4]. Moreover, in the setting of the above lemma,

$$\|\mu\|_{\text{lambda}} := \|P(\mu + 2\rho_c - \rho)\|. \quad (3.12)$$

We note that in Vogan’s original definition [V3, Def. 5.4.1], there is a square on the RHS. Here for convenience of comparing the lambda-norm and the norm of
the infinitesimal character, we remove this square. This modification will give us no trouble in applying the theory of [V3].

**Lemma 3.3.3.** For any \( \mu \in i(t_0)^* \) which is dominant integral for \( \Delta^+(t, t^c) \), and for any \( \Phi \subseteq \Delta^+(p, t^c) \), we have

\[
\| \mu - \rho_n + \langle \Phi \rangle + \rho_c \| \geq \| \mu \|_{\text{lambda}}.
\]

The equality holds if and only if

\[
P(\mu + 2\rho_c - \rho) - (\mu + 2\rho_c - \rho) = \langle \Phi \rangle.
\]

**Proof.** By Lemma 3.3.1, we can assume that \( \mu + 2\rho_c \) is dominant for \( \Delta^+(g, t^c) \). Then

\[
\| \mu \|_{\text{lambda}} = \| P(\mu + 2\rho_c - \rho) \|.
\]

On the other hand, for any subset \( \Phi \subseteq \Delta^+(p, t^c) \),

\[
\mu - \rho_n + \langle \Phi \rangle + \rho_c = \mu + 2\rho_c - \rho + \langle \Phi \rangle.
\]

Hence it boils down to show

\[
\| \mu + 2\rho_c - \rho + \langle \Phi \rangle \|^2 \geq \| P(\mu + 2\rho_c - \rho) \|^2.
\]

Indeed,

\[
\| \mu + 2\rho_c - \rho + \langle \Phi \rangle \|^2 - \| P(\mu + 2\rho_c - \rho) \|^2
\]

\[
= \| \langle \Phi \rangle - \sum_i c_i \beta_i \|^2 + 2\langle \langle \Phi \rangle - \sum_i c_i \beta_i, P(\mu + 2\rho_c - \rho) \rangle
\]

\[
= \| \langle \Phi \rangle - \sum_i c_i \beta_i \|^2 + 2\langle \langle \Phi \rangle, P(\mu + 2\rho_c - \rho) \rangle,
\]

where we use [Lemma 3.3.1, (b), part (i)] at the first step, and [Lemma 3.3.1, (b), part (ii)] at the second step. Now we are done since [Lemma 3.3.1, (a)] gives

\[
\langle \langle \Phi \rangle, P(\mu + 2\rho_c - \rho) \rangle \geq 0.
\]
Proposition 3.3.4. For any $\mu \in i(t_0)^*$ which is dominant integral for $\Delta^+(\mathfrak{k}, \mathfrak{t}^\circ)$, we have
\[ \|\mu\|_{\text{spin}} \geq \|\mu\|_{\lambda}. \] (3.13)

Proof. This is immediate from the previous lemma and (3.11) of Lemma 3.2.5. \qed

3.4 Application to tempered representations

In this section, we will apply Proposition 3.3.4 to tempered representations.

In the book [V3], Vogan used standard modules to give the Langlands classification. Let us briefly recall this theory as follows.

Firstly, let us recall how to attach a set of $\theta$-stable data $(q, H, \delta, \nu)$ to an irreducible $(\mathfrak{g}, K)$ module $X$. Let $\pi$ be a fixed lambda-lowest $K$-type ([V3, Def. 5.4.1]) of $X$ with highest weight $\mu \in \widehat{T}^c$. Note that by [V3, Cor. 6.5.15], the multiplicity of $\pi$ in $X$ is one. Then via [V3, Prop. 5.3.3], we can associate a unique element $\lambda \in i(t_0)^+$ to $\mu$. Hence one can use it to define a $\theta$-stable parabolic subalgebra $q = \mathfrak{t} \oplus \mathfrak{u}$. Note that $t_0$ is quasisplit by [V3, Lemma 5.3.23]. By [V3, Prop. 5.4.2] (which says $R^G_\mu = R^L_\mu$) and [V3, Cor. 5.1.10], we see that there is a natural bijection between $A^{L\cap K}(\mu)$ and $A^K(\mu)$. Let $\pi^0 \in A^{L\cap K}(\mu)$ correspond to $\pi \in A^K(\mu)$ and let
\[ \pi^0 = \pi^0 \otimes [\wedge^R(\mathfrak{u} \cap \mathfrak{p})]^* \in \widehat{K}_L. \]

Then by [V3, Cor. 5.4.7], $\pi^0$ is a fine representation of $K_L$ with respect to $L$ and has multiplicity one in $H^{R}(\mathfrak{u}, X)$. Let $Y$ be the unique subquotient of $H^{R}(\mathfrak{u}, X)$ containing $\pi^0$. Let $H = TA$ be a $\theta$-stable maximally split Cartan subgroup of $L$. Suppose $\delta \in \widehat{T}$ occurs in $\pi^0|_T$. Then $\delta$ is fine with respect to $L$ and $\pi^0 \in A_L(\delta)$. 

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By [V3, Th. 4.4.8],

\[ Y \cong \pi_L(\delta \otimes \nu)(\pi^0), \] 

for some \( \nu \in \hat{A} \).

Then [V3, Cor. 5.4.9] says that \((q, H, \delta, \nu)\) is a set of \(\theta\)-stable data corresponding to \(X\) and all such sets arise in this way. Moreover, by [V3, Th. 6.5.12], \((q, H, \delta, \nu)\) is a set of \(\theta\)-stable data for \(G\) in the sense of [V3, Def. 6.5.1].

Conversely, suppose we have a set of \(\theta\)-stable data for \(G\) ([V3, Def. 6.5.1]) at hand, say, \((q, H, \delta, \nu)\). Choose a minimal parabolic subgroup \(P = TAN\) of \(L\), in a way such that \(\nu\) is negative with respect to \(N\). Let \(X_L(P, \delta \otimes \nu)\) be the \((t, K_L)\) module of the principal series \(\text{Ind}^L_T\)\(_{\text{AN}}(\delta \otimes \nu)\). Then as introduced in [V3, Def. 6.5.2], the standard \((g, K)\) module with parameters \((q, H, \delta, \nu)\) is

\[ X(q, \delta \otimes \nu) := \mathcal{R}_q^S(X_L(P, \delta \otimes \nu)). \]

Recall from [V3, Th. 6.5.9] that the set of lambda-lowest \(K\)-types of \(X(q, \delta \otimes \nu)\) is \(A(q, \delta)\), which has a natural one-one correspondence with \(A_L(\delta)\) (cf. [V3, Def. 6.5.5]). Moreover, the correspondence is given by the functor \(\mathcal{L}_S^K\). Now fix any \(\pi^L \in A_L(\delta)\), let \(\pi^G \in A(q, \delta)\) be the counterpart of \(\pi^L\). Let \(\mu^L \in \hat{T}^e\) be the highest weight of \(\pi^L\), then

\[ \mu^G := \mu^L + 2\rho(u \cap p, t^c) \in \hat{T}^e \]

is the highest weight of \(\pi^G\). Let \(\lambda^L \in t^*\) be the differential of \(\delta\), and let

\[ \lambda^G = \lambda^L + \rho(\Delta(u, t)). \]

By [V3, Lemma 6.5.4], \(\lambda^L\) (resp., \(\lambda^G\)) is the weight associated to \(\mu^L\) (resp., \(\mu^G\)) via [V3, Prop. 5.3.3]. Indeed, we have

\[ \lambda^L = \mu^L + 2\rho_c^L - \rho^L + \sum_i c_i \beta_i, \]  

(3.14)

\[ \lambda^G = \mu^G + 2\rho_c - \rho + \sum_i c_i \beta_i. \]  

(3.15)
Moreover, the infinitesimal character of $X(q, \delta \otimes \nu)$ is $\Lambda = (\lambda^G, \nu) \in \mathfrak{h}^*$ by [V3, Th. 6.5.9].

Finally, we note that the standard modules lead to Langlands classification, cf. [V3, Th. 6.5.12].

Now we are ready to apply Proposition 3.3.4 to tempered representations.

**Proposition 3.4.1.** In the above setting, if $\nu \in \hat{A}$ is unitary, then any $K$-type $\pi_G$ in $X(q, \delta \otimes \nu)$ which is not lambda-lowest can not contribute to $H_D(X(q, \delta \otimes \nu))$.

**Proof.** By Proposition 3.3.4, we have

$$\|\pi_G\|_{\text{spin}} \geq \|\pi_G\|_{\text{lambda}} > \|\pi^G\|_{\text{lambda}} = \|\lambda^G\| \geq \|\lambda^G + \nu\|.$$  

Now it is direct from Proposition 2.2.2 that $\pi_G$ can not contribute to $H_D(X(q, \delta \otimes \nu))$. $\square$

**Corollary 3.4.2.** Let $X$ be any irreducible tempered $(\mathfrak{g}, K)$ module. Then only the lambda-lowest $K$-types can contribute to $H_D(X)$.

**Proof.** This is direct from Proposition 3.4.1 and [SV, Th. 4.3 and Cor. 4.4]. $\square$
Chapter 4

Spin-norm and unitarily small \(K\)-types

4.1 Salamanca-Riba and Vogan’s conjectures

In [SV], Salamanca-Riba and Vogan propose a conjecture on the classification of the unitary dual of \(G\). They decompose \(\Pi_u(G)\) into disjoint subsets with a (very explicit) discrete parameter set \(\Lambda_u\):

\[
\Pi_u(G) = \bigcup_{\lambda_u \in \Lambda_u} \Pi_u^{\lambda_u}(G).
\]

Moreover, they propose the following conjecture

[SV, Conj. 0.6] The cohomological induction functor \(L_S(\lambda_u)\) implements a bijection from \(\Pi_u^{\lambda_u}(G(\lambda_u))\) onto \(\Pi_u^{\lambda_u}(G)\).

Therefore, the problem of classifying \(\Pi_u(G)\) would be reduced to the case \(G(\lambda_u) = G\). Recall that in [SV, Def. 6.1], a \(K\)-type \(\delta\) is called unitarily small if its highest weight \(\mu\) satisfies \(G(\lambda_u(\mu)) = G\). [SV, Conj. 5.7] further reduces the classification of \(\Pi_u(G)\) to that of the \(u\)-small ones. Finally, they reduce [SV,
Conj. 5.7] to [SV, Conj. 7.13], which desires to sharpen Parthasarathy’s Dirac inequality.

In [SV, §6, §7], a series of characterizations of unitarily small $K$-types are given. Among these characterizations, we single out the following one. Let

$$R(\Delta(p, t^c)) = \left\{ \sum_{\alpha \in \Delta(p, t^c)} b_{\alpha} \alpha, 0 \leq b_{\alpha} \leq 1 \right\},$$

which is a convex set and invariant under the action of $W(K)$. Then by the equivalence of (a) and (c) in [SV, Cor. 6.12], we have

$$\delta \in \hat{K} \text{ is unitarily small if and only if its highest weight belongs to } R(\Delta(p, t^c)).$$

(4.2)

### 4.2 Spin-norms of unitarily small $K$-types

**Lemma 4.2.1.** The convex set $R(\frac{1}{2}\Delta(p, t^c))$ is the convex hull of all the extremal weights of the spin representation $S$.

**Proof.** Since any extremal weight of $S$ must belong to $R(\frac{1}{2}\Delta(p, t^c))$, the containment $\subseteq$ is obvious. Noticing that $R(\frac{1}{2}\Delta(p, t^c))$ is a convex set invariant under $W(K)$, to have $\subseteq$, it suffices to show that all the extremal points of $R(\frac{1}{2}\Delta(p, t^c))$ which are dominant for $\Delta^+(t, t^c)$ belong to the RHS. However, these points are exactly all the highest weights (without multiplicity) of $S$, and we are done. $\square$

**Lemma 4.2.2.** For any point $\lambda \in R(\frac{1}{2}\Delta(p, t^c))$, we always have

$$\|\lambda + \rho_c\| \leq \|\rho\|.$$

**Proof.** By the previous lemma, we need only to consider the case that $\lambda$ is an extremal weight of $S$. Since $\rho_c$ is dominant for $\Delta^+(t, t^c)$, to have an upper bound
for $\|\lambda + \rho_c\|$, one can further focus on the case that $\lambda$ is a highest weight of $S$, say $\lambda = w\rho - \rho_c$. Then $\lambda + \rho_c = w\rho$ and the conclusion is now obvious. \hfill \Box

**Proposition 4.2.3.** Let $\delta_\lambda$ be any unitarily small $K$-type with highest weight $\lambda$. Then

$$\|\rho_c\| \leq \|\delta_\lambda\|_{\text{spin}} \leq \|\rho\|.$$  

**Proof.** The first inequality is obvious from (3.10). We note that $\|\delta_\lambda\|_{\text{spin}} = \|\rho_c\|$ if and only if $\lambda = w\rho - \rho_c$ for some $w \in W(g, t^c)^1$. For the second inequality, we use [SV, Th. 6.7(f)], which says that there is a positive system $(\Delta^+)'(g, t^c)$ containing $\Delta^+(t, t^c)$ such that

$$\lambda = \sum_{\beta \in (\Delta^+)'(p, t^c)} c_\beta \beta, \ c_\beta \in [0, 1].$$

Then, as noted in the proof of Lemma 3.2.5,

$$-\rho'_n = -\sum_{\beta \in (\Delta^+)'(p, t^c)} \frac{1}{2} \beta$$

is a lowest weight of $S$. Therefore,

$$\|\delta_\lambda\|_{\text{spin}} \leq \|\{\lambda - \rho'_n\} + \rho_c\|.$$  

Note that $\lambda - \rho'_n \in R(\frac{1}{2}\Delta(p, t^c))$, which is $W(K)$ invariant. Hence $\{\lambda - \rho'_n\} \in R(\frac{1}{2}\Delta(p, t^c))$. Therefore, by the previous lemma, we have

$$\|\{\lambda - \rho'_n\} + \rho_c\| \leq \|\rho\|.$$  

\hfill \Box

Let us illustrate the above stuff via a specific example.

**Example 4.2.4.** Take $G = Sp(4, \mathbb{R})$, $K = U(2)$, then $T^c = U(1) \times U(1)$, and

$$\hat{T}^c \cong \mathbb{Z}^2 \subseteq \mathbb{R}^2 \cong i(t_0^c)^*.$$
The set of roots of $T^c$ in $\mathfrak{g}$ is

$$\Delta(\mathfrak{g}, t^c) = \{ (\pm 2, 0), (0, \pm 2), \pm (1, 1), \pm (1, -1) \};$$

all but the last pair are noncompact. As a positive root in $K$ we choose $(1, -1) = 2\rho_c$; the dominant weights $\mu_{(p,q)}$ are then parameterized by decreasing pairs of integers $(p, q)$. We write $\delta_{(p,q)}$ for the irreducible representation of $U(2)$ with highest weight $\mu_{(p,q)}$. Note that there are four choices of positive systems of $\Delta(p, t^c)$ compatible with $\Delta^+(t, t^c)$. We fix one firstly, say,

$$\Delta^+(p, t^c) = \{ (1, 1), (2, 0), (0, 2) \}.$$

Then it is easy to calculate that $\rho = (2, 1)$, and

$$W(\mathfrak{g}, t^c) = \{ w_0 = e, w_1 = s_{(0,2)}, w_2 = s_{(1,1)}s_{(0,2)}, w_3 = s_{(2,0)}s_{(1,1)}s_{(0,2)} \},$$

where $e$ stands for the identity element, $s_{(0,2)}$ stands for the reflection along the root $(0, 2)$, and so on. Then all the four choices can be described as follows:

$$w_0 \leftrightarrow \Delta^+(p, t^c) = \{ (1, 1), (2, 0), (0, 2) \} : \rho_n = w_0 \rho - \rho_c = \left( \frac{3}{2} \right) \left( \frac{3}{2} \right)$$

$$w_1 \leftrightarrow (\Delta^+)^{(1)}(p, t^c) = \{ (1, 1), (2, 0), (0, -2) \} : \rho_n^{(1)} = w_1 \rho - \rho_c = \left( \frac{3}{2} \right) \left( -\frac{1}{2} \right)$$

$$w_2 \leftrightarrow (\Delta^+)^{(2)}(p, t^c) = \{ (-1, -1), (2, 0), (0, -2) \} : \rho_n^{(2)} = w_2 \rho - \rho_c = \left( \frac{1}{2} \right) \left( -\frac{3}{2} \right)$$

$$w_3 \leftrightarrow (\Delta^+)^{(3)}(p, t^c) = \{ (-1, -1), (-2, 0), (0, -2) \} : \rho_n^{(3)} = w_3 \rho - \rho_c = \left( -\frac{3}{2} \right) \left( -\frac{3}{2} \right).$$

The spin representation $S$ is decomposed as

$$S = E_{\left( \frac{3}{2}, \frac{3}{2} \right)} \oplus E_{\left( \frac{3}{2}, -\frac{3}{2} \right)} \oplus E_{\left( \frac{3}{2}, -\frac{3}{2} \right)} \oplus E_{\left( -\frac{3}{2}, -\frac{3}{2} \right)}$$

with corresponding lowest weights

$$\left( \frac{3}{2}, \frac{3}{2} \right), \left( -\frac{1}{2}, \frac{3}{2} \right), \left( -\frac{3}{2}, \frac{1}{2} \right), \left( -\frac{3}{2}, -\frac{3}{2} \right).$$

As a set, they are precisely

$$\{-\rho_n, -\rho_n^{(1)}, -\rho_n^{(2)}, -\rho_n^{(3)}\}.$$
We note further that to get the convex set $R(\frac{1}{2} \Delta(p, t^c))$, one can just link all the extremal weights of $S$

$$\{\rho_n = -\rho_n^{(3)}, \rho_n^{(1)}, \rho_n^{(2)}, \rho_n^{(3)} = -\rho_n^{(1)}, -\rho_n^{(2)}\}$$

consecutively. Doubling it, we get $R(\Delta(p, t^c))$. Using the line $p = q$ to cut it, one can easily obtain all the 25 unitarily small $K$-types as follows:

$$\{\delta_{(p,q)} \mid -3 \leq q \leq p \leq 3, p - q \leq 4\}.$$

### 4.3 Some toy examples

In this section, we will answer Problem B affirmatively on some toy examples. For these examples, we have (1.3).

**Example 4.3.1.** Take $G = SL(2, \mathbb{R})$, $K = SO(2) = T^c$, so that

$$\hat{K} = \hat{T}^c \cong \mathbb{Z} \subseteq \mathbb{R} \cong i(t_0^c)^*.$$

All the $K$-types $\delta_p$ are parameterized by integers $p$. The set of roots of $T^c$ in $\mathfrak{g}$ is

$$\Delta(\mathfrak{g}, t^c) = \{\pm 2\}.$$

We take $\Delta^+(p, t^c) = \{2\}$, then $\rho = \rho_n = 1, \rho_c = 0, W(\mathfrak{g}, t^c)^1 \cong \mathbb{Z}_2$ and

$$S = E_1 \oplus E_{-1}.$$ 

It is easy to calculate that $\{\delta_0, \delta_{\pm 1}, \delta_{\pm 2}\}$ are all the unitarily small $K$-types. Moreover, we have

$$\|\delta_{\pm p}\|_{\text{spin}} = (p - 1)\|\rho\|, \forall p \geq 1.$$ 

Hence (1.3) holds for $SL(2, \mathbb{R})$.

**Example 4.3.2.** Let $G$ be the real $G_2$, which is split. Then $K \cong SU(2) \times SU(2)$ and $T^c \cong U(1) \times U(1)$. We fix

$$\Delta^+(\mathfrak{t}, t^c) = \{\alpha_1, 3\alpha_1 + 2\alpha_2\},$$
where \( \alpha_1 = (1, -1, 0), \alpha_2 = (-2, 1, 1) \). Then \( \rho_c = 2\alpha_1 + \alpha_2 \), the two fundamental weights are

\[
\varpi_1 = \left(\frac{1}{2}, -\frac{1}{2}, 0\right), \varpi_2 = \left(-\frac{1}{2}, -\frac{1}{2}, 1\right).
\]

Let \( \delta_{(p,q)} \) be the \( K \)-type with highest weight \( p\varpi_1 + q\varpi_2 \), where \( p, q \) are non-negative integers. We choose

\[
\Delta^+(p, t_c) = \{\alpha_2,\alpha_1 + \alpha_2, 2\alpha_1 + \alpha_2, 3\alpha_1 + \alpha_2\}.
\]

Then \( \rho = 5\alpha_1 + 3\alpha_2, \rho_n = 3\alpha_1 + 2\alpha_2 \), \( W(g, t_c)^1 = \{e, s_{\alpha_2}, s_{\alpha_1 + \alpha_2} s_{\alpha_2}\} \), and

\[
S = E_{2\varpi_2} \oplus E_{3\varpi_1 + \varpi_2} \oplus E_{4\varpi_1}.
\]

One easily obtains all the 29 unitarily small \( K \)-types as follows

\[
\{\delta_{(p,q)} | 0 \leq p, 0 \leq q, p + q \leq 8, \frac{1}{3}p + q \leq 4\}.
\]

By Proposition 4.2.3, to verify (1.3), it suffice to check that

\[
\|\delta'\|_{\text{spin}} > \|\rho\| = \sqrt{14},
\]

where \( \delta' \) is any non-unitarily small \( K \)-type. Indeed, we can exhaust all these \( K \)-types in the following fashion:

- \( \|\delta_{(0,4+k)}\|_{\text{spin}} = \sqrt{\frac{3}{2}k^2 + 9k + 14} > \sqrt{14}, \forall k \geq 1.\)
- \( \|\delta_{(1,4+k)}\|_{\text{spin}} = \sqrt{\frac{3}{2}k^2 + 9k + \frac{31}{2}} > \sqrt{14}, \forall k \geq 0.\)
- \( \|\delta_{(2,4+k)}\|_{\text{spin}} = \sqrt{\frac{3}{2}k^2 + 9k + 18} > \sqrt{14}, \forall k \geq 0.\)
- \( \|\delta_{(3+k,3+k)}\|_{\text{spin}} = \sqrt{\frac{3}{2}k^2 + 6k + 14} > \sqrt{14}, \forall k \geq 1.\)

By similar calculations which we omit here, one can verify that

\[
\|\delta_{(4,3+k)}\|_{\text{spin}} > \sqrt{14}, \forall k \geq 0; \|\delta_{(5,3+k)}\|_{\text{spin}} > \sqrt{14}, \forall k \geq 0;
\]

\[
\|\delta_{(6,2+k)}\|_{\text{spin}} > \sqrt{14}, \forall k \geq 1; \|\delta_{(7,1+k)}\|_{\text{spin}} > \sqrt{14}, \forall k \geq 1;
\]

\[
\|\delta_{(8,k)}\|_{\text{spin}} > \sqrt{14}, \forall k \geq 1; \|\delta_{(p,k)}\|_{\text{spin}} > \sqrt{14}, \forall k \geq 0, \forall p \geq 9.
\]
4.4 More examples

As one may has already noticed, in Example 4.3.2, it is an awkward computation to show that any non-unitarily small $K$-type has a spin-norm bigger than $\|\rho\|$, and it is hard to imagine that this fashion can be carried out in general. The first target of this section is to handle this point.

Recall that Proposition 4.2.3 tells us that any unitarily small $K$-type has a spin-norm $\leq \|\rho\|$. Hence to answer Problem B, it suffices to take care of all the $K$-types whose spin-norms are $\leq \|\rho\|$. Now let us show that these $K$-types are finite. (This conclusion is not surprising at all. What we will actually do is to give an algorithm to exhaust all these $K$-types.) Here, for simplicity, we assume that the roots of $\Delta^+(\mathfrak{k}, t^c)$ span $t^*_R = i(t^c_0)^*$, which is met when $G/K$ is not Hermitian symmetric.

Lemma 4.4.1. There are finitely many $K$-types whose spin-norms are $\leq \|\rho\|$.

Proof. Let $\{\varpi_1, \cdots, \varpi_l\}$ be the fundamental weights for $\Delta^+(\mathfrak{k}, t^c)$. Note that they have pairwise non-negative inner products. Let $L$ denote the cardinality of the set $W(\mathfrak{g}, t^c)^1$. Let

$$\rho_n^{(i)} = a_1^{(i)} \varpi_1 + \cdots + a_l^{(i)} \varpi_l, 0 \leq i \leq L - 1, a_j^{(i)} \in \mathbb{N}$$

be all the weights $w\rho - \rho_c, w \in W(\mathfrak{g}, t^c)^1$. Let

$$\mu = a_1 \varpi_1 + \cdots + a_l \varpi_l, a_j \in \mathbb{N}$$

be the highest weight of a $K$-type. Then by (3.10), we have

$$\|\mu\|_{\text{spin}}^2 = \min_{0 \leq i \leq L-1} \|\{\mu - \rho_n^{(i)}\} + \rho_c\|^2$$

$$\geq \min_{0 \leq i \leq L-1} \|\{\mu - \rho_n^{(i)}\}\|^2 + \|\rho_c\|^2$$

$$= \min_{0 \leq i \leq L-1} \|\mu - \rho_n^{(i)}\|^2 + \|\rho_c\|^2.$$
Therefore, it suffices to show that there are all but finitely many $\mu$ such that

\[
\min_{0 \leq i \leq L-1} \|\mu - \rho_n^{(i)}\|^2 > \|\rho\|^2 - \|\rho_c\|^2. \tag{4.3}
\]

We claim that for any fixed $j \in \{1, \cdots, l\}$, there exists a $M_j \in \mathbb{N}$ such that if $a_j \geq M_j$, then no matter what allowable values $a_k (k \neq j)$ may take, one always has (4.3). Indeed, let us start with one $a_j \geq \max_{0 \leq i \leq L-1} a_j^{(i)}$, then

\[
\|\mu - \rho_n^{(i)}\|^2 = \|(a_j - a_j^{(i)}) \varpi_j + \sum_{k \neq j} (a_k - a_k^{(i)}) \varpi_k\|^2
\geq (a_j - a_j^{(i)})^2 \|\varpi_j\|^2 + 2 (a_j - a_j^{(i)}) \sum_{k \neq j} \langle \varpi_j, (a_k - a_k^{(i)}) \varpi_k\rangle
\geq (a_j - a_j^{(i)})^2 \|\varpi_j\|^2 - 2 (a_j - a_j^{(i)}) \sum_{k \neq j} a_k^{(i)} \langle \varpi_j, \varpi_k\rangle,
\]

which is a quadratic equation in terms of $a_j$. Existence of the desired $M_j$ is then obvious and we have the claim. Therefore, for $\|\mu\|_{\text{spin}} \leq \|\rho\|$ to happen, one must have

\[
a_1 < M_1, \cdots, a_l < M_l,
\]

and the desired conclusion is now immediate. $\square$

Using Lemma 4.4.1, we are able to find that (1.3) holds for type F II and and type E IV.

**Example 4.4.2.** Let $G$ be the real F II. For its basic data, one can refer to [Kn, page 717]. Here

\[
K \simeq \text{Spin}(9),
\]

and we can choose the simple roots for $\Delta^+(\mathfrak{f}, \mathfrak{t}^c)$ as follows:

\[
\{2\alpha_1 + 2\alpha_2 + \alpha_3, \alpha_4, \alpha_3, \alpha_2\},
\]

where $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ are as on [Kn, page 691]. We fix a $\Delta^+(\mathfrak{g}, \mathfrak{t}^c)$ so that it has simple roots $\{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$. Then

\[
\rho_c = (\frac{7}{2}, \frac{5}{2}, \frac{3}{2}, \frac{1}{2}), \rho_n = (2, 0, 0, 0),
\]

and the desired conclusion is now immediate.
and it is easy to calculate that

$$|W(g, \mathfrak{t}^c)^1| = \frac{2^7 \cdot 3^2}{4! \cdot 2^4} = 3.$$  

Moreover, one can determine that

- $\rho_n^{(1)} = \rho_n - \alpha_1 = (\frac{3}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2})$; simple roots for $(\Delta^+)^{(1)}(g, \mathfrak{t}^c)$:
  
  $\{-\alpha_1, \alpha_1 + \alpha_2, \alpha_3, \alpha_4\}$.  

- $\rho_n^{(2)} = \rho_n - \alpha_1 - (\alpha_1 + \alpha_2) = (1, 1, 1, 0)$; simple roots for $(\Delta^+)^{(2)}(g, \mathfrak{t}^c)$:
  
  $\{-\alpha_1 - \alpha_2, \alpha_2, \alpha_4, 2\alpha_1 + 2\alpha_2 + \alpha_3\}$.

Finally, one can use Lemma 4.4.1 to find all the $K$-types with spin-norms $\leq \|\rho\|$.

Indeed, we can take

$$M_1 = 7, M_2 = 5, M_3 = 4, M_4 = 7,$$

and a direct calculation gives all these 27 $K$-types. Moreover, all of them are unitarily small. Hence (1.3) holds for $F$ II.

**Example 4.4.3.** Let $G$ be the real $E$ IV. For its basic data, one can refer to [Kn, page 710]. Here

$$\mathfrak{k}_0 = \mathfrak{f}_4,$$

and we can choose the simple roots for $\Delta^+(\mathfrak{k}, \mathfrak{t}^c)$ as follows:

$$\{\alpha_1, \alpha_2, \alpha_3, \alpha_4\},$$

where $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ are as on [Kn, page 691]. We fix a $\Delta^+(g, \mathfrak{t}^c)$ so that it has simple roots $\{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$. Then

$$\rho_c = (\frac{11}{2}, \frac{5}{2}, \frac{3}{2}, \frac{1}{2}); \rho_n = (\frac{5}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}),$$

and it is obvious that

$$W(g, \mathfrak{t}^c)^1 = \{e\}.$$
Finally, one can use Lemma 4.4.1 to find all the $K$-types with spin-norms $\leq \|\rho\|$.

Indeed, we can take

$$M_1 = 9, M_2 = 6, M_3 = 4, M_4 = 7,$$

and a direct calculation gives all these 37 $K$-types. Moreover, all of them are unitarily small. Hence (1.3) holds for $E$ IV.
Chapter 5

Reduction along a pencil:
complex classical group case

5.1 Preparations

Let $G$ be complex, viewed as a real Lie group. Note that $\Delta(g, h^c)$ consists of complex roots appearing in pairs. Therefore, we always have

$$\Delta(f, t^c) = \Delta(p, t^c).$$

Since a choice $\Delta^+(f, t^c)$ of positive systems of $\Delta(f, t^c)$ has been fixed, to be compatible, one can only choose the positive root system of $\Delta(p, t^c)$ as

$$\Delta^+(p, t^c) = \Delta^+(f, t^c).$$

Now $W(g, t^c)^1 = \{e\}$ and

$$S = 2^{[l_0/2]} E_{\rho^c},$$

where $l_0 = \dim a^c = \dim t^c$. Thus, Lemma 3.2.5 simplifies as follows.

Lemma 5.1.1. Let $G$ be complex. Let $\delta_\mu$ be the $K$-type with highest weight $\mu$. Then
1) \( \| \delta_\mu \|_{\text{spin}} = \| \{ \mu - \rho_c \} + \rho_c \|. \)

2) \( \| \delta_\mu \|_{\text{lambda}} = \| \mu \|. \)

3) \( \| \delta_\mu \|_{\text{spin}} \geq \| \mu \|, \text{ and the equality happens if and only if } \mu \text{ is regular.} \)

4) \( \| \delta_0 \|_{\text{spin}} = \| \delta_{2\rho_c} \|_{\text{spin}} = 2 \| \rho_c \|, \| \delta_{\rho_c} \|_{\text{spin}} = \| \rho_c \|. \)

**Proof.** For (1), the identity follows from (3.10) of Lemma 3.2.5. Notice that 
\( S = 2^{[h_0/2]} E_{\rho_c} \) has lowest weight \(-\rho_c\). For (2), it is obvious that \( \mu + 2\rho_c \) is dominant for \( \Delta^+ \langle g, t^c \rangle \). Then use the definition of lambda-norm (cf. Lemma 3.3.1 and (3.12)), we have 
\[
\| \delta_\mu \|_{\text{lambda}} = \| P(\mu + 2\rho_c - \rho) \| = \| P(\mu) \| = \| \mu \|.
\]
Combining (2) and Proposition 3.3.4 gives the first part of (3); the remaining part of (3) is then obvious. (4) is immediate from (1). \( \square \)

**Remark 5.1.2.** We note that \( \delta_{\rho_c} \) is the unique \( K \)-type with the minimal spin-norm \( \| \rho_c \| \). Therefore, [BP, Conj. 3.4] can be rephrased as: let \( X \) be any irreducible unitary representation with infinitesimal character \( \rho_c \). Suppose that \( H_D(X) \neq 0 \) (which then must be contributed from its spin-lowest \( K \)-type \( \delta_{\rho_c} \)), then the \( K \)-type \( \delta_{\rho_c} \) must occur with multiplicity one. Stated in this way, one would like to imagine that: for any irreducible unitary representation with non-zero Dirac cohomology (which is contributed exactly from its spin-lowest \( K \)-types), can we conclude that each spin-lowest \( K \)-type must occur with multiplicity one?

For the fixed \( \Delta^+ \langle t, t^c \rangle \), let \( \{ \alpha_1, \ldots, \alpha_l \} \) be corresponding simple roots, and let \( \{ \varpi_1, \ldots, \varpi_l \} \) be the corresponding fundamental weights. We use \( \Lambda^+_+ \) to denote all the non-negative integer combinations of \( \alpha_1, \ldots, \alpha_l \), and use \( \Lambda^+ \) to denote all those of \( \varpi_1, \ldots, \varpi_l \).
Lemma 5.1.3. Let \( \delta_\lambda \) be the \( K \)-type with highest weight \( \lambda \). Then it is unitarily small if and only if \( \langle \lambda - 2\rho_c, \varpi_i \rangle \leq 0, \ 1 \leq i \leq l \).

Proof. By [SV, Prop. 1.10], \( R(\Delta(p,t^c)) \) is the convex hull of the \( W(K) \) orbit of \( 2\rho_c \). Therefore, in the dominant Weyl chamber for \( \Delta^+(\mathfrak{t},t^c) \), there is only one extremal point of the convex set \( R(\Delta(p,t^c)) \), namely \( 2\rho_c \). Hence, \( \delta_\lambda \) is unitarily small if and only if \( \lambda \) lives in the domain of the dominant Weyl chamber enclosed inside by the \( l \) hyperplanes passing through \( 2\rho_c \) with normal vectors \( \varpi_i \), that is, if and only if \( \langle \lambda - 2\rho_c, \varpi_i \rangle \leq 0 \).

Remark 5.1.4. This lemma is a special case of [SV, Th. 6.7 (a), (d)]. We include a short proof here for completeness.

Lemma 5.1.5. Let \( \delta_\lambda \) be any unitarily small \( K \)-type with highest weight \( \lambda \). Then

\[
\|\rho_c\| \leq \|\delta_\lambda\|_{\text{spin}} \leq 2\|\rho_c\|.
\]

Proof. This is a special case of Proposition 4.2.3.

Example 5.1.6. Let \( G \) be the complex \( G_2 \). Let \( K \) be its maximal compact subgroup. Then \( \Delta(\mathfrak{t},t^c) = \Delta(p,t^c) \), both of type \( G_2 \). We fix

\[ \Delta^+(\mathfrak{t},t^c) = \{ \alpha_1, \alpha_2, \alpha_1 + \alpha_2, 3\alpha_1 + 2\alpha_2, 2\alpha_1 + \alpha_2, 3\alpha_1 + \alpha_2 \} \]

where \( \alpha_1 = (1,-1,0), \alpha_2 = (-2,1,1) \). The two fundamental weights are

\[ \varpi_1 = 2\alpha_1 + \alpha_2, \varpi_2 = 3\alpha_1 + 2\alpha_2. \]

Let \( \delta_{(p,q)} \) be the \( K \)-type with highest weight \( p\varpi_1 + q\varpi_2 \), where \( p, q \) are non-negative integers. All the 14 unitarily small \( K \)-types are as follows:

\[ \{ \delta_{(0,0)}, \delta_{(1,0)}, \delta_{(2,0)}, \delta_{(0,1)}, \delta_{(4,0)}, \delta_{(5,0)}, \delta_{(0,1)}, \delta_{(1,1)}, \delta_{(2,1)}, \delta_{(3,1)}, \delta_{(0,2)}, \delta_{(1,2)}, \delta_{(2,2)}, \delta_{(0,3)} \} \]

Moreover, it is direct to verify that

\[
\|\mu\| > 2\|\rho_c\|, \quad (5.1)
\]
where $\mu$ is the highest weight of any non-unitarily small $K$-type. Therefore, (1.3) follows from Lemma 5.1.1 (3) and Lemma 5.1.5.

Let $\beta$ be the largest root in $\Delta^+(\mathfrak{k}, \mathfrak{t}^c)$. Then $\beta$ is the highest weight of the $\mathfrak{p}$ representation of $K$. This chapter is devoted to prove

**Proposition 5.1.7.** Suppose $G$ is complex of type $A$, $B$, $C$ or $D$. Let $\beta$ be the largest positive root of $\Delta^+(\mathfrak{k}, \mathfrak{t}^c)$. Then we have

$$\|\mu\|_{\text{spin}} > \|\mu - \beta\|_{\text{spin}},$$

where $\mu$ is any non-unitarily small $K$-type such that $\mu - \beta$ is dominant.

Let us mention that (5.2) can be reduced further.

Actually, by (1) of Lemma 5.1.1,

$$\|\mu\|_{\text{spin}}^2 = \|\{\mu - \rho_c\} + \rho_c\|^2$$

$$= \|\rho_c\|^2 + \|\{\mu - \rho_c\}\|^2 + 2\langle \rho_c, \{\mu - \rho_c\} \rangle$$

$$= \|\rho_c\|^2 + \|\mu - \rho_c\|^2 + 2\langle \rho_c, \{\mu - \rho_c\} \rangle.$$

Similarly, we have

$$\|\mu - \beta\|_{\text{spin}}^2 = \|\rho_c\|^2 + \|\mu - \beta - \rho_c\|^2 + 2\langle \rho_c, \{\mu - \beta - \rho_c\} \rangle.$$

Therefore, to arrive at (5.2), it suffices to prove

$$\|\mu - \rho_c\|^2 > \|\mu - \beta - \rho_c\|^2$$

(5.3)

and

$$\langle \rho_c, \{\mu - \rho_c\} \rangle > \langle \rho_c, \{\mu - \beta - \rho_c\} \rangle.$$  

(5.4)

In the remaining part of this chapter, we will prove (5.3) and (5.4) case by case.
5.2 Type A

In the section, we take \( G = SL(n + 1, \mathbb{C}) \). Then \( \beta = e_1 - e_{n+1} \) is the largest root. Let \( \mu = (a_1, a_2, \cdots, a_{n+1}) = m_1 \varpi_1 + m_2 \varpi_2 + \cdots + m_n \varpi_n \) be a dominant weight. Here all the \( m_i \)'s are non-negative integers. Then

\[
a_i = \sum_{j=i}^{n} m_j + a_{n+1}, 1 \leq i \leq n; \quad a_{n+1} = -\frac{1}{n+1} \sum_{j=1}^{n} jm_j.
\]

We note that although the \( a_i \)'s are not necessarily integers, the difference between any two of them must be an integer.

**Lemma 5.2.1.** If \( a_1 - a_{n+1} = m_1 + m_2 + \cdots + m_n \leq n + 1 \), then \( \mu \) must be unitarily small.

**Proof.** According to Lemma 5.1.3, it boils down to verify that under the above assumption, we have \( \langle \mu - 2\rho_c, \varpi_i \rangle \leq 0, 1 \leq i \leq n \). Indeed, it is easy to calculate that

\[
\langle \varpi_j, \varpi_i \rangle = \begin{cases} 
  j(1 - \frac{i}{n+1}) & \text{if } j \leq i; \\
  i(1 - \frac{j}{n+1}) & \text{if } j > i.
\end{cases}
\]

Therefore,

\[
\langle \mu - 2\rho_c, \varpi_i \rangle = \frac{n+1-i}{n+1} \sum_{j=1}^{i} j(m_j - 2) + i \sum_{j=i+1}^{n} \frac{n+1-j}{n+1} (m_j - 2).
\]

Note that \( m_i \) has the largest coefficient. Hence the largest value of the above expression occurs when \( m_i = n+1 \) and \( m_j = 0 \), for any \( j \neq i \). Hence

\[
\langle \mu - 2\rho_c, \varpi_i \rangle \leq \frac{1}{n+1} \left[ i(n+1)(n+1-i) - (n+1-i) \sum_{j=1}^{i} 2j - i \sum_{j=i+1}^{n} 2(n+1-j) \right] \\
= \frac{1}{n+1} \left[ i(n+1)(n+1-i) - (n+1-i)i(i+1) - i(n-i)(n+1-i) \right] \\
= \frac{1}{n+1} \left[ i(n+1)(n+1-i) - i(n+1-i)(n+1) \right] = 0.
\]

\(\square\)
Now we can easily deduce (5.3) from the above lemma. Indeed, since
\[
\rho_c = \left(\frac{n}{2}, \frac{n}{2} - 1, \cdots, -\frac{n}{2}\right)
\]
\[
\mu - \rho_c = (a_1 - \frac{n}{2}, a_2 + 1 - \frac{n}{2}, \cdots, a_{n+1} + \frac{n}{2}),
\]
we have
\[
\|\mu - \rho_c\|^2 - \|\mu - \beta - \rho_c\|^2 = 2\langle\beta, \mu - \rho_c\rangle - \|\beta\|^2
\]
\[
= 2(a_1 - a_{n+1} - n) - 2
\]
\[
= 2(a_1 - a_{n+1}) - 2(n + 1) > 0
\]
since \(\mu\) is non-unitarily small.

Now let us show (5.4). Indeed,
\[
\rho_c = (n, n - 1, \cdots, 1, 0) - \nu
\]
\[
\mu - \rho_c = (a_1 - n, a_2 - (n - 1), \cdots, a_n - 1, a_{n+1}) + \nu
\]
\[
\mu - \beta - \rho_c = (a_1 - 1 - n, a_2 - (n - 1), \cdots, a_n - 1, a_{n+1} + 1) + \nu,
\]
where \(\nu = \frac{n}{2}(1, 1, \cdots, 1, 1)\).

Therefore,
\[
\langle\{\mu - \rho_c\}, \rho_c\rangle = \langle\{\mu - \rho_c - \nu\} + \nu, (n, n - 1, \cdots, 1, 0) - \nu\rangle
\]
\[
= \langle\{\mu - \rho_c - \nu\}, (n, n - 1, \cdots, 1, 0)\rangle + A,
\]
where
\[
A = \langle\{\mu - \rho_c - \nu\}, -\nu\rangle + \langle\nu, (n, n - 1, \cdots, 1, 0)\rangle + \langle\nu, -\nu\rangle
\]
\[
= \left(\frac{n}{2}\right)^2(n + 1) - \frac{n}{2}(a_1 + \cdots + a_{n+1}).
\]

Similarly, one can calculate that
\[
\langle\{\mu - \beta - \rho_c\}, \rho_c\rangle = \langle\{\mu - \beta - \rho_c - \nu\}, (n, n - 1, \cdots, 1, 0)\rangle + A.
\]
Since $\mu$ is assumed to be non-unitarily small, we have $a_1 - a_{n+1} > n + 1$. Since the difference $a_1 - a_{n+1}$ must be an integer, we actually have $a_1 - a_{n+1} \geq n + 2$. That is,

$$a_1 - 1 - n \geq a_{n+1} + 1.$$ 

Now let us partition the set $B := \{a_2 - (n - 1), \ldots, a_n - 1\}$ into three parts

$$B \cap [a_1 - n, +\infty) : b_1 \geq \cdots \geq b_{i-1},$$

$$B \cap [a_{n+1} + 1, a_1 - n] : c_1 \geq \cdots \geq c_{j-1},$$

$$B \cap (-\infty, a_{n+1}] : d_1 \geq \cdots \geq d_{k-1},$$

where $i, j, k$ are positive integers and $i + j + k = n + 2$. Here we note that the difference between $a_1 - n$ and any element $x \in B$ must be an integer. Therefore, if $x \geq a_1 - n$ fails, then one must have $x \leq a_1 - n - 1$. Similarly, if $x \geq a_{n+1} + 1$ fails, then one must have $x \leq a_{n+1}$. Hence, the three sets above indeed partition the set $B$.

Now it is easy to see that

$$\{\mu - \rho_c - \nu\} = (b_1, \cdots, b_{i-1}, a_1 - n, c_1, \cdots, c_{j-1}, a_{n+1}, d_1, \cdots, d_{k-1}),$$

$$\{\mu - \beta - \rho_c - \nu\} = (b_1, \cdots, b_{i-1}, a_1 - 1 - n, c_1, \cdots, c_{j-1}, a_{n+1} + 1, d_1, \cdots, d_{k-1}).$$

Pairing with $(n, n - 1, \cdots, 1, 0)$, the above two terms give a difference

$$(n + 1 - i) - (n + 1 - (i + j)) = j > 0.$$ 

Therefore, we have (5.4) and Proposition 5.1.7 is proved for type A.

### 5.3 Type B
In this section, we take $G = \text{Spin}(2n + 1, \mathbb{C})$, $n \geq 2$. Then $\beta = e_1 + e_2$ is the largest root. Let $\mu = (a_1, a_2, \cdots, a_n) = m_1 \varpi_1 + m_2 \varpi_2 + \cdots + m_n \varpi_n$ be a dominant weight. Here all the $m_i$'s are non-negative integers. Then

$$a_i = \sum_{j=i}^{n-1} m_j + \frac{m_n}{2}, 1 \leq i \leq n-1; \ a_n = \frac{m_n}{2}.$$ 

We note that the $a_i$'s are integers or half-integers simultaneously.

**Lemma 5.3.1.** If $a_1 + a_2 = m_1 + 2m_2 + \cdots + 2m_{n-1} + m_n \leq 2n - 1$, then $\mu$ must be unitarily small.

**Proof.** According to Lemma 5.1.3, it boils down to verify that under the above assumption, we have $\langle \mu - 2\rho_c, \varpi_i \rangle \leq 0$, $1 \leq i \leq n$.

**Case I** $1 \leq i \leq n - 1$. Then it is easy to calculate that

$$\langle \varpi_j, \varpi_i \rangle = \begin{cases} 
    j & \text{if } j < i; \\
    i & \text{if } i \leq j \leq n - 1; \\
    \frac{i}{2} & \text{if } j = n.
\end{cases}$$

Therefore,

$$\langle \mu - 2\rho_c, \varpi_i \rangle = \sum_{j=1}^{i-1} j(m_j - 2) + i \sum_{j=i}^{n-1} (m_j - 2) + \frac{i}{2}(m_n - 2)$$

$$= (m_1 + \cdots + (i-1)m_{i-1}) + i(m_i + \cdots + m_{n-1} + \frac{m_n}{2}) - (2ni - i^2).$$

- If $i = 1$, we have

$$\langle \mu - 2\rho_c, \varpi_1 \rangle = (m_1 + \cdots + m_{n-1} + \frac{m_n}{2}) - (2n - 1)$$

$$\leq (m_1 + 2m_2 + \cdots + 2m_{n-1} + m_n) - (2n - 1) \leq 0.$$  

- If $2 \leq i \leq n - 1$, we have

$$\langle \mu - 2\rho_c, \varpi_i \rangle \leq i(m_1 + m_2 + \cdots + m_{n-1} + \frac{m_n}{2}) - (2ni - i^2)$$

$$\leq \frac{2n-1}{2}i - (2ni - i^2) = i(-n - \frac{1}{2} + i) < 0.$$
Case II \( i = n \). Then it is easy to calculate that

\[
\langle \varpi_j, \varpi_n \rangle = \begin{cases} 
\frac{j}{2} & \text{if } 1 \leq j \leq n - 1; \\
\frac{n}{4} & \text{if } j = n.
\end{cases}
\]

Therefore,

\[
\langle \mu - 2\rho_c, \varpi_n \rangle = \sum_{j=1}^{n-1} \frac{j}{2}(m_j - 2) + \frac{n}{4}(m_n - 2)
\]

\[
\leq \frac{n}{4}(m_1 + 2m_2 + \cdots + 2m_{n-1} + m_n) - \frac{n^2}{2}
\]

\[
\leq \frac{1}{4}[n(2n - 1) - 2n^2] = -\frac{n}{4} < 0.
\]

Now we can easily deduce (5.3) from the above lemma. Indeed, since

\[
\rho_c = (n - \frac{1}{2}, n - \frac{3}{2}, n - \frac{5}{2}, \cdots, \frac{1}{2})
\]

\[
\mu - \rho_c = (a_1 + \frac{1}{2} - n, a_2 + \frac{3}{2} - n, a_3 + \frac{5}{2} - n, \cdots, a_n - \frac{1}{2}),
\]

we have

\[
\|\mu - \rho_c\|^2 - \|\mu - \beta - \rho_c\|^2 = 2\langle \beta, \mu - \rho_c \rangle - \|\beta\|^2
\]

\[
= 2(a_1 + a_2 - (2n - 2)) - 2
\]

\[
= 2(a_1 + a_2 - (2n - 1)) > 0
\]

since \( \mu \) is non-unitarily small.

Now let us show (5.4). Since \( \mu \) is non-unitarily small, by the above lemma, we have

\[
a_1 + a_2 \geq 2n. \tag{5.5}
\]

Moreover, note that

\[
\mu - \rho_c = (a_1 + \frac{1}{2} - n, a_2 + \frac{3}{2} - n, a_3 + \frac{5}{2} - n, \cdots, a_n - \frac{1}{2})
\]

\[
\mu - \beta - \rho_c = (a_1 - \frac{1}{2} - n, a_2 + \frac{1}{2} - n, a_3 + \frac{5}{2} - n, \cdots, a_n - \frac{1}{2}).
\]
Case I \( a_1 - \frac{1}{2} - n \geq 0 \) and \( a_2 + \frac{1}{2} - n \geq 0 \). Then (5.4) is obvious.

Case II \( a_1 - \frac{1}{2} - n \geq 0 \) and \( a_2 + \frac{1}{2} - n < 0 \). Therefore, we have \( a_2 + \frac{1}{2} - n \leq -\frac{1}{2} \) and

\[
a_2 + \frac{3}{2} - n \leq \frac{1}{2} \leq a_1 + \frac{1}{2} - n.
\]

It is easy to see that due to (5.5), we must have \( a_2 + \frac{3}{2} - n < a_1 + \frac{1}{2} - n \). Similarly, one can see that \( -(a_2 + \frac{3}{2} - n) < a_1 + \frac{1}{2} - n \). Hence we have

\[
|a_2 + \frac{3}{2} - n| < a_1 + \frac{1}{2} - n.
\]

We note that the difference of the above two terms must be an integer. Therefore,

\[
|a_2 + \frac{3}{2} - n| + 1 \leq a_1 + \frac{1}{2} - n.
\]

- If \( a_2 + \frac{3}{2} - n = \frac{1}{2} \), then (5.4) is obvious since

\[
|a_2 + \frac{1}{2} - n| = |a_2 + \frac{3}{2} - n| = \frac{1}{2}.
\]

- If \( a_2 + \frac{3}{2} - n \leq 0 \), then

\[
|a_2 + \frac{1}{2} - n| = |a_2 + \frac{3}{2} - n| + 1 \leq a_1 + \frac{1}{2} - n.
\]

Now let us partition the set \( B := \{|a_3 + \frac{5}{2} - n|, \ldots, |a_n - \frac{1}{2}|\} \) into three parts

\[
B \cap [a_1 + \frac{1}{2} - n, +\infty) : b_1 \geq \cdots \geq b_{i-1}
\]

\[
B \cap (a_2 + \frac{3}{2} - n, a_1 + \frac{1}{2} - n) : c_1 \geq \cdots \geq c_{j-1}
\]

\[
B \cap (-\infty, a_2 + \frac{3}{2} - n] : d_1 \geq \cdots \geq d_{k-1},
\]

where \( i, j, k \) are positive integers and \( i + j + k = n + 1 \).

Here we note that the difference between \( a_1 + \frac{1}{2} - n \) and any element \( x \in B \) must be an integer. Therefore, if \( x < a_1 + \frac{1}{2} - n \) holds, then one must have \( x \leq a_1 - \frac{1}{2} - n \). Similarly, if \( x > |a_2 + \frac{3}{2} - n| \) holds, then one must have \( x \geq |a_2 + \frac{3}{2} - n| + 1 = |a_2 + \frac{1}{2} - n| \).
Now it is easy to see that
\[
\{\mu - \rho_c\} = (b_1, \cdots, b_{i-1}, a_1 + \frac{1}{2} - n, c_1, \cdots, c_{j-1}, |a_2 + \frac{3}{2} - n|, d_1, \cdots, d_{k-1})
\]
\[
\{\mu - \beta - \rho_c\} = (b_1, \cdots, b_{i-1}, a_1 - \frac{1}{2} - n, c_1, \cdots, c_{j-1}, |a_2 + \frac{1}{2} - n|, d_1, \cdots, d_{k-1})
\]
Pairing with \(\rho_c = (n - \frac{1}{2}, n - \frac{3}{2}, \cdots, \frac{1}{2})\), the above two terms give a difference
\[
(n - \frac{2i-1}{2}) - (n - \frac{2(i+j)-1}{2}) = j > 0.
\]
Therefore, we have (5.4).

**Case III** \(a_1 - \frac{1}{2} - n < 0\) and \(a_2 + \frac{1}{2} - n < 0\). This case can not happen due to (5.5).

**Case IV** \(a_1 - \frac{1}{2} - n < 0\) and \(a_2 + \frac{1}{2} - n \geq 0\). Noticing (5.5) and the fact that \(a_1\) and \(a_2\) are both integers or half-integers, it is easy to see that \(a_1 = a_2 = n\). Then (5.4) is obvious.

Now all the cases are done, and in sum, we have (5.4). Thus, Proposition 5.1.7 is proved for type B.

### 5.4 Type C

In this section, we take \(G = Sp(n, \mathbb{C})\), \(n \geq 3\). Then \(\beta = 2e_1\) is the largest root. Let \(\mu = (a_1, a_2, \cdots, a_n) = m_1 \varpi_1 + m_2 \varpi_2 + \cdots + m_n \varpi_n\) be a dominant weight. Here all the \(m_i\)'s are non-negative integers. Then
\[
a_i = \sum_{j=1}^{n} m_j, 1 \leq i \leq n.
\]

**Lemma 5.4.1.** If \(a_1 = m_1 + \cdots + m_n \leq n + 1\), then \(\mu\) must be unitarily small.

**Proof.** According to Lemma 5.1.3, it boils down to verify that under the above assumption, we have \((\mu - 2\rho_c, \varpi_i) \leq 0, 1 \leq i \leq n\).
It is easy to calculate that
\[
\langle \varpi_j, \varpi_i \rangle = \begin{cases} 
  j & \text{if } j < i; \\
  i & \text{if } j \geq i.
\end{cases}
\]

Therefore,
\[
\langle \mu - 2\rho_c, \varpi_i \rangle = \sum_{j=1}^{i-1} j(m_j - 2) + i \sum_{j=i}^{n} (m_j - 2)
\]
\[
= \sum_{j=1}^{i-1} jm_j + i \sum_{j=i}^{n} m_j - (2(n+1)i - i^2 - i)
\]
\[
\leq i(m_1 + \cdots + m_n) - (2(n+1)i - i^2 - i)
\]
\[
\leq i(n+1) - (2(n+1)i - i^2 - i) = i(i-n) \leq 0.
\]

Now we can easily deduce (5.3) from the above lemma. Indeed, since
\[
\rho_c = (n, n-1, \cdots, 1)
\]
\[
\mu - \rho_c = (a_1 - n, a_2 + 1 - n, \cdots, a_n - 1),
\]
we have
\[
\|\mu - \rho_c\|^2 - \|\mu - \beta - \rho_c\|^2 = 2\langle \beta, \mu - \rho_c \rangle - \|\beta\|^2
\]
\[
= 4(a_1 - n) - 4
\]
\[
= 4(a_1 - (n + 1)) > 0
\]
since \(\mu\) is non-unitarily small.

Now let us show (5.4). Since \(\mu\) is non-unitarily small, by the above lemma, we have
\[
a_1 \geq n + 2.
\]
Moreover, note that

\[ \mu - \rho \ = \ (a_1 - n, \ a_2 + 1 - n, \ldots, a_n - 1) \]
\[ \mu - \beta - \rho \ = \ (a_1 - 2 - n, \ a_2 + 1 - n, \ldots, a_n - 1). \]

Since \( a_1 - 2 - n \geq 0 \), (5.4) is obvious. Thus, Proposition 5.1.7 is proved for type C.

5.5 Type D

In this section, we take \( G = \text{Spin}(2n, \mathbb{C}), \ n \geq 4 \). Then \( \beta = e_1 + e_2 \) is the largest root. Let \( \mu = (a_1, a_2, \ldots, a_n) = m_1 \varpi_1 + m_2 \varpi_2 + \cdots + m_n \varpi_n \) be a dominant weight. Here all the \( m_i \)'s are non-negative integers. Then

\[ a_i = \sum_{j=i}^{n-2} \frac{m_{j-1} + m_n}{2}, \ 1 \leq i \leq n-2; \ a_{n-1} = \frac{m_{n-1} + m_n}{2}; \ a_n = \frac{-m_{n-1} + m_n}{2}. \]

We note that the \( a_i \)'s are integers or half-integers simultaneously.

**Lemma 5.5.1.** If \( a_1 + a_2 = m_1 + 2m_2 + \cdots + 2m_{n-2} + m_{n-1} + m_n \leq 2n - 2 \), then \( \mu \) must be unitarily small.

**Proof.** According to Lemma 5.1.3, it boils down to verify that under the above assumption, we have \( \langle \mu - 2\rho, \varpi_i \rangle \leq 0, \ 1 \leq i \leq n \).

**Case I** \( 1 \leq i \leq n-2 \). Then it is easy to calculate that

\[ \langle \varpi_j, \varpi_i \rangle = \begin{cases} 
    j & \text{if } j < i; \\
    i & \text{if } i \leq j \leq n - 2; \\
    \frac{i}{2} & \text{if } j = n - 1 \text{ or } n.
\end{cases} \]
Therefore,
\[ \langle \mu - 2 \rho_c, \varpi_i \rangle = \sum_{j=1}^{i-1} j(m_j - 2) + i \sum_{j=i}^{n-2} (m_j - 2) + \frac{i}{2}(m_{n-1} - 2) + \frac{i}{2}(m_n - 2) \]
\[ = \sum_{j=1}^{i-1} jm_j + i \left( \sum_{j=i}^{n-2} m_j + \frac{m_{n-1} + m_n}{2} \right) - (2ni - i^2 - i). \]

- If \( i = 1 \), we have
  \[ \langle \mu - 2 \rho_c, \varpi_1 \rangle = (m_1 + \cdots + m_{n-2} + \frac{m_{n-1} + m_n}{2}) - (2n - 2) \]
  \[ \leq m_1 + 2(m_2 + \cdots + 2m_{n-2} + \frac{m_{n-1} + m_n}{2}) - (2n - 2) \leq 0. \]

- If \( 2 \leq i \leq n - 2 \), we have
  \[ \langle \mu - 2 \rho_c, \varpi_i \rangle \leq \frac{i}{2}(m_1 + 2(m_2 + \cdots + m_{n-2} + \frac{m_{n-1} + m_n}{2}) - (2ni - i^2 - i) \]
  \[ \leq \frac{i}{2}(2n - 2) - (2ni - i^2 - i) = i(i - n) < 0. \]

**Case II** \( i = n - 1 \). Then it is easy to calculate that
\[ \langle \varpi_j, \varpi_{n-1} \rangle = \begin{cases} \frac{j}{2} & \text{if } 1 \leq j \leq n - 2; \\ \frac{n}{4} & \text{if } j = n - 1; \\ \frac{n^2}{4} & \text{if } j = n. \end{cases} \]

Therefore,
\[ \langle \mu - 2 \rho_c, \varpi_{n-1} \rangle = \sum_{j=1}^{n-2} \frac{j}{2}(m_j - 2) + \frac{n}{4}(m_{n-1} - 2) + \frac{n-2}{4}(m_n - 2) \]
\[ \leq \frac{n}{4}(m_1 + 2m_2 + \cdots + 2m_{n-2} + m_{n-1} + m_n) - \frac{n^2 - n}{2} \]
\[ \leq \frac{n(2n - 2)}{4} - \frac{n^2 - n}{2} = 0. \]

**Case III** \( i = n \). Then it is easy to calculate that
\[ \langle \varpi_j, \varpi_n \rangle = \begin{cases} \frac{j}{2} & \text{if } 1 \leq j \leq n - 2; \\ \frac{n^2}{4} & \text{if } j = n - 1; \\ \frac{n}{4} & \text{if } j = n. \end{cases} \]
Therefore,
\[
\langle \mu - 2\rho_c, \omega_{n-1} \rangle = \sum_{j=1}^{n-2} \frac{j}{2} (m_j - 2) + \frac{n-2}{4} (m_{n-1} - 2) + \frac{n}{4} (m_n - 2)
\]
\[
\leq \frac{n}{4} (m_1 + 2m_2 + \cdots + 2m_{n-2} + m_{n-1} + m_n) - \frac{n^2 - n}{2}
\]
\[
\leq \frac{n(2n-2)}{4} - \frac{n^2 - n}{2} = 0.
\]

Now we can easily deduce (5.3) from the above lemma. Indeed, since
\[
\rho_c = (n-1, n-2, \cdots, 1, 0)
\]
\[
\mu - \rho_c = (a_1 - n + 1, a_2 - n + 2, \cdots, a_{n-1} - 1, a_n),
\]
we have
\[
\|\mu - \rho_c\|^2 - \|\mu - \beta - \rho_c\|^2 = 2\langle \beta, \mu - \rho_c \rangle - \|\beta\|^2
\]
\[
= 2(a_1 + a_2 - (2n - 3)) - 2
\]
\[
= 2(a_1 + a_2 - (2n - 2)) > 0
\]
since \(\mu\) is non-unitarily small.

Now let us show (5.4). Since \(\mu\) is non-unitarily small, by the above lemma, we have
\[
a_1 + a_2 \geq 2n - 1. \tag{5.6}
\]

Moreover, note that
\[
\mu - \rho_c = (a_1 - n + 1, a_2 - n + 2, a_3 - n + 3, \cdots, a_{n-1} - 1, a_n),
\]
\[
\mu - \beta - \rho_c = (a_1 - n, a_2 - n + 1, a_3 - n + 3, \cdots, a_{n-1} - 1, a_n).
\]

**Case I** \(a_1 - n < 0\) and \(a_2 - n + 1 \geq 0\). In this case, due to (5.6), we must have
\[
a_1 = a_2 = n - \frac{1}{2}.
\]
Then (5.5) is obvious.

**Case II** \( a_1 - n < 0 \) and \( a_2 - n + 1 < 0 \). This case can not happen due to (5.6).

**Case III** \( a_1 - n \geq 0 \) and \( a_2 - n + 1 \geq 0 \). In this case, (5.6) is obvious.

**Case IV** \( a_1 - n \geq 0 \) and \( a_2 + n - 1 < 0 \). Due to (5.6), we must have \( a_1 > n \).

Similarly,

\[
0 < |a_2 - n + 1| = -(a_2 - n + 1) \leq a_1 - n.
\]

- If \( a_2 - n + 2 > 0 \), then \( a_2 \geq n - \frac{3}{2} \). While on the other hand, \( a_2 - n + 1 < 0 \) implies that \( a_2 \leq n - \frac{3}{2} \). Therefore, we must have \( a_2 = n - \frac{3}{2} \). Then (5.4) is obvious.

- If \( a_2 - n + 2 \leq 0 \), then

\[
|a_2 - n + 2| + 1 = |a_2 - n + 1| \leq a_1 - n < a_1 - n + 1.
\]

Now let us partition the set \( B := \{|a_3 - n + 3|, \ldots, |a_{n-1} - 1|, |a_n|\} \) into three parts

\[
B \cap [a_1 - n + 1, +\infty) : b_1 \geq \cdots \geq b_{i-1}
\]

\[
B \cap [a_2 - n + 1, a_1 - n + 1) : c_1 \geq \cdots \geq c_{j-1}
\]

\[
B \cap (-\infty, |a_2 - n + 1|) : d_1 \geq \cdots \geq d_{k-1},
\]

where \( i, j, k \) are positive integers and \( i + j + k = n + 1 \).

Here we note that the difference between \( a_1 - n + 1 \) and any element \( x \in B \) must be an integer. Therefore, if \( x < a_1 - n + 1 \) holds, then one must have \( x \leq a_1 - n \). Similarly, if \( x < |a_2 - n + 1| \) holds, then one must have \( x \leq |a_2 - n + 1| - 1 = |a_2 - n + 2| \).
Now it is easy to see that

\[
\{\mu - \rho_c\} = (b_1, \cdots, b_{i-1}, a_1 - n + 1, c_1, \cdots, c_{j-1}, |a_2 - n + 2|, d_1, \cdots, d_{k-1}),
\]
\[
\{\mu - \beta - \rho_c\} = (b_1, \cdots, b_{i-1}, a_1 - n, c_1, \cdots, c_{j-1}, |a_2 - n + 1|, d_1, \cdots, d_{k-1}).
\]

Pairing with \(\rho_c = (n - 1, n - 2, \cdots, 1, 0)\), the above two terms give a difference

\[
(n - i) - (n - (i + j)) = j > 0.
\]

Therefore, we have (5.4).

Now all the cases are done, and in sum, we have (5.4). Thus, Proposition 5.1.7 is proved for type D.
Chapter 6

Complex group representations

Let $G$ be complex, and adopt the same setting as in §5.1.

6.1 Zhelobenko’s classification

Let $W = W(\mathfrak{g}_0, \mathfrak{h}_0^c)$, and we identify

$$\mathfrak{h}^c \cong \mathfrak{h}_0^c \times \mathfrak{h}_0^c, \mathfrak{t}^c \cong \{(x, -x) : x \in \mathfrak{h}_0^c\}, \mathfrak{a}^c \cong \{(x, x) : x \in \mathfrak{h}_0^c\}. \quad (6.1)$$

Let $(\lambda_L, \lambda_R) \in (\mathfrak{h}_0^c)^* \times (\mathfrak{h}_0^c)^*$ be such that $\lambda_L - \lambda_R$ is a weight of a finite dimensional holomorphic representation of $G$. We can view $\lambda_L - \lambda_R$ as a weight of $T^c$ and $\lambda_L + \lambda_R$ as a character of $A^c$, and form the principal series

$$X(\lambda_L, \lambda_R) := \text{Ind}_{B}^{G}[\mathbb{C}_{\lambda_L - \lambda_R} \otimes \mathbb{C}_{\lambda_L + \lambda_R} \otimes 1]_{K-\text{finite}}. \quad (6.2)$$

**Theorem 6.1.1.** (Zhelobenko, cf. [Zh] or [BP, Th. 2.1]) The $K$-type with extremal weight $\lambda_L - \lambda_R$ occurs with multiplicity one in $X(\lambda_L, \lambda_R)$. Let $J(\lambda_L, \lambda_R)$ be the unique subquotient of $X(\lambda_L, \lambda_R)$ containing this $K$-type.

1) Every $X \in \Pi_a(G)$ is of the form $J(\lambda_L, \lambda_R)$.  

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2) Two such modules $J(\lambda_L, \lambda_R)$ and $J(\lambda'_L, \lambda'_R)$ are equivalent if and only if there exists $w \in W$ such that $w\lambda_L = \lambda'_L$ and $w\lambda_R = \lambda'_R$.

**Remark 6.1.2.** Note that $J(\lambda_L, \lambda_R)$ is finite dimensional if and only if both $\lambda_L$ and $\lambda_R$ are regular and integral, and belong to the same Weyl chamber. In this case, we have

$$J(\lambda_L, \lambda_R) \mid_K = V(\{\lambda_L\} - \rho_c) \otimes V(\{\lambda_R\} - \rho_c)^c. \quad (6.3)$$

Due to (2) of Theorem 6.1.1, we may and we will require that $\lambda_L$ is dominant. Choose $w \in W$ such that $w(\lambda_L - \lambda_R)$ is dominant. Let

$$\lambda := w(\lambda_L - \lambda_R), \nu := w(\lambda_L + \lambda_R). \quad (6.4)$$

Let $\overline{X}(\lambda, \nu)$ be the unique subquotient of $X(\lambda_L, \lambda_R)$ containing the $K$-type with highest weight $\lambda$. Then by (2) of Theorem 6.1.1, we have

$$\overline{X}(\lambda, \nu) \simeq J(\lambda_L, \lambda_R). \quad (6.5)$$

In the remaining part of this thesis, we will always refer to $J(\lambda_L, \lambda_R)$ as the $\overline{X}(\lambda, \nu)$ defined as above.

### 6.2 $K$-types pattern of a principal series

**Lemma 6.2.1.** All the $K$-types occurring in the principal series $X(\lambda_L, \lambda_R)$ (see (6.2)) are contained in

$$(\lambda_0 + \Lambda^+_t) \cap \Lambda^+, \quad \text{where } \lambda_0 \text{ denotes the unique dominant element to which } \lambda_L - \lambda_R \text{ is conjugate.}$$

**Proof.** Since for any $w \in W$, $X(\lambda_L, \lambda_R)$ and $X(w\lambda_L, w\lambda_R)$ have the same composition factors and multiplicities, we may assume that $\lambda_L - \lambda_R$ is dominant. Then $\lambda_0 = \lambda_L - \lambda_R$. Note that $M = Z_K(a) = T^c$. By Frobenius reciprocity, the
$K$-type $E_\lambda$ appears in $X(\lambda_L, \lambda_R)$ if and only if $E_\lambda|_{T^c}$ contains the $T^c$-type $\lambda_0$, if and only if $\lambda_0$ is a weight of $E_\lambda$. Therefore, the desired result follows from the highest weight theorem.

\[ \square \]

6.3 A description of all u-small $X \in \Pi_a(G)$

Lemma 6.3.1. For any $\lambda_0 \in \Lambda^+$ which is not unitarily small, there is no unitarily small $K$-type living on $(\lambda_0 + \Lambda^+_L) \cap \Lambda^+$.

Proof. Take any element $\lambda \in (\lambda_0 + \Lambda^+_L) \cap \Lambda^+$, say

$$\lambda = \lambda_0 + \sum_{i=1}^{l} n_i \alpha_i, n_i \in \mathbb{N}.$$ 

Since $\lambda_0$ is not unitarily small, by Lemma 5.1.3, we can find $1 \leq j \leq l$ such that

$$\langle \lambda_0 - 2 \rho_c, \varpi_j \rangle > 0.$$ 

Then

$$\langle \lambda - 2 \rho_c, \varpi_j \rangle = \langle \lambda - \lambda_0, \varpi_j \rangle + \langle \lambda_0 - 2 \rho_c, \varpi_j \rangle$$ 

$$= \frac{n_j}{2} ||\alpha_j||^2 + \langle \lambda_0 - 2 \rho_c, \varpi_j \rangle \geq \langle \lambda_0 - 2 \rho_c, \varpi_j \rangle > 0.$$ 

Therefore, $\lambda$ must be non-unitarily small by Lemma 5.1.3. \[ \square \]

Now we are able to describe all u-small $X \in \Pi_a(G)$ using Zhelobenko’s classification.

Proposition 6.3.2. Up to equivalence, all the u-small $X \in \Pi_a(G)$ are exhausted by

$$\{ X(\lambda, \nu) \mid \lambda \text{ is the highest weight of a unitarily small } K\text{-type} \}.$$
Proof. By Theorem 6.1.1 and (6.5), any $X \in \Pi_a(G)$ must be equivalent to certain $\overline{X}(\lambda, \nu)$ with $\lambda$ dominant integral. Now by Lemma 6.2.1 and Lemma 6.3.1, $\overline{X}(\lambda, \nu)$ is u-small if and only if $\lambda$ is unitarily small. 

6.4 Necessary conditions for $J(\lambda_L, \lambda_R)$ to have non-zero Dirac cohomology

This section, due to Barbasch and Pandžić [BP, page 5], is devoted to deduce necessary conditions for $J(\lambda_L, \lambda_R)$ to have non-zero Dirac cohomology. Indeed, take any $\tilde{K}$-type with highest weight $\tau$ in $H_D(J(\lambda_L, \lambda_R))$. Then by Theorem 2.2.1, we should have

$$w_1\lambda_L + w_2\lambda_R = 0, w_1\lambda_L - w_2\lambda_R = \tau + \rho_c, \text{ for some } w_1, w_2 \in W.$$ 

Since $\lambda_L$ is assumed to be dominant, the above conditions simplify as

$$\lambda_R = -s\lambda_L, 2\lambda_L = \lambda + \nu = \tau + \rho_c, \text{ for some } s \in W. \quad (6.6)$$

In particular, $2\lambda_L$ must be dominant integral and regular, and

$$(2\lambda_L, 0) \in (t^c)^* + (a^c)^*$$

is the infinitesimal character of $J(\lambda_L, \lambda_R)$. Moreover, if $J(\lambda_L, \lambda_R)$ is further assumed to admit a non-degenerate Hermitian form, then as deduced on [BP, page 5], $s \in W$ must be an involution.

6.5 A geometric description of the bottom layer $K$-types of $J(\lambda_L, -s\lambda_L)$

Starting with an irreducible representation $J(\lambda_L, \lambda_R)$, one can associate a set of $\theta$-stable data $(q, H, \delta, \nu')$ to it, and construct a corresponding standard module
$X(q, \delta \otimes \nu')$. Then $J(\lambda_L, \lambda_R)$ is located as a sub-representation of $X(q, \delta \otimes \nu')$. This theory of Vogan [V3] is briefly recalled in §4.3. Now let us specialize it to $\overline{X}(\lambda, \nu) \simeq J(\lambda_L, -s\lambda_L)$, where $s \in W$, $2\lambda_L$ is dominant integral regular, $\lambda := \{\lambda_L + s\lambda_L\}$ is dominant integral for $\Delta^+(\mathfrak{k}, t_c)$ (see (6.5) and (6.6)):

a) As mentioned in (2) Lemma 5.1.1, the weight associated to $\lambda \in i(\mathfrak{t}_0^c)^*$ via [V3, Prop. 5.3.3] is $\lambda$ itself. So the $\theta$-stable parabolic subalgebra $q = \mathfrak{t} + \mathfrak{u}$ is defined by $\lambda$. In particular, we see that $L$ is complex, and $H^c = T^c A^c$ is the unique $\theta$-stable Cartan subgroup of $L$. Thus $H = H^c$, $T = T^c$, $A = A^c$. Moreover, $\delta \in \widehat{T}^c$ is $\lambda - 2\rho(\mathfrak{u} \cap \mathfrak{p}) = \lambda - \rho(\mathfrak{u})$. Let $Y$ be the unique subquotient of $H^R(\mathfrak{u}, \overline{X}(\lambda, \nu))$ containing the $K_L$-type $\lambda - \rho(\mathfrak{u})$. Then by [V3, Th. 4.4.8],

$$Y \cong \overline{\pi}(\delta \otimes \nu')$$ for some $\nu' \in \widehat{A}^c$.

b) Now let us figure out the parameters $(\lambda'_L, \lambda'_R)$ of $\overline{\pi}(\delta \otimes \nu')$, where $\lambda'_L$ is dominant. Since the infinitesimal character of $\overline{X}(\lambda, \nu) \simeq J(\lambda_L, -s\lambda_L)$ is $2\lambda_L$, the infinitesimal character of $Y$ is $2\lambda_L - \rho(\mathfrak{u})$. Then, similar to the deduction in (6.6), we have

$$2\lambda'_L = 2\lambda_L - \rho(\mathfrak{u}).$$

On the other hand, $\lambda - \rho(\mathfrak{u})$ is the highest weight of the lambda-lowest $K$-type of $\overline{\pi}(\delta \otimes \nu')$. Since $\lambda - \rho(\mathfrak{u})$ is perpendicular to every root in $\Delta(\mathfrak{i}, \mathfrak{h}^c)$, we thus have

$$\lambda'_L - \lambda'_R = w_1(\lambda - \rho(\mathfrak{u})) = \lambda - \rho(\mathfrak{u}),$$

where $w_1 \in W_L = W(\mathfrak{t}_0, \mathfrak{h}^c_0)$. Therefore,

$$\lambda'_L + \lambda'_R = 2\lambda'_L - (\lambda'_L - \lambda'_R) = 2\lambda_L - \lambda.$$

To sum up, the set of $\theta$-stable data associated to $\overline{X}(\lambda, \nu)$ can be chosen as

$$(q, H^c, \lambda - \rho(\mathfrak{u}), 2\lambda_L - \lambda).$$

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c) As deduced in (6.6), the infinitesimal character of $\overline{X}(\lambda, \nu) \simeq J(\lambda_L, -s\lambda_L)$ is

$$(2\lambda_L, 0) \in (t^c)^* + (a^c)^*.$$ 

Since $\lambda_L$ is assumed to be dominant, the “good range” condition (3.1) is satisfied. Thus, Lemma 3.1.1 applies.

d) Now we can construct the standard module

$$R^S_q(X_L(\lambda - \rho(u), 2\lambda_L - \lambda)),$$

and it is obvious that (6.7) holds. By Lemma 6.2.1, the $K$-types of the minimal principal series $X_L(\lambda - \rho(u), 2\lambda_L - \lambda)$ are contained in

$$(\lambda - \rho(u) + \Lambda^+_{r,L}) \cap \Lambda^+_{L}.$$ 

Here $\Lambda^+_{r,L}$ is the set of all non-negative integer combinations of $\Delta^+ (l \cap k, t^c)$, and $\Lambda^+_{L}$ denotes all the corresponding dominant weights. Therefore, as noted in Remark 3.1.2, the bottom layer $K$-types of $X(q, \delta \otimes \nu)$ are contained in

$$(\lambda + \Lambda^+_{r,L}) \cap \Lambda^+.$$ 

Here, recall that $\Lambda^+$ denotes all the dominant weights corresponding to $\Delta^+ (k, t^c)$. In particular, we see that all these bottom layer $K$-types sit on the hyperplane passing through $\lambda$ and being perpendicular to $\lambda$ since

$$\Delta^+ (l \cap k, t^c) = \{ \alpha \in \Delta^+ (k, t^c) \mid \langle \lambda, \alpha \rangle = 0 \}.$$ 

Thus we arrive at

**Proposition 6.5.1.** Let $\overline{X}(\lambda, \nu) \simeq J(\lambda_L, -s\lambda_L)$ (see (6.5)) be an irreducible representation of a complex group $G$, where $s \in W(g_0, h_0^c)$, $2\lambda_L$ is dominant integral regular for $\Delta^+ (k, t^c)$, and $\lambda := \{ \lambda_L + s\lambda_L \}$ (the unique dominant element to which $\lambda_L + s\lambda_L$ is conjugate) is dominant integral for $\Delta^+ (k, t^c)$. Let $q = l + u$
be the $\theta$-stable parabolic subalgebra defined by $\lambda$. Then the set of $\theta$-stable data associated to $\mathcal{X}(\lambda,\nu)$ can be chosen as $(q,H^c = T^c A^c, \lambda - \rho(u), 2\lambda_L - \lambda)$, and we have

$$\mathcal{X}(\lambda,\nu) \simeq \mathcal{R}_q^S(\mathcal{X}_L(\lambda - \rho(u), 2\lambda_L - \lambda)).$$

(6.7)

Moreover, all the bottom layer $K$-types of the RHS are contained in the $\Delta^+(\mathfrak{k},t^c)$-dominant members of

$$\lambda + \Lambda^+_{r,L},$$

(6.8)

where $\Lambda^+_{r,L}$ is the set of all non-negative integer combinations of $\Delta^+(\mathfrak{t},t^c)$. In particular, all of them sit on the hyperplane passing through $\lambda$ and being perpendicular to $\lambda$. Finally, only these $K$-types can contribute to $H_D(\mathcal{X}(\lambda,\nu))$. 

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Chapter 7

Problem A for complex $F_4$

This Chapter is devoted to answer Problem A for complex $F_4$. We choose $\Delta^+(t,t^c)$, the corresponding fundamental weights $\{\varpi_1, \varpi_2, \varpi_3, \varpi_4\}$, etc., as [Kn, p. 691]. We will refer to a dominant weight by its coordinates in the basis $\{\varpi_1, \varpi_2, \varpi_3, \varpi_4\}$. For example, $[7,1,1,1]$ means $7\varpi_1 + \varpi_2 + \varpi_3 + \varpi_4$. We will also freely refer to a $K$-type by its highest weight.

Recall that if Problem B has an affirmative answer, then so does Problem A. Hence we start with Problem B.

7.1 Reduction by spin-norm

By Lemma 5.1.5, to answer Problem B, it suffices to consider all the unitarily small $K$-types, and all the non-unitarily small $K$-types whose spin-norms are less than or equal to $2\|\rho_c\| = 2\sqrt{39}$. Let us denote the first set by $UWeights$, and the latter by $NUWeights$. Using (3) of Lemma 5.1.1, we see that both of them are contained in the circular region with radius $2\|\rho_c\|$ and centered at the origin.
Therefore, we can easily identify these $K$-types. Actually, we have

$\#\text{UWeights} = 451$, $\#\text{NUWeights} = 20$.

The set $\text{NUWeights}$ being non-empty means (1.3) on longer holds. It is the existence of these $\text{NUWeights}$ that make the situation subtle while interesting. And our job is to handle the 451 families of irreducible representations containing a unitarily small $K$-type as the lambda-lowest $K$-type (cf. Proposition 6.3.2).

Some of these families are fairly easy to handle. Indeed, let us denote the minimal spin-norm of these 20 members in $\text{NUWeights}$ by $\text{MinNUSpinNorm}$, which equals to $\sqrt{141}$ and happens at $[7, 1, 1, 1]$. Collect the elements in $\text{UWeights}$ whose spin-norms are bigger than or equal to $\text{MinNUSpinNorm}$ as $\text{BigUWeights}$. It suffices to take care of these families. Moreover, if $\lambda$ is regular, then by (3) of Lemma 5.1.1 and Lemma 6.2.1, we have

$$\|\lambda\|_{\text{spin}} = \|\lambda\| < \|\mu\| \leq \|\mu\|_{\text{spin}},$$

where $\mu$ is any non-unitarily small $K$-type of $X(\lambda, \nu)$. So Problem B has an affirmative answer for $X(\lambda, \nu)$ when $\lambda$ is regular, no matter which value $\nu$ takes.

Therefore, it suffices to consider the irregular members of $\text{BigUWeights}$ (denoted by $\text{IrBigUWeights}$). For complex $F_4$, we have

$\#\text{BigUWeights} = 69$, $\#\text{IrBigUWeights} = 57$.

In particular, $[0, 0, 0, 0]$ belongs to $\text{IrBigUWeights}$. So up to now, answer for the spherical family remains unknown.

### 7.2 Reduction by one pencil

Note that Proposition 2.5.1 does not tell us the starting $K$-types of all the pencils in $X(\lambda, \nu)$. Therefore, it does not tells us the precise $K$-types pattern of
$X(\lambda, \nu)$. However, it does tell us that each $K$-type living on the pencil

$$\{\lambda + n\beta \mid n \in \mathbb{N}\}$$

occurs in $X(\lambda, \nu)$.

Now for any unitarily small $K$-type $\lambda$, let us denote by $P_\lambda$ the minimal spin-norm of all the unitarily small $K$-types living on the pencil (7.1), and denote by $N_\lambda$ the minimal spin-norm of all the non-unitarily small $K$-types living on the root lattice $(\lambda + \Lambda_+^+) \cap \Lambda^+$. If

$$N_\lambda > P_\lambda$$

holds, we call $\lambda$ determined; otherwise, we call $\lambda$ undetermined.

By Lemma 6.2.1, Problem B has an affirmative answer for $X(\lambda, \nu)$ when $\lambda$ is determined, no matter which value $\nu$ takes. We collect all the undetermined elements in $\text{IrBigUWeights}$ as $\text{UUWeights}$. It remains to consider these families, and we have

$$\#\text{UUWeights} = 12.$$

Let us present them as follows:

$$[0, 0, 0, 8], [0, 7, 0, 0], [2, 6, 0, 0], [5, 4, 0, 0], [6, 0, 0, 5], [7, 0, 0, 4],$$

$$[8, 0, 0, 3], [8, 2, 0, 0], [9, 0, 0, 2], [9, 0, 1, 0], [10, 0, 0, 1], [11, 0, 0, 0].$$

In particular, the trivial $K$-type is determined. Thus we have

**Corollary 7.2.1.** Let $G$ be the complex $F_4$. Then for any spherical unitary representation $X$ of $G$, only its unitarily small $K$-types can contribute to $H_D(X)$.

**Remark 7.2.2.** A vivid explanation is that along the pencil $\{0 + n\beta \mid n \in \mathbb{N}\}$, the spin-norm decreases firstly and then increases. In particular, one can find a unitarily small $K$-type living on this pencil whose spin-norm is small enough (say, smaller than $N_0$). This also explains that all the 12 members of $\text{UUWeights}$ are very close to the boundary of the convex hull $R(\Delta(p, t'))$. 
7.3 Reduction by Dirac cohomology

For the remaining 12 families, instead of studying Problem B, let us consider Problem A directly. That is, we try to answer Problem A for the representations in

\[ \{ \overline{X}(\lambda, \nu) \text{ unitary} \mid \lambda \text{ is a undetermined unitarily small } K\text{-type} \}. \quad (7.3) \]

Suppose this is false. Then we can find a undetermined unitarily small \( K \)-type \( \lambda \) and a parameter \( \nu \) such that

\[ \overline{X}(\lambda, \nu) \text{ is unitary} \quad (7.4) \]

and

at least one non-unitarily small \( K \)-type of it contributes to \( H_D(\overline{X}(\lambda, \nu)) \).

\[ (7.5) \]

As deduced in §6.4, necessary conditions for (7.4) and (7.5) to happen are

\[ \lambda_R = -s\lambda_L, \lambda = \{ \lambda_L + s\lambda_L \}, \quad (7.6) \]

where \( s \in W \) is an involution; and

\[ 2\lambda_L = \{ \mu - \rho_c \} + \rho_c, \quad (7.7) \]

where \( \mu \) is a non-unitarily small \( K \)-type living on \( (\lambda + \Lambda^+_r) \cap \Lambda^+ \); and

\[ \|2\lambda_L\| \leq P_\lambda. \quad (7.8) \]

The requirements (7.6), (7.7) and (7.8) allows fairly few candidate representations to survive. We call them undetermined representations, and collect them as \( UR \). For complex \( F_4 \), we have two such representations:

\[ \begin{array}{cc}
\lambda_L & -s\lambda_L \\
[5/2, 1/2, 1/2, 3/2] & [5/2, 1/2, 1/2, -13/2] \\
[2, 3/2, 1/2, 1/2] & [2, -11/2, 1/2, 1/2]
\end{array} \]
7.4 Reduction by tracing the bottom layer $K$-types

For each of the two undetermined representations in the previous section, it is direct to check that $\lambda_L + s\lambda_L$ is dominant. Then one can easily reformulate them according to (6.4) as follows:

$$\lambda = \lambda_L + s\lambda_L, \nu = \lambda_L - s\lambda_L.$$ 

Thus, the two undetermined representations are

$$X([0, 0, 0, 8], [5, 1, 1, -5]), X([0, 7, 0, 0], [4, -4, 1, 1]).$$

When one applies Proposition 6.5.1 to them, the parameters on the Levi level are $\lambda - \rho(u)$, and $2\lambda_L - \lambda = \lambda + \nu - \lambda = \nu$, respectively. Thus, we have

$$X(\lambda, \nu) \simeq R^S_q(X_L(\lambda - \rho(u), \nu)).$$

Now let us do non-unitarity test for them by tracing certain bottom layer $K$-types:

- Up to a center, the lambda-lowest $K$-type $\lambda = [0, 0, 0, 8]$ defines $L$ of type $C_3$, and the simple roots for $\Delta^+(l \cap \mathfrak{t}, \mathfrak{t}^c)$ are $\{\alpha_1, \alpha_2, \alpha_3\}$ with the corresponding largest root $2\alpha_1 + 2\alpha_2 + \alpha_3$. Note that

$$X_{C_3}([0, 0, 0], [5, 1, 1])$$

is infinite dimensional by Remark 6.1.2. Thus, by Proposition 2.5.1 and Remark 3.1.2, the bottom layer $K$-type (which is unitarily small)

$$\lambda + (2\alpha_1 + 2\alpha_2 + \alpha_3) = [2, 0, 0, 7]$$

must occur in $X(\lambda, \nu)$. But this $K$-type has spin-norm $7\sqrt{3}$, which is smaller than $\|2\lambda_L\| = \sqrt{155}$. Therefore, by the Dirac inequality, $X(\lambda, \nu)$ is not unitary.
• Up to a center, the lambda-lowest $K$-type $\lambda = [0, 7, 0, 0]$ defines $L$ of type $A_1 \times A_2$, and the simple roots for $\Delta^+(\mathfrak{f} \cap \mathfrak{k}, \mathfrak{t}^c)$ are $\{\alpha_1\} \times \{\alpha_3, \alpha_4\}$. Note that $\alpha_1$ and $\alpha_3 + \alpha_4$ are the corresponding largest roots. Since

$$\mathbb{X}_{A_2}([0, 0], [1, 1])$$

is infinite dimensional by Remark 6.1.2, a similar argument as above shows that the bottom layer $K$-type (which is unitarily small)

$$\lambda + (\alpha_3 + \alpha_4) = [0, 5, 1, 1]$$

must occur in $\mathbb{X}(\lambda, \nu)$. But this $K$-type has spin-norm $5\sqrt{6}$, which is smaller than $\|2\lambda_L\| = \sqrt{153}$. Therefore, by the Dirac inequality, $\mathbb{X}(\lambda, \nu)$ is not unitary.

Since both of the two undetermined representations are not unitary, we answer Problem A affirmatively for complex $F_4$.

### 7.5 Further remarks

**Proposition 7.5.1.** In the spherical unitary dual of complex $F_4$, there are only finitely many representations which can have non-zero Dirac cohomology.

**Proof.** As deduced in §6.4, necessary conditions for $\mathbb{X}(0, \nu)$ to be unitary and with non-zero Dirac cohomology are

$$0 = \lambda_L + s\lambda_L, \nu = \lambda_L - s\lambda_L,$$

where $s \in W$ is an involution and $2\lambda_L$ is dominant integral and regular; Corollary 7.2.1 further requires that

$$\nu = \{\lambda - \rho_c\} + \rho_c,$$
where $\lambda$ is the highest of a unitarily small $K$-type; and

$$\|\nu\| \leq P_0.$$  \hfill (7.11)

Now it is obvious that (7.9), (7.10) and (7.11) allow only finitely many choices for $\nu$. \hfill \Box

**Remark 7.5.2.** Except for the trivial representation, all the possible choices of $\nu$ can be easily calculated as follows:

$$[2, 2, 1, 1], [1, 2, 1, 1], [2, 1, 1, 1], [2, 1, 1, 2], [1, 1, 2, 1], [1, 1, 1, 2], [1, 1, 1, 1], [3, 1, 1, 1].$$

Therefore, all the spherical unitary representations of complex $F_4$ which can have non-zero Dirac cohomology should occur in the above list. In particular, they are *fairly few*. After doing a careful unitarity test, one can locate them precisely. However, a more interesting question is that among the unitary dual, what is the role played by those with non-zero Dirac cohomology? For this direction, we refer the reader to [BP, Conj. 1.1].

### Table 1. Basic information of complex type E

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<td>$4\sqrt{29}$</td>
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<td>$2|\rho_c|$</td>
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<td>$\sqrt{798}$</td>
<td>$4\sqrt{155}$</td>
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</table>
A second remark is that the approach of answering Problem A for complex $F_4$ shall also work for complex $G$ of type $E$. Part of such information is listed in Table 1. In particular, we always have $P_0 < \text{Min} \text{NUSpinNorm}$. Therefore, $P_0 < N_0$, and we have

**Corollary 7.5.3.** Let $G$ be complex of type $E$. Then for any spherical unitary representation $X$ of $G$, only its unitarily small $K$-types can contribute to $H_D(X)$. 


Chapter 8

Problem A for complex $E_6$

This section is devoted to answer Problem A for complex $E_6$. We choose $\Delta^+(\mathfrak{t}, \mathfrak{t}^e)$, the corresponding fundamental weights $\{\varpi_1, \cdots, \varpi_6\}$, etc., as [Kn, p. 687]. We will refer to a dominant weight by its coordinates in the basis $\{\varpi_1, \cdots, \varpi_6\}$. For example, $[0, 0, 0, 0, 9, 0]$ means $9\varpi_5$. We will also freely refer to a $K$-type by its highest weight.

We separate the 162 undetermined representations (cf. Table 1) into two parts according to whether $\lambda + s\lambda_L$ is dominant or not.

8.1 $\lambda + s\lambda_L$ is dominant

There are 114 undetermined representations such that $\lambda + s\lambda_L$ is dominant. For 112 of them, we can do non-unitarity test in the same fashion as §7.4. That is, in the following table, $\lambda'$ (if there exists) is a unitarily small $K$-type occurring in $\overline{X}(\lambda, \nu)$ (guaranteed by Proposition 2.5.1, Remark 3.1.2, Remark 6.1.2 and Proposition 6.5.1) such that $\|\lambda'\|_{\text{spin}} < \|\lambda + \nu\|$. Therefore, by the Dirac inequality, $\overline{X}(\lambda, \nu)$ is not unitary.
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70
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[2, 2, 0, 0, 7, 0] & [-1, -1, 1, 1, -3, 4] & [1, 1, 1, 1, 6, 0] \\
[2, 3, 0, 0, 6, 0] & [-1, -1, 1, 1, -3, 4] & [1, 2, 1, 1, 5, 0] \\
[2, 5, 0, 0, 5, 0] & [-1, -1, 1, 1, -3, 4] & [1, 4, 1, 1, 4, 0] \\
[3, 0, 0, 6, 0, 0] & [-2, 2, 4, -4, 1, 1] & [3, 0, 0, 5, 1, 1] \\
[3, 0, 0, 6, 0, 0] & [-1, 4, 2, -4, 1, 1] & [3, 0, 0, 5, 1, 1] \\
[3, 0, 2, 6, 0, 0] & [0, 1, -1, 1, -3, 4] & [3, 1, 1, 1, 5, 0] \\
[3, 0, 3, 0, 5, 0] & [0, 1, -1, 1, -3, 4] & [3, 1, 2, 1, 4, 0] \\
[3, 2, 0, 5, 0, 0] & [-2, 0, 4, -3, 1, 1] & [3, 2, 0, 4, 1, 1] \\
[3, 3, 0, 0, 6, 0] & [-1, -1, 1, 1, -3, 4] & [2, 2, 1, 1, 5, 0] \\
[3, 4, 0, 0, 5, 0] & [-1, -1, 1, 1, -3, 4] & [2, 3, 1, 1, 4, 0] \\
[4, 0, 0, 0, 7, 0] & [-2, 1, 1, 2, -5, 4] & [3, 1, 1, 0, 6, 0] \\
[4, 2, 0, 0, 6, 0] & [-1, -1, 1, 1, -3, 4] & [3, 1, 1, 1, 5, 0] \\
[4, 4, 0, 0, 5, 0] & [-1, -1, 1, 1, -3, 4] & [3, 3, 1, 1, 4, 0] \\
[4, 5, 0, 0, 4, 0] & [-1, -1, 1, 1, -3, 4] & [3, 4, 1, 1, 3, 0] \\
[5, 0, 0, 0, 7, 0] & [-2, 1, 1, 2, -5, 4] & [4, 1, 1, 0, 6, 0] \\
[8, 0, 0, 0, 0, 8] & [-7, 1, 5, 1, -5] & [7, 0, 0, 1, 0, 7] \\
[8, 0, 0, 0, 0, 8] & [-5, 1, 1, 5, -7] & [7, 0, 0, 1, 0, 7] \\
[8, 0, 0, 0, 0, 8] & [-5, 5, 1, 1, -5] & [7, 0, 0, 1, 0, 7] \\
[10, 0, 0, 0, 0, 0] & [-8, 1, 1, 1, 1, 7] & [9, 0, 0, 0, 1, 0] \\
[11, 0, 0, 0, 0, 0] & [-9, 1, 1, 2, 2, 4] & [10, 0, 0, 0, 1, 0] \\
[11, 0, 0, 0, 0, 0] & [-9, 2, 2, 1, 1, 5] & [10, 0, 0, 0, 1, 0] \\
\end{array}\]
\[
\begin{array}{ccc}
\lambda & \nu & \lambda' \\
[11,0,0,0,0] & [-8,1,1,1,2,5] & [10,0,0,0,1,0] \\
[11,0,0,0,0] & [-8,1,1,1,1,7] & [10,0,0,0,1,0] \\
[12,0,0,0,0,0] & [-9,1,1,2,2,4] & [11,0,0,0,1,0] \\
[12,0,0,0,0,0] & [-9,2,2,1,1,5] & [11,0,0,0,1,0] \\
[12,0,0,0,0,0] & [-9,1,1,1,5,1] & [11,0,0,0,1,0] \\
[12,0,0,0,0,0] & [-8,1,1,1,2,5] & [11,0,0,0,1,0] \\
[12,0,0,0,0,0] & [-8,1,1,1,1,7] & [11,0,0,0,1,0] \\
\end{array}
\]

Among all the 114 undetermined representations above, it now suffices to look more carefully at the following two:

\[\overline{X}([0,0,0,9,0],[1,1,1,1,-5,4]), \overline{X}([0,0,9,0,0,0],[4,1,-5,1,1,1]).\]

Due to the symmetry of the root system of \( E_6 \), it suffices to consider the first one. As one can calculate, \( \mu = [1,1,1,1,4,4] \) is the unique non-unitarily small \( K \)-type living on \((\lambda + \Lambda^+) \cap \Lambda^+\) and being conjugate to the infinitesimal character. Thus by Lemma 3.1.1, if it occurs in \( \overline{X}(\lambda, \nu) \), it should lie in its bottom layer. Remark 3.1.2 allows us to trace the bottom layer \( K \)-types of \( \overline{X}(\lambda, \nu) \) from the Levi level. By Proposition 6.5.1, we have

\[\overline{X}(\lambda, \nu) \simeq R_q^S(\overline{X}_L(\lambda - \rho(u), \nu)).\]

Thus, up to a center character, we meet the following representations on the two simple factors of \( L \):

\[\overline{X}_{A_4}([0,0,0,0],[1,1,1,1]) \text{ and } \overline{X}_{A_1}([0],[4]) = J_{A_1}([2],[2]).\]

By Remark 6.1.2, the \( K \)-types of \( J_{A_1}([2],[2]) \) are given by the decomposition

\[V(\varpi_6) \otimes V(\varpi_6)^c = V(0) \oplus V(2\varpi_6) = V(0) \oplus V(\alpha_6).\] (8.1)

Thus we see that \( \mu \) does not occur in \( \overline{X}(\lambda, \nu) \) since

\[\mu - \lambda = 2\alpha_1 + 2\alpha_2 + 3\alpha_3 + 3\alpha_4 + 2\alpha_6.\]
8.2 $\lambda_L + s\lambda_L$ is not dominant

There are 48 undetermined representations such that $\lambda_L + s\lambda_L$ is not dominant. We will refer to the parameters $(\lambda_L, -s\lambda_L)$ as $(\lambda := \{\lambda_L + s\lambda_L\}, 2\lambda_L - \lambda)$.

For 34 of them, we can test the non-unitarity in the same fashion as §7.4.

<table>
<thead>
<tr>
<th>$\lambda$</th>
<th>$2\lambda_L - \lambda$</th>
<th>$\lambda'$</th>
</tr>
</thead>
<tbody>
<tr>
<td>[0, 0, 1, 0, 8, 0]</td>
<td>[3, 1, 0, 1, −5, 4]</td>
<td>[0, 1, 0, 1, 7, 0]</td>
</tr>
<tr>
<td>[0, 0, 2, 0, 7, 0]</td>
<td>[1, 2, −1, 1, −4, 4]</td>
<td>[0, 1, 1, 1, 6, 0]</td>
</tr>
<tr>
<td>[0, 0, 7, 0, 1, 2]</td>
<td>[4, 2, −4, 1, 0, −1]</td>
<td>[0, 1, 6, 1, 0, 2]</td>
</tr>
<tr>
<td>[0, 0, 7, 0, 1, 3]</td>
<td>[4, 2, −4, 1, 0, −1]</td>
<td>[0, 1, 6, 1, 0, 3]</td>
</tr>
<tr>
<td>[0, 0, 7, 0, 2, 0]</td>
<td>[4, 2, −4, 1, −1, 1]</td>
<td>[0, 1, 6, 1, 1, 0]</td>
</tr>
<tr>
<td>[0, 0, 8, 0, 0, 2]</td>
<td>[4, 2, −6, 2, 1, −1]</td>
<td>[0, 1, 7, 0, 1, 1]</td>
</tr>
<tr>
<td>[0, 0, 8, 0, 1, 0]</td>
<td>[4, 1, −5, 1, 0, 3]</td>
<td>[0, 1, 7, 1, 0, 0]</td>
</tr>
<tr>
<td>[0, 1, 0, 0, 8, 0]</td>
<td>[1, 0, 3, 1, −6, 4]</td>
<td>[1, 0, 0, 1, 7, 0]</td>
</tr>
<tr>
<td>[0, 1, 6, 0, 0, 5]</td>
<td>[4, 0, −4, 1, 2, −2]</td>
<td>[0, 0, 5, 1, 1, 4]</td>
</tr>
<tr>
<td>[0, 1, 7, 0, 0, 1]</td>
<td>[6, 0, −5, 1, 1, 0]</td>
<td>[0, 0, 6, 1, 1, 0]</td>
</tr>
<tr>
<td>[0, 1, 7, 0, 0, 3]</td>
<td>[4, 0, −4, 1, 2, −2]</td>
<td>[0, 0, 6, 1, 1, 2]</td>
</tr>
<tr>
<td>[0, 1, 8, 0, 0, 0]</td>
<td>[4, 0, −6, 1, 3, 1]</td>
<td>[0, 0, 7, 1, 0, 1]</td>
</tr>
<tr>
<td>[0, 1, 8, 0, 0, 1]</td>
<td>[4, 1, −5, 1, 1, 1]</td>
<td>[0, 0, 7, 1, 1, 0]</td>
</tr>
<tr>
<td>[0, 1, 8, 0, 0, 1]</td>
<td>[6, 0, −5, 1, 1, 0]</td>
<td>[0, 0, 7, 1, 1, 0]</td>
</tr>
<tr>
<td>[0, 3, 0, 1, 6, 0]</td>
<td>[2, −1, 1, 0, −3, 4]</td>
<td>[1, 3, 1, 0, 6, 0]</td>
</tr>
<tr>
<td>[0, 3, 6, 1, 0, 0]</td>
<td>[4, −1, −3, 0, 1, 2]</td>
<td>[0, 3, 6, 0, 1, 1]</td>
</tr>
<tr>
<td>[0, 4, 0, 1, 5, 0]</td>
<td>[2, −1, 1, 0, −3, 4]</td>
<td>[1, 4, 1, 0, 5, 0]</td>
</tr>
<tr>
<td>[0, 4, 5, 1, 0, 0]</td>
<td>[4, −1, −3, 0, 1, 2]</td>
<td>[0, 4, 5, 0, 1, 1]</td>
</tr>
<tr>
<td>[0, 4, 6, 0, 0, 1]</td>
<td>[4, −2, −4, 2, 1, 0]</td>
<td>[0, 3, 5, 1, 1, 0]</td>
</tr>
<tr>
<td>[0, 10, 0, 0, 0, 2]</td>
<td>[4, −6, 1, 1, 1, 1]</td>
<td>[1, 9, 0, 0, 1, 1]</td>
</tr>
</tbody>
</table>
Thus 6.5.1) such that

For the remaining 12 undetermined representations, there is only one non-unitarily small $K$-type $\mu$ living on $(\lambda + \Lambda^+_c) \cap \Lambda^+$ such that

$$\{\mu - \rho_c\} + \rho_c = 2\lambda_L.$$
But one can calculate that $\mu - \lambda$ involves certain simples root $\alpha_i$ satisfying $\langle \lambda, \alpha_i \rangle > 0$. Therefore, by (6.8) of Proposition 6.5.1, $\mu$ does not occur in $X(\lambda, \nu)$.

<table>
<thead>
<tr>
<th>$\lambda$</th>
<th>$2\lambda_L - \lambda$</th>
<th>$\mu - \lambda$</th>
</tr>
</thead>
<tbody>
<tr>
<td>[0, 0, 6, 1, 0, 4]</td>
<td>[4, 1, -3, 0, 1, -1]</td>
<td>$2\alpha_1 + \alpha_2 + \alpha_4 + \alpha_5$</td>
</tr>
<tr>
<td>[0, 0, 7, 1, 0, 2]</td>
<td>[4, 1, -3, 0, 1, -1]</td>
<td>$2\alpha_1 + \alpha_2 + \alpha_4 + \alpha_5$</td>
</tr>
<tr>
<td>[0, 1, 8, 0, 0, 1]</td>
<td>[4, 0, -4, 1, 1, 0]</td>
<td>$2\alpha_1 + \alpha_2 + 2\alpha_4 + 2\alpha_5 + \alpha_6$</td>
</tr>
<tr>
<td>[0, 2, 1, 0, 7, 0]</td>
<td>[1, -1, 0, 1, -3, 4]</td>
<td>$\alpha_1 + \alpha_3 + \alpha_4 + 2\alpha_6$</td>
</tr>
<tr>
<td>[0, 2, 7, 0, 1, 0]</td>
<td>[4, -1, -3, 1, 0, 1]</td>
<td>$2\alpha_1 + \alpha_4 + \alpha_5 + \alpha_6$</td>
</tr>
<tr>
<td>[0, 3, 1, 0, 6, 0]</td>
<td>[1, -1, 0, 1, -3, 4]</td>
<td>$\alpha_1 + \alpha_3 + \alpha_4 + 2\alpha_6$</td>
</tr>
<tr>
<td>[0, 3, 6, 0, 1, 0]</td>
<td>[4, -1, -3, 1, 0, 1]</td>
<td>$2\alpha_1 + \alpha_4 + \alpha_5 + \alpha_6$</td>
</tr>
<tr>
<td>[0, 5, 1, 0, 5, 0]</td>
<td>[1, -1, 0, 1, -3, 4]</td>
<td>$\alpha_1 + \alpha_3 + \alpha_4 + 2\alpha_6$</td>
</tr>
<tr>
<td>[0, 5, 5, 0, 1, 0]</td>
<td>[4, -1, -3, 1, 0, 1]</td>
<td>$2\alpha_1 + \alpha_4 + \alpha_5 + \alpha_6$</td>
</tr>
<tr>
<td>[1, 1, 0, 0, 8, 0]</td>
<td>[0, 0, 1, 1, -4, 4]</td>
<td>$\alpha_1 + \alpha_2 + 2\alpha_3 + 2\alpha_4 + 2\alpha_6$</td>
</tr>
<tr>
<td>[2, 0, 0, 1, 7, 0]</td>
<td>[-1, 1, 1, 0, -3, 4]</td>
<td>$\alpha_2 + \alpha_3 + \alpha_4 + 2\alpha_6$</td>
</tr>
<tr>
<td>[4, 0, 0, 1, 6, 0]</td>
<td>[-1, 1, 1, 0, -3, 4]</td>
<td>$\alpha_2 + \alpha_3 + \alpha_4 + 2\alpha_6$</td>
</tr>
</tbody>
</table>
Chapter 9

Proof of Theorem III

This Chapter is devoted to prove Theorem III. We do it by two steps.

9.1 Proof from $G$-level to $L$-level

Any $\tilde{K}$-type of $H_D(L_S(Z))$ (if there exists) must be $\gamma_G = \lambda + \rho(u) - \rho_c$ (cf. (6.6)). Assume that the $K$-type $\mu_G$ of $L_S(Z)$ gives such a contribution. Then Lemma 3.1.1 says that $\mu_G$ must be a bottom layer $K$-type. That is, we can find a $K_L$-type $\mu_L$ of $Z$ such that

$$\mu_G = \mu_L + 2\rho(u \cap p) = \mu_L + \rho(u).$$

Moreover, the proof of Lemma 3.1.1 says that

$$w^{-1}(\mu_G - \rho_c) = \gamma_G = \mu_G - \rho_c + \langle \Phi \rangle,$$

where $\Phi \subseteq \Delta^+(I \cap p, t^c)$ and $w \in W(t, t^c)$. That is,

$$\gamma_G - w \gamma_G = \langle \Phi \rangle.$$
By (1.8), $\gamma_G$ is dominant integral for $\Delta^+(\mathfrak{t}, \mathfrak{t}^c)$. Then Lemma 9.1.1 says that there exists $w_1 \in W(\mathfrak{l} \cap \mathfrak{t}, \mathfrak{t}^c)$ such that $w_1 \gamma_G = w \gamma_G$. Therefore,

$$\gamma_G - w_1 \gamma_G = \langle \Phi \rangle.$$ 

Note that $\rho(u \cap p)$ is fixed by $w_1$. So we have

$$(\gamma_G - \rho(u \cap p)) - w_1 (\gamma_G - \rho(u \cap p)) = \langle \Phi \rangle. \quad (9.1)$$

Note also that

$$\gamma_G - \rho(u \cap p) = \mu_G - \rho_c + \langle \Phi \rangle - \rho(u \cap p)$$

$$= \mu_L + \rho(u \cap p) - \rho_c + \langle \Phi \rangle$$

$$= \mu_L + \rho(u \cap \mathfrak{t}) - \rho_c + \langle \Phi \rangle$$

$$= \mu_L - \rho^L_c + \langle \Phi \rangle := \gamma_L$$

is dominant for $\Delta^+(\mathfrak{l} \cap \mathfrak{t}, \mathfrak{t}^c)$, since $\rho(u \cap p)$ is perpendicular to every root in $\Delta^+(\mathfrak{l} \cap \mathfrak{t}, \mathfrak{t}^c)$. Now (9.1) reads as

$$w_1^{-1} (\mu_L - \rho^L_c) = \gamma_L = \mu_L - \rho^L_c + \langle \Phi \rangle.$$

Therefore, $F_{\gamma_L}$ is the PRV component of $F_{\mu_L} \otimes F_{\rho_c}$. Moreover, we have

$$\|\mu_G\|_{\text{spin}} = \|\gamma_G + \rho_c\| = \|\lambda + \rho(u)\|.$$ 

Therefore, by Lemma 3.1.3, we have

$$\|\mu_L\|_{\text{spin}} = \|\gamma_L + \rho^L_c\| = \|\lambda\|.$$ 

Hence the $K_L$-type $\mu_L$ contributes $\gamma_L$ to $H_D(Z)$. Finally, as noted in Remark 3.1.2, the multiplicity of $\mu_L$ in $Z$ equals to that of $\mu_G$ in $L_S(Z)$.

**Lemma 9.1.1.** Let $\Lambda \in (\mathfrak{t}^c)^*$ be dominant for $\Delta^+(\mathfrak{t}, \mathfrak{t}^c)$. Let $w \in W(\mathfrak{t}, \mathfrak{t}^c)$ be such that $\Lambda - w \Lambda$ is a linear combination of roots in $\Delta^+(\mathfrak{l} \cap \mathfrak{p}, \mathfrak{t}^c) = \Delta^+(\mathfrak{l} \cap \mathfrak{t}, \mathfrak{t}^c)$. Then there exists $w_1 \in W(\mathfrak{l} \cap \mathfrak{t}, \mathfrak{t}^c)$ such that $w_1 \Lambda = w \Lambda$. 

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Proof. Let us denote by $\Pi(\mathfrak{k}, \mathfrak{t}^c)$ the simple roots for $\Delta^+(\mathfrak{k}, \mathfrak{t}^c)$. Let $w = s_r \cdots s_1$ be a reduced expression of simple reflections corresponding to simple roots $\alpha_i \in \Pi(\mathfrak{k}, \mathfrak{t}^c)$. Then,

$$\Lambda - w\Lambda = \sum_{i=1}^{r} (s_{i-1} \cdots s_1 \Lambda - s_i \cdots s_1 \Lambda) = 2 \sum_{i=1}^{r} \frac{\langle s_{i-1} \cdots s_1 \Lambda, \alpha_i \rangle}{\langle \alpha_i, \alpha_i \rangle} \alpha_i = 2 \sum_{i=1}^{r} \frac{\langle \Lambda, s_1 \cdots s_{i-1}(\alpha_i) \rangle}{\langle \alpha_i, \alpha_i \rangle} \alpha_i.$$

Here, $s_1 \cdots s_{i-1}(\alpha_i) \in \Delta^+(\mathfrak{k}, \mathfrak{t}^c)$, so each coefficient is a non-negative real number. Up to now, this is the content of [HKP, Lemma 2.3].

To arrive at the desired conclusion, let us do induction on $r$. There are two cases:

Case 1. If $\alpha_r \in \Delta^+(\mathfrak{k} \cap \mathfrak{t}, \mathfrak{t}^c)$, then

$$\Lambda - s_{r-1} \cdots s_1 \Lambda = (\Lambda - w\Lambda) - 2 \frac{\langle \Lambda, s_1 \cdots s_{r-1}(\alpha_r) \rangle}{\langle \alpha_r, \alpha_r \rangle} \alpha_r,$$

which is again a linear combination of roots in $\Delta^+(\mathfrak{k} \cap \mathfrak{t}, \mathfrak{t}^c)$. Applying the induction hypothesis, we have

$$s_{r-1} \cdots s_1 \Lambda = w'_1 \Lambda$$

for some $w'_1 \in W(\mathfrak{k} \cap \mathfrak{t}, \mathfrak{t}^c)$. Thus we can take $w = s_rw'_1$.

Case 2. If $\alpha_r \notin \Delta^+(\mathfrak{k} \cap \mathfrak{t}, \mathfrak{t}^c)$, then we must have

$$\langle \Lambda, s_1 \cdots s_{r-1}(\alpha_r) \rangle = 0,$$

since $\Lambda - w\Lambda$ is a linear combination of roots in $\Delta^+(\mathfrak{k} \cap \mathfrak{t}, \mathfrak{t}^c)$. Therefore,

$$w\Lambda = s_{r-1} \cdots s_1 \Lambda,$$

and the desired conclusion follows from the induction hypothesis on $s_{r-1} \cdots s_1 \Lambda$. 

\qed
9.2 Proof from $L$-level to $G$-level

Any $\widetilde{K}_L$-type of $H_D(Z)$ (if there exists) must be $\gamma_L = \lambda - \rho_c^L$ (cf. (6.6)). Assume that the $K_L$-type $\mu_L$ of $Z$ gives such a contribution. Then

$$w^{-1}(\mu_L - \rho_c^L) = \gamma_L = \mu_L - \rho_c^L + \langle \Phi \rangle,$$

where $\Phi \subseteq \Delta^+(l \cap p, t^c)$ and $w \in W(l \cap \mathfrak{k}, t^c)$. Adding $\rho(u \cap p)$ to both sides, we have

$$w^{-1}(\mu_G - \rho_c) = \mu_G - \rho_c + \langle \Phi \rangle := \gamma_G.$$ (9.2)

Here $\mu_G := \mu_L + 2\rho(u \cap p) = \mu_L + \rho(u)$. Note that by (1.8),

$$\gamma_G = \gamma_L + \rho(u \cap p) = \lambda + \rho(u) - \rho_c$$

is dominant integral for $\Delta^+(\mathfrak{k}, t^c)$. Thus, by Lemma 9.2.1, $\mu_G$ is dominant for $\Delta^+(\mathfrak{k}, t^c)$. Therefore, Remark 3.1.2 says that $\mu_G$ is the bottom layer $K$-type associated to $\mu_L$, and the multiplicity of $\mu_G$ in $L_S(Z)$ equals to that of $\mu_L$ in $Z$. Note that $E_{\gamma_G}$ is the PRV component of $E_{\mu_G} \otimes E_{\rho_c}$ by (9.2). Moreover, we have

$$\|\mu_L\|_{\text{spin}} = \|\gamma_L + \rho_c^L\| = \|\lambda\|.$$ 

Now, by Lemma 3.1.3, we have

$$\|\mu_G\|_{\text{spin}} = \|\gamma_G + \rho_c\| = \|\lambda + \rho(u)\|.$$ 

Hence the $\widetilde{K}$-type $\mu_G$ contributes $\gamma_G$ to $H_D(L_S(Z))$.

**Lemma 9.2.1.** Let $\gamma_L = \mu_L - \rho_c^L + \langle \Phi \rangle$, where $\Phi \subseteq \Delta^+(l \cap p, t^c) = \Delta^+(l \cap \mathfrak{k}, t^c)$. Suppose that $\gamma_L + \rho(u \cap p)$ is dominant for $\Delta^+(\mathfrak{k}, t^c)$. Then $\mu_L + 2\rho(u \cap p)$ is dominant for $\Delta^+(\mathfrak{k}, t^c)$.

**Proof.** Denote by $\Pi(\mathfrak{k}, t^c)$ the simple roots for $\Delta^+(\mathfrak{k}, t^c)$. If $\mu_L + 2\rho(u \cap p)$ is not dominant for $\Delta^+(\mathfrak{k}, t^c)$, we can find $\alpha \in \Pi(\mathfrak{k}, t^c) \setminus \Delta^+(l \cap \mathfrak{k}, t^c)$ such that

$$\langle \mu_L + 2\rho(u \cap p), \hat{\alpha} \rangle < 0.$$
On the other hand, we have

$$\langle \mu_L - \rho^L_c + \langle \Phi \rangle + \rho(u \cap p), \tilde{\alpha} \rangle \geq 0.$$  

Taking the difference of the two equations above, we have

$$\langle \langle \Phi \rangle - \rho^L_c - \rho(u \cap p), \tilde{\alpha} \rangle > 0.$$  

That is,

$$\langle \langle \Phi \rangle, \tilde{\alpha} \rangle > \langle \rho^L_c + \rho(u \cap p), \tilde{\alpha} \rangle = \langle \rho^L_c + \rho(u \cap t), \tilde{\alpha} \rangle = \langle \rho_c, \tilde{\alpha} \rangle = 1,$$

which is impossible since the LHS is \( \leq 0 \). \( \square \)
Bibliography


