Market model for portfolio credit derivatives

by

Hu Zhiwei

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Market model for portfolio credit derivatives

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Abstract

This thesis develops a market model with jump-diffusion dynamics for pricing portfolio credit derivatives. The state variables of the market model are mean loss rates, and the model can be calibrated to credit default swap rates quoted by market. The market model could be extended to incorporate either a Cox process where the default time is thought of the first jump time or the jump-diffusion process introduced by [25]. With our arbitrage-free market model, the Black’s formula for CDS option can be justified. By combining the market model with copula models of default times, we can price portfolio credit derivatives like CDOs, using Monte Carlo simulation. The implied base correlations are computed for different CDO equity tranches. In addition, some techniques for initial data processing and the possibility of adopting $t$-copula as a potential stochastic copula model have also been discussed.
Chapter 1

Introduction

The importance of the credit derivatives market has increased remarkably during recent years. Investors of credit markets have witnessed a rapid growth of liquidity in credit default swaps (CDS), options on the credit default swaps (CDS options), tranches of collateralized debt obligation (CDO), and even some exotic portfolio credit derivatives like bespoke single-tranche CDOs and CDOs of CDOs (so-called CDO$^2$).

For the pricing of single-name CDS options, a standard market model is commonly intended to be a model enabling to define the implied volatility of a market option, positing a lognormal dynamics of the underlying under an equivalent pricing measure. [27], [2], [20], and [17], among others, have led to the adoption of Black’s formula as the market standard.

The modeling of dependent defaults is essential for the pricing of CDO tranches. The survival-time copula models represented by [22], [13], [1] and [10] is well received in market application. The Gaussian copula model formulated by [13] and later improved by [1] has also become the market standard.

However, there are some major disadvantages remaining. The absence of a unified pricing framework for CDS and CDO markets attributes to the detachment of the single-name credit derivatives market from the portfolio credit derivatives market. Moreover, the Gaussian copula model, which is used to describe default time dependency, has no capability to utilize either spread dynamics or spread correlations observed in the CDS markets for CDO pricing.

To overcome these disadvantages, Ho and Wu introduced a unified framework for
the arbitrage pricing of CDS options and CDOs in [16]. They developed a market model with forward credit spreads, which, differing from structural models with firm values, takes observable quantities as the state variables. Such a model can also be applied to price CDO tranches, using Monte Carlo simulations. The new model has the capability to utilize the dynamically evolving CDS rates, CDS rate correlations and implied CDS option volatilities as well as to price default-time correlations.

There still are several drawbacks in the market model in [16]. Firstly, it does not explicitly include the term of jump to default. This is not mathematically precise. Secondly, the CDS rates is not always changing continuously. The jumps of CDS rates can be easily observed in the market. In this thesis, We develop an extend market model that includes both the jump to default term and jumps of CDS rates. The adoption of Merton’s formula can also be justified under this extended market model, follow which similar results of CDS option price could be derived.

Including the information of default probabilities and CDS rate correlations, the market model in [16] could be applied to price default time correlations for CDO tranches, using Monte Carlo simulation. In their paper, the implied compound correlations have been backed out as an important input for pricing a portfolio credit risk. However, for some mezzanine tranches, the implied compound correlation is not well defined. In addition, the compound correlation smile causes difficulty for interpolation. Based on these considerations, We derive the base correlation as an alternative market-implied correlation measure in my thesis, removing some of the drawbacks.

The rest of the thesis is structured as follow. Chapter 2 presents the foundations of the unified framework, demonstrates the extended market model with mean loss rates and develops the method for CDS option pricing. In Chapter 3, We illustrate how the market model and its dynamics can be utilized to price CDO tranches and how to derive the implied base correlation. Details of initial data processing and results of my work are also presented in this chapter. The paper is concluded in Chapter 4. The proofs of some theorems are provided in the Appendix.
Chapter 2

Market Model For Single Name Credit Derivatives

2.1 Risky Bonds and Forward Spreads

In this thesis, all models, without indication otherwise, are in the filtered probability space, \((\Omega, (\mathcal{F}_t)_{t \geq 0}, \mathbb{Q})\), where \(\mathbb{Q}\) is the risk-neutral measure. All stochastic processes are adapted to \((\mathcal{F}_t)_{t \geq 0}\). The default time is modeled as the first jump time of a Cox process with an intensity process, \(\tilde{\lambda}(t)\). Although it may be against some empirical findings in [8], we assume independence between credit spreads and US Treasury yields and independence between hazard rate and recovery rate of the credit.

A defaultable coupon bond pays regular coupons until a default occurs or the maturity arrives. When a default happens, the bond holder will receive a payment that consists of a fraction of both principal and accrued interest. Without loss of generality, we make two assumptions, one is the final payment is made at the next coupon date following default and the other is the last coupon accrues until the final payment date. Let \(c\) be the coupon rate of a risky bond with tenor\(^1\) \([T_m, T_n]\), \(\tau\) be the default time and \(R_\tau\) be the recovery rate at the default time. Defined in [16], the bond is priced as the expectation of discounted cash

\(^1\)The coupon dates are \(\{T_j\}_{j=m+1}^n\). The bond is said to be “forward starting” if \(m > 0\).
flow under risk neutral measure:

\[
B_c(t) = \sum_{j=m}^{n-1} E_t^Q \left[ \frac{B(t)}{B(T_{j+1})} \left\{ \Delta T_j c 1_{\{\tau > T_{j+1}\}} + R_\tau (1 + \Delta T_j c) 1_{\{T_j < \tau \leq T_{j+1}\}} \right\} \right] \\
+ E_t^Q \left[ \frac{B(t)}{B(T_n)} 1_{\{\tau > T_n\}} \right] \\
= \sum_{j=m}^{n-1} Z_{j+1}(t) \Delta T_j c E_t^{Q_{j+1}} \left[ 1_{\{\tau > T_{j+1}\}} + R_\tau 1_{\{T_j < \tau \leq T_{j+1}\}} \right] \\
+ \sum_{j=m}^{n-1} Z_{j+1}(t) E_t^{Q_{j+1}} \left[ R_\tau 1_{\{T_j < \tau \leq T_{j+1}\}} \right] + Z_n(t) E_t^{Q_n} \left[ 1_{\{\tau > T_n\}} \right], \quad (2.1)
\]

where \( B(t) = e^{\int_0^t r_s ds} \) is the money market account, \( Z_j(t) \) is the price of risk-free zero-coupon bond with notional value $1 and maturity \( T_j \), and \( Q_j \) is the \( T_j \) forward measure.

Also in [16], the so called C-strip zero-coupon bonds are defined, and they are backed by coupons of risky bonds:

\[
\tilde{Z}_j(t) = E_t^Q \left[ \frac{B(t)}{B(T_j)} \left\{ 1_{\{\tau > T_j\}} + R_\tau 1_{\{T_{j-1} < \tau \leq T_j\}} \right\} \right] \\
= Z_j(t) E_t^{Q_j} \left[ 1_{\{\tau > T_j\}} + R_\tau 1_{\{T_{j-1} < \tau \leq T_j\}} \right] \\
\triangleq Z_j(t) D_j(t). \quad (2.2)
\]

This C-strip zero-coupon bonds play an important role in the construction of our market model. However, the information of risky zero-coupon bonds is not available as they are not traded in real market. Nonetheless, using coupon bond prices together with CDS rates, the prices of risky zero-coupon bonds can be backed out.

Before introducing the definition of forward spread, we need to introduce the “risky forward rates” defined in [26] and [6]. A risky forward rate is defined as the fair rate on a defaultable loan for a future period of time \( (T_j, T_{j+1}] \). According to arbitrage pricing theory [14], the risky forward rate, denoted as \( \hat{f}_j(t) \) make the present value of case flows of the risky loan equal to zero:

\[
0 = E_t^Q \left[ \frac{B(t)}{B(T_j)} 1_{\{\tau > T_j\}} \right] - E_t^Q \left[ \frac{B(t)}{B(T_{j+1})} (1 + \Delta T_j \hat{f}_j(t)) 1_{\{\tau > T_{j+1}\}} + R_\tau 1_{\{T_j < \tau \leq T_{j+1}\}} \right] \\
= Z_j(t) E_t^{Q_j} \left[ 1_{\{\tau > T_j\}} \right] - Z_{j+1}(t) (1 + \Delta T_j \hat{f}_j(t)) D_{j+1}(t) \\
\triangleq Z_j(t) \Lambda_j(t) - Z_{j+1}(t) (1 + \Delta T_j \hat{f}_j(t)) D_{j+1}(t), \quad (2.3)
\]
where \( \Lambda_j(t) \) is the \( Q_j \) probability of survival until \( T_j \). It is easy for us to obtain:

\[
\hat{f}_j(t) = \frac{1}{\Delta T_j} \left[ Z_j(t) \frac{\Lambda_j(t)}{Z_{j+1}(t) \frac{D_{j+1}(t)}{D_j(t)}} - 1 \right]
\]

\[
= \frac{1}{\Delta T_j} \left[ \frac{Z_j(t)}{Z_{j+1}(t)} \frac{\Lambda_j(t)}{D_j(t)} - 1 \right]
\]

Comparing this result with default-free forward rate for the period \((T_j, T_{j+1}]\) defined by

\[ f_j(t) = \frac{1}{\Delta T_j} \left( \frac{Z_j(t)}{Z_{j+1}(t)} - 1 \right), \quad t \leq T_j, \]

we find out that \( f_j(t) \) is the lower bound of \( \hat{f}_j(t) \) and they equal to each other when the recovery rate \( R_r = 1 \).

Now, the two building blocks of the market model could be developed intuitively. The “forward spread” is defined as the difference between a risky forward rate and its corresponding risk-free forward rate, that is:

\[
S_j(t) = \hat{f}_j(t) - f_j(t)
\]

\[
= (1 + \Delta T_j \hat{f}_j(t)) \frac{E^{Q_{j+1}}_t \left[ (1 - R_r)1_{\{T_j < \tau \leq T_{j+1}\}} \right]}{\Delta T_j D_{j+1}(t)}
\]

\[
\equiv (1 + \Delta T_j f_j(t)) H_j(t), \quad j = 1, 2, \ldots \quad (2.5)
\]

where \( H_j(t) \) can be explained as the discrete-tenor version of “mean loss rate” that was introduced in [9]. Above equation can be transformed as

\[
1 + \Delta T_j \hat{f}_j(t) = (1 + \Delta T_j f_j(t))(1 + \Delta T_j H_j(t)), \quad j = 1, 2, \ldots
\]

Forward spread \( S_j(t) \) and mean loss rate \( H_j(t) \) are two important notions in our market model. Equation (2.5) demonstrate the relation between them. Hence, once given the term structure of \( H_j(t) \), the corresponding term structure of \( S_j(t) \) could be obtained without difficulties using Itô’s lemma. Consequently, for the rest of this paper, we only focus on the discussion of modeling with mean loss rate \( H_j(t) \). The parallel extension for forward spread \( S_j(t) \) could be found in [16].

\(^2\text{It is also the } Q \text{ probability of survival until } T_j \text{ due to the independence between US Treasury yields and the default probability.}\)
2.2 Credit Default Swap Rates

Credit default swaps (CDS) are the workhorses of the credit derivatives market. There are three parties involved in a CDS: the reference entity, the protection buyer and the protection seller. If the reference entity experiences a default event before the maturity date, $T$, the protection seller makes a default payment to the protection buyer, which cover the loss on a security issued by the reference entity; this part of a CDS is called the protection leg. For compensation, the protection buyer makes a periodic premium payment to the protection seller (the fee leg); after the default of the reference entity, premium payments stop.

In this section, we consider the swaps for fixed-rate bonds. Without loss of generality, we assume that the notional value of the bonds to be $1$, and the coupon rate to be $c$. The protection payment is $(1 - R_\tau)(1 + \Delta Tc)$. A swap rate, which denoted by $\bar{s}_{m,n}(t)$, should equalize the expected present value of protection leg and fee leg under risk neutral measure. The value of the fee leg is

$$PV_{fee} = \bar{s}_{m,n}(t) \sum_{j=m}^{n-1} \Delta T_j Z_{j+1}(t) E^Q_{t+1} \left[ I_{\{\tau > T_j+1\}} + R_\tau 1_{\{T_j < \tau \leq T_{j+1}\}} \right]$$

$$= \bar{s}_{m,n}(t) \sum_{j=m}^{n-1} \Delta T_j \bar{Z}_{j+1}(t). \quad (2.6)$$

Notice that this fee leg is a tradable annuity backed by risky zero-coupon bonds, analogous to the fixed leg of default-free swaps. Meanwhile, the present value of the protection leg can be written as

$$PV_{prot} = (1 + \Delta Tc) \sum_{j=m}^{n-1} Z_{j+1}(t) E^Q_{t+1} \left[ (1 - R_\tau) 1_{\{T_j < \tau \leq T_{j+1}\}} \right]. \quad (2.7)$$

Therefore, we obtain

$$\bar{s}_{m,n}(t) = (1 + \Delta Tc) \frac{\sum_{j=m}^{n-1} Z_{j+1}(t) E^Q_{t+1} \left[ (1 - R_\tau) 1_{\{T_j < \tau \leq T_{j+1}\}} \right]}{\sum_{j=m}^{n-1} \Delta T_j \bar{Z}_{j+1}(t)}$$

$$= (1 + \Delta Tc) \sum_{j=m}^{n-1} \bar{\omega}_j H_j(t), \quad (2.8)$$

where

$$\bar{\omega}_j(t) = \frac{\Delta T_j \bar{Z}_{j+1}(t)}{\sum_{j=m}^{n-1} \Delta T_j \bar{Z}_{j+1}(t)}.$$

6
In reality, the swaps for fixed-rate bonds with coupon rate $c = 0$ dominate the liquidity of single-name credit derivatives markets. Thereafter, the CDS rates are expressed as:

$$s_{m,n}(t) = \sum_{j=m}^{n-1} \bar{\omega}_j H_j(t).$$

One advantage of using the mean loss rate, compared with other CDS rate formula (e.g., [27], [6]), is that the CDS rate could be simply expressed in the form of weighted average of the mean loss rate, $H_j(t)$, which is analogous to the swap rate in LIBOR market model.

### 2.3 CDS Options

A CDS option is an option to buy or sell a particular credit default swap on a particular reference entity at a particular future time $T$. The cost of the option would be paid up-front. Like CDS forwards, CDS options are usually structured so that they will cease to exist if the reference entity defaults before option matures. Denoting the predetermined swap rate by $\bar{s}^*$, and spot CDS rate for a tenor $[T_m, T_n]$ at maturity $T$ by $s_{m,n}(T)$. Then, the gain or loss for a long position at maturity $T$ of the CDS option is

$$1_{\{\tau > T\}} (s_{m,n}(T) - \bar{s}^*)^+ \sum_{j=m}^{n-1} \Delta T_j \hat{Z}_j(T).$$

The standard market practice is to define the implied volatility of a market option, posit a lognormal dynamics of the underlying under an equivalent pricing measure. The first important contribution to the development of models of this kind of credit derivatives is given in [26]. The model leads to for Black Scholes formulas but differs from standard market models, since it is based on using probability measures that are not equivalent to the risk neutral probability measure. In [17] Black and Scholes formulas for CDS options are also tested on market data.

Parallel to [27], the default swap measure, $\mathbb{Q}^D$, is defined by [16] as

$$\frac{d\mathbb{Q}^D}{d\mathbb{Q}} \bigg|_{\mathcal{F}_t} = \frac{A^D_{m,n}(t)}{A^D_{m,n}(0)} \left/ \frac{B(t)}{B(0)} \right. \triangleq \bar{L}(t),$$
where

\[ A_{m,n}^D(t) = 1_{\{\tau > t\}} \sum_{j=m}^{n-1} \Delta T_j \tilde{Z}_{j+1}(t). \]

Under this new measure, we can express the price of the CDS options as

\[ V = E_t^Q \left[ \frac{B(t)}{B(T)} A_{m,n}^D(T) (\bar{s}_{m,n}(T) - \bar{s}^*)^+ \right] \]

\[ = A_{m,n}^D(t) E_t^Q \left[ \left( \frac{A_{m,n}(T)}{A_{m,n}(t)} \right) \left( \frac{B(T)}{B(t)} \right) (\bar{s}_{m,n}(T) - \bar{s}^*)^+ \right] \]

\[ = A_{m,n}^D(t) E_t^{Q_D} \left[ (\bar{s}_{m,n}(T) - \bar{s}^*)^+ \right]. \] (2.11)

The process \( \bar{L}(t) \) is a nonnegative \( Q \)-martingale with initial value one, but it is not strictly positive: \( \bar{L}(t) \) jumps to zero at default. This means that the measure \( Q^D \) attaches a weight of zero to all events that involve default before \( T \):

\[ P^D(\tau \leq T) = E_t^{Q_D} \left[ 1_{\{\tau < T\}} \right] = E_t^Q \left[ \bar{L}(T) 1_{\{\tau < T\}} \right] = 0. \] (2.12)

The default swap measure \( Q^D \) is not equivalent to \( Q \) any more, but it is absolutely continuous with respect to \( Q \), so Girsanov’s theorem can still be applied. We can interpret this default swap measure as a measure that is conditioned on survival until \( T \). Besides the conclusion that \( Q^D \)-default probability is zero, it is also easy to see that the CDS rate \( \bar{s}_{m,n}(t) \) is a martingale:

\[ E_t^{Q_D} [\bar{s}_{m,n}(T)] = \bar{s}_{m,n}(t), \quad \forall T \in (t, T_m]. \] (2.13)

Now, two cases of the CDS rate dynamics could be discussed. Firstly, assuming \( \bar{s}_{m,n}(t) \) is lognormal martingale under \( Q^D \):

\[ d\bar{s}_{m,n}(t) = \bar{s}_{m,n}(t) \sigma_{m,n} \cdot dW_t^D, \] (2.14)

where \( W_t^D \) is a Brownian motion under \( Q^D \) and \( \sigma_{m,n} \) is the swap-rate volatility.\(^3\)

The Black and Scholes formulas for CDS options can then be justified by this lognomality assumption:

\[ V = A_{m,n}^D(t) \left[ \bar{s}_{m,n}(t) N(d_1) - \bar{s}^* N(d_2) \right], \] (2.15)

with

\[ d_{1,2} = \frac{\ln(\bar{s}_{m,n}(t)/\bar{s}^*) \pm \frac{1}{2} \sigma_{m,n}^2 (T - t)}{\sigma_{m,n} \sqrt{T - t}}. \]

\(^3\)The Brownian motions and corresponding volatilities are always assumed vectors in my thesis without specification.
Secondly, assuming $\bar{s}_{m,n}(t)$ is a lognormal jump-diffusion process:

$$
\frac{d\bar{s}_{m,n}(t)}{\bar{s}_{m,n}(t)} = \bar{\sigma}_{m,n} \cdot dW_t^D - (e^{Y_{N(t)}} - 1)dN(t) - (k - 1)\lambda dt,
$$

where $N(t)$ is a Poisson process with intensity $\lambda$ (triggering the jumps in $\bar{s}_{m,n}$), $Y_n \sim N(\mu, \nu^2)$ are independent normally distributed random variables and $k = e^{\mu + \nu^2/2}$. At each jump of $N(t)$, a random number $Y_n$ is drawn and the new CDS rate jumps to $e^{Y_n}$ times the old CDS rate $\bar{s}_{m,n}(t) = e^{Y_n} \bar{s}_{m,n}(t-)$. The well known result of [25] gives us the option pricing formula:

$$
V(t) = \sum_{n=0}^{\infty} e^{\lambda(T-t)} \lambda^n(T-t)^n \frac{n!}{n!} C_{BS}(\bar{s}_{m,n}(t), T, \bar{s}^*, r_n, \sigma_{n}^2)
$$

where $C_{BS}()$ is the classical Black-Scholes formula, $r_n = -\lambda k + \frac{1}{T-t} n \ln(1 + k)$ and $\sigma_{n}^2 = \bar{\sigma}_{m,n}^2 + n\nu^2/(T-t)$.

### 2.4 The Extended Market Model

In [16], the market model with mean loss rates does not explicitly include the term of jump to default. This is not mathematically precise because the dynamics of any entity’s mean loss rate may not evolve forever. There should be a stop time. In addition, the CDS rate of a given entity is not always changing continuously. For example, the CDS rates of General Motors have jumped several times in the past five years. Therefore, a market model which has the capability to describe the non-continuous movements of CDS rate and to stop at some time is required.

In this section, we introduce the extended market model with mean loss rates based on the work of [16]. This model integrates two jump terms into the diffusion process of mean loss rates. One is a Cox process that describes the jump to default of a specific entity, the other is a lognormal jump process with time varying intensities that captures the jumps of CDS rates observed in the market. The model is established as follow:

$$
\frac{dH_j(t)}{H_j(t-)} = \mu_j^H(t) \cdot dt + \sigma_j^H(t) \cdot dW_t + d \left[ \sum_{i=1}^{N(t)} (Y_{ji} - 1) \right] + (\hat{Y}_j(t) - 1)d\hat{N}_t
$$

where $W_t$ a standard Brownian motion under risk neutral measure $Q$, $\mu_j^H(t), \sigma_j^H(t)$ deterministic, $N(t)$ is a Poisson process with intensity $\lambda(t)$, and $Y_{ji}$ is lognormal
distributed with parameters $m_j(t)$ and $v_j(t)$; and $\tilde{N}(t)$ is a Cox process with intensity $\tilde{\lambda}(\bar{s}_{m,n}(t))$ and $\tilde{Y}_j(t) \in (0, \infty]$. $W_t, N(t), Y_{ji}, \tilde{N}(t)$ and $\tilde{Y}_j(t)$ are independent.

If the intensity $\lambda(t)$ and $\tilde{\lambda}(\bar{s}_{m,n}(t))$ both equal to zero, this model is reduced to the market model in [16]:

$$
\frac{dH_j}{H_j} = \mu^H_j \, dt + \sigma^H_j \cdot dW_t,
$$

(2.16)

where $\sigma^H_j$ are constant vectors and $W_t$ is a multi-dimensional Brownian motion under $Q$. Under the assumption that the recovery rate is time stationary, $E^Q[R_\tau|T_{j-1} < \tau \leq T_j] = R = \text{constant}$, [16] derives that

$$
\mu^H_j(t) = \sigma^H_j(t) \sum_{k=\eta(t)}^j \Delta T_k H_k(t) \left( \sigma^H_k(t) - \left( \frac{\tilde{H}_k(t)}{H_k(t)} \right) \sigma^H_{k-1}(t) \right),
$$

(2.17)

where

$$
\tilde{H}_j(t) = \frac{1}{\Delta T_j} \left( \frac{D_j(t)}{D_{j+1}(t)} - 1 \right).
$$

If the intensity $\lambda(t)$ is zero but the intensity $\tilde{\lambda}(\bar{s}_{m,n}(t))$ is not, that would be a diffusion process coupled with a Cox process. A Cox process is a generalization of the Poisson process in which the intensity is allowed to be random. The default time can be thought of as the first jump time of a Cox process with intensity process $\lambda(\bar{s}_{m,n}(s))$. We define the default time $\tau$ as:

$$
\tau = \inf \left\{ t : \int_0^t \tilde{\lambda}(\bar{s}_{m,n}(s)) \, ds \geq E_1 \right\},
$$

where $E_1$ is a unit exponential random variable. From the definition, the survival probability under the risk neutral measure of an entity could be derived as:

$$
P(\tau > t|\{(\bar{s}_{m,n}(s))_{0 \leq s \leq t}\}) = \exp \left( - \int_0^t \tilde{\lambda}(\bar{s}_{m,n}(s)) \, ds \right) \quad t \in [0, T],
$$

$$
P(\tau > t) = E \left[ \exp \left( - \int_0^t \tilde{\lambda}(\bar{s}_{m,n}(s)) \, ds \right) \right] \quad t \in [0, T].
$$

What we have modeled above is the first jump of a Cox process. Now the model of mean loss rate $H_j(t)$ can be written as:

$$
\frac{dH_j(t)}{H_j(t-)} = \mu^H_j(t) \, dt + \sigma^H_j(t) \cdot dW_t + (Y_j(t) - 1) d\tilde{N}_t,
$$

(2.18)
where \( \tilde{N}_t \) is a Cox process and \( Y_j(t) \in (0, \infty] \). When the first jump of \( \tilde{N}_t \) happens at time \( t_0 \), the entity corresponding to this mean loss rate defaults immediately.

As demonstrated previously, the default probability of a company under \( Q^D \) is zero. As a result, there is no jump term appear in the term structure of \( H_j(t) \) after changing from risk neutral measure \( Q \) to default swap measure \( Q^D \). Using frozen coefficients technique, an approximate swap-rate process is derived by [16]:

\[
d\bar{s}_{m,n}(t) = \bar{s}_{m,n}(t)\bar{\sigma}_{m,n} \cdot dW^D_t ,
\]

(2.19)

where

\[
\bar{\sigma}_{m,n} \triangleq \sum_{j=m}^{n-1} \tilde{\alpha}_j \sigma_j^H , \quad \tilde{\alpha}_j \triangleq \frac{\partial \tilde{s}_{m,n}(0)}{\partial H_j} \frac{H_j(0)}{\bar{s}_{m,n}(0)} \approx \tilde{\omega}_j \frac{H_j(0)}{\bar{s}_{m,n}(0)} ,
\]

(2.20)

and \( W^D_t \) is a multi-dimensional Brownian motion under \( Q^D \), defined by:

\[
dW^D_t = dW_t - \sum_{j=m}^{n-1} \tilde{\omega}_j \tilde{\gamma}_{j+1}(t)dt ,
\]

(2.21)

and \( \tilde{\gamma}_{j+1} \) is the volatility of \( \tilde{Z}_{j+1}(t) \).

In the more general case, both the intensity \( \lambda(t) \) and the intensity \( \tilde{\lambda}(\tilde{s}_{m,n}(t)) \) are not zero, which is the market model introduced at the beginning of this section. The no arbitrage condition of this model under the risk neutral measure can also be derived.

To make the process of measure changing clear, we write the jump term that represents CDS rates jumps in the form of Marked Point Process (MPP), following the work in [12].

\[
\sum_{i=1}^{N(t)} (Y_{ji} - 1) = J_j(t) = \sum_{i=1}^{N(t)} K_j(X_i, \tau_i) ,
\]

(2.22)

where \( J_j(t) \) is a jump process constructed by a MPP, \( K_j(x, t) \). To make the equation hold, we have:

\[
K_j(x, t) = e^{\nu_j(t)} x^{\nu_j(t)} - 1 ,
\]

where \( x \) is standard lognormal distributed with density function \( f(., t) \) and the mark has the following arrival rate under risk neutral measure:

\[
\nu_Q(dx, t) = \lambda(t) f(x, t)dx .
\]
The conclusion is given by the following theorem:

**Theorem 2.1** The no arbitrage condition of the market model

\[
\frac{dH_j(t)}{H_j(t^-)} = \mu^H_j(t) dt + \sigma^H_j(t) \cdot dW_t + dJ_j(t) + (\tilde{Y}_j(t) - 1)d\tilde{N}_t
\]

is

\[
\mu^H_j(t) = \sigma^H_j(t) \sum_{k=\eta(t)}^{j} \frac{\Delta T_k H_k(t^-)}{1 + \Delta T_k H_k(t^-)} \left( \sigma^H_k(t) - \frac{H_k(t^-)}{H_k(t^-)} \sigma^H_{k-1}(t) \right) - \int_{0}^{+\infty} K_j(x, t) \prod_{k=\eta(t)}^{j} \frac{1 + \Delta T_k H_k(t^-)}{1 + \Delta T_k H_k(t^-)(1 + K_j(x, t))} \nu_Q(dx,t)
\]

(2.23)

The proof of this theorem is given in the appendix.

Next, the dynamics of the market model with mean loss rates under the default swap measure is derived.

**Theorem 2.2** The dynamics of the market model with mean loss rate under the default swap measure is

\[
\frac{dH_j(t)}{H_j(t^-)} = \tilde{\mu}^H_j(t) dt + \sigma^H_j(t) \cdot dW^D_t + dJ_j(t),
\]

where

\[
J_j(t) = \sum_{i=1}^{N(t)} K_j(X_i, \tau_i).
\]

(2.24)

The intensity of the marked point process is given by

\[
\nu_{Q^D}(dx,t) = \sum_{k=m}^{n-1} \tilde{\omega}_k(t) \prod_{j=\eta(t)}^{k} \frac{(1 - \frac{R}{1-R} \Delta T_{j-1} H_{j-1}(t^-)(1 + K_{j-1}(x, t)))}{(1 - \frac{R}{1-R} \Delta T_{j-1} H_{j-1}(t^-))} \cdot \frac{(1 + \Delta T_j H_j(t^-))}{(1 + \Delta T_j H_j(t^-)(1 + K_j(x, t)))} \lambda(t)f(x, t) dx.
\]

(2.25)

Recalling that the CDS rate is martingale under the default swap measure $Q^D$
and the CDS option payoff is transformed into a simple European payoff on the CDS rate. Therefore, a possible model of CDS rates could be as follow:

\[
\frac{d\hat{s}_{m,n}(t)}{\hat{s}_{m,n}(t-) = \hat{\sigma}_{m,n}(t) \cdot dW_t^D - \hat{\lambda}(t)E \left[ \hat{Y}_j - 1 \right] dt + d \left[ \sum_{j=1}^{\hat{N}(t)} (\hat{Y}_j - 1) \right], \tag{2.26}
\]

where \(W_t^D\) is a standard Brownian motion under default swap measure \(Q^D\), \(\sigma_{m,n}(t)\) is deterministic, \(\hat{N}(t)\) is a Poisson process with intensity \(\hat{\lambda}(t)\), and \(\hat{Y}_j\) is independent with lognormal density \(f(\cdot, t)\), which is parameterized through the mean \(\hat{m}(t)\) and standard deviation \(\hat{\upsilon}(t)\). This model is similar to the model introduced in [25]. The difference lies in the assumption of the coefficients: in Merton’s model, all the coefficients are fixed, yet in the above model of CDS rates, the coefficients are allowed to be time varying. Given the above dynamics of underlying, [12] gives the pricing formula of the European option:

\[
E^{Q^D}[(\hat{s}_{m,n}(T) - \hat{s}^*)^+] = \hat{s}_{m,n}(t)D_1 - \hat{s}^*D_2 \tag{2.27}
\]

\[
D_1 = \frac{1}{2} + \frac{1}{\pi} \int_{0}^{\infty} \frac{e^{F_1(s)} \sin(F_2(s) - s \ln(\hat{s}^*/\hat{s}_{m,n}(t)))}{s} ds
\]

\[
D_2 = \frac{1}{2} + \frac{1}{\pi} \int_{0}^{\infty} \frac{e^{F_3(s)} \sin(F_4(s) - s \ln(\hat{s}^*/\hat{s}_{m,n}(t)))}{s} ds \tag{2.28}
\]

and

\[
F_1(s) = \Delta T \sum_{i=1}^{n} \lambda_i [e^{\hat{m}_i + \hat{\upsilon}_i^2 (1-s^2)/2} \cos(\hat{w}_i s) - 1] - \lambda_i \hat{m}_i - (\hat{\sigma}_{m,n}^i)^2 s^2 / 2
\]

\[
F_2(s) = \Delta T \sum_{i=1}^{n} \lambda_i e^{\hat{m}_i + \hat{\upsilon}_i^2 (1-s^2)/2} \sin(\hat{w}_i s) + \hat{\alpha}_{m,n}^i s + (\hat{\sigma}_{m,n}^i)^2 s
\]

\[
F_3(s) = \Delta T \sum_{i=1}^{n} \lambda_i e^{-\hat{\upsilon}_i^2 s^2 / 2} \cos(\hat{m}_i s) - 1] - (\hat{\sigma}_{m,n}^i)^2 s^2 / 2
\]

\[
F_4(s) = \Delta T \sum_{i=1}^{n} \lambda_i e^{-\hat{\upsilon}_i^2 s^2 / 2} \sin(\hat{m}_i s) + \hat{\alpha}_{m,n}^i s \tag{2.29}
\]

where \(\hat{w}_i = \hat{m}_i + \hat{\upsilon}_i^2\)

By now, we have already derived the dynamics of mean loss rates under the default swap measure and quoted the pricing formula for European options given by [12], whose underlying is CDS rates. Next, I will establish the relation between CDS rates and mean loss rates under the default swap measure by approximating
the CDS rate as a linear combination of mean loss rates and using the frozen coefficients technique. The result is given by the following proposition.

**Proposition 2.3** Under the default swap measure, the coefficients of the CDS rates model in (2.27) can be derived from the mean loss rates model in Theorem 2.2 by the following equation:

\[
\hat{\sigma}_{m,n}(t) = \frac{1}{\hat{s}_{m,n}(0)} \left[ \sum_{j=m}^{n-1} \sum_{k=m}^{n-1} \bar{\omega}_j(0) \bar{\omega}_k(0) \sigma_j^H(t) \cdot \sigma_k^H(t) H_j(0) H_k(0) \right]^{1/2},
\]  

(2.30)

\[
\hat{\lambda}(t) = \int_{R_+} \sum_{k=m}^{n-1} \bar{\omega}_k(0) \prod_{j=\eta(t)}^{k} \left( 1 - \frac{\bar{R}}{1-R} \Delta T_{j-1} H_{j-1}(0)(1 + K_{j-1}(x,t)) \right) \left( 1 + \Delta T_j H_j(0) \right) \lambda(t)f(x,t) \, dx,
\]  

(2.31)

\[
\hat{m} = \log(1 + \hat{\mu}) - \hat{\sigma}^2/2, \quad \hat{\sigma} = \left[ \log\left( \frac{I_2/\hat{s}_{m,n}(0) + 1 + 2\hat{\mu}}{(1 + \hat{\mu})^2} \right) \right]^{1/2},
\]  

(2.32)

and

where \( \hat{\mu} \) and \( I_2 \) are defined in the proof.

**Proof:** By using the frozen coefficients technique, we have

\[
\hat{s}_{m,n}(t) \approx \sum_{j=m}^{n-1} \bar{\omega}_j(0) H_j(t),
\]  

(2.34)

Ito’s lemma gives us the diffusion term of the right side of the above equation:

\[
\sum_{j=m}^{n-1} \bar{\omega}_j(0) H_j(t) \sigma_j^H(t) \cdot dW^D_t,
\]

which has quadratic variation

\[
\sum_{j=m}^{n-1} \sum_{k=m}^{n-1} \bar{\omega}_j(0) \bar{\omega}_k(0) \sigma_j^H(t) \cdot \sigma_k^H(t) H_j(0) H_k(t) \, dt.
\]

This diffusion term in (2.24) has a quadratic variation \( \hat{s}_{m,n}(t)^2 \hat{\sigma}_{m,n}(t)^2 \, dt \). Fixing the mean loss rates at time zero and matching quadratic variations, we obtain:

\[
\hat{\sigma}_{m,n}(t) = \frac{1}{\hat{s}_{m,n}(0)} \left[ \sum_{j=m}^{n-1} \sum_{k=m}^{n-1} \bar{\omega}_j(0) \bar{\omega}_k(0) \sigma_j^H(t) \cdot \sigma_k^H(t) H_j(0) H_k(0) \right]^{1/2}.
\]  

(2.35)
Next we focus on jump contribution. Notice that while the mean loss rates that contribute to CDS rate can jump with different magnitudes, they all jump at the same times as they are driven by the same marked point process. As the result, the total arrival rate of the jumps of $\sum_{j=m}^{n-1} \tilde{\omega}_j(0) H_j(t)$ is the integral of $\nu_Q(dx, t)$ through $x \in \mathbb{R}^d$. As the result dependent on the path of $H_j$, so we fix the mean loss rates $H_j$ to obtain

$$\hat{\lambda}(t) = \int_{R^+} \sum_{k=m}^{n-1} \tilde{\omega}_k(0) \prod_{j=\eta(t)}^{k} \frac{(1 - \frac{R}{1-R} \Delta T_{j-1} H_{j-1}(0)(1 + K_{j-1}(x, t)))}{(1 - \frac{R}{1-R} \Delta T_{j-1} H_{j-1}(0))} \frac{(1 + \Delta T_j H_j(0))}{(1 + \Delta T_j H_j(0)(1 + K_j(x, t)))} \lambda(t) f(x, t) dx \ .$$

(2.36)

We can identify $\hat{f}$ by approximately matching its first two moments with those of the jump size of the swap rate, conditional on being at a jump time. The conditional jump probability density is

$$\nu_Q(dx, t) \approx \nu_Q(dx, t) \hat{\lambda}(t) f(x, t) dx \ .$$

(2.37)

Therefore, at a jump time $\tau$ of $\tilde{s}_{m,n}(t)$ we have

$$E_Q^D[\tilde{s}_{m,n}(\tau) - \tilde{s}_{m,n}(\tau-) \mid \tau, H(\tau-)] \approx \int_{R^+} \sum_{j=m}^{n-1} \tilde{\omega}_j(\tau-) H_j(\tau-) K_j(x, \tau) \nu_Q(dx, \tau) \lambda(\tau) \ ,$$

(2.38)

which motivates fixing the rates and weights at time zero on the right side above, writing $\nu_Q$ explicitly and defining

$$I_1(t) \equiv \int_{R^+} \sum_{j=m}^{n-1} \tilde{\omega}_j H_j(0) K_j(x, t) \prod_{j=\eta(t)}^{k} \frac{(1 - \frac{R}{1-R} \Delta T_{j-1} H_{j-1}(0)(1 + K_{j-1}(x, t)))}{(1 - \frac{R}{1-R} \Delta T_{j-1} H_{j-1}(0))} \frac{(1 + \Delta T_j H_j(0))}{(1 + \Delta T_j H_j(0)(1 + K_j(x, t)))} \lambda(t) f(x, t) dx \ .$$

(2.39)

For $\tilde{s}_{m,n}$, we write

$$E_Q^D[\tilde{s}_{m,n}(s) - \tilde{s}_{m,n}(s-) \mid s, \tilde{s}_{m,n}(s-)] = \tilde{s}_{m,n}(s-) \int_0^{\infty} (y-1) \hat{f}(y, s) dy = \tilde{s}_{m,n}(s-) \hat{\mu}(s) \ .$$

(2.40)
With $\hat{s}_{m,n}$ fixed at time 0,

$$\hat{\mu}(t) \equiv \frac{I_1(t)}{\hat{s}_{m,n}(0)}.$$  \hspace{1cm} (2.41)

We proceed in the same way for deriving the second moment. Define

$$I_2(t) \equiv \int_{R^+} \left( \sum_{j=m}^{n-1} \bar{\omega}_j H_j(0) K_j(x,t) \right)^2 \prod_{j=n(t)}^{k} \frac{(1 - \frac{R}{1-R} \Delta T_{j-1} H_{j-1}(0)(1 + K_{j-1}(x,t)))}{(1 - \frac{R}{1-R} \Delta T_{j-1} H_{j-1}(0))} \frac{(1 + \Delta T_j H_j(0)) \lambda(t)}{(1 + \Delta T_j H_j(0)(1 + K_j(x,t)) \lambda(t)} f(x,t) dx ,$$  \hspace{1cm} (2.42)

and match it to the right hand side of

$$E^{Q^D}[(\hat{s}_{m,n}(s) - \hat{s}_{m,n}(s-))^2|s, \hat{s}_{m,n}(s-)] = \hat{s}_{m,n}(s-)^2 \int_{0}^{\infty} (y - 1)^2 \hat{f}(y, s) dy .$$

With $\hat{s}_{m,n}(s-)$ replaced by $\hat{s}_{m,n}(0)$, We again get

$$e^{\hat{\psi}(t)}(1 + \hat{\mu}(t))^2 - 2\hat{\mu}(t) - 1 = \frac{I_2(t)}{\hat{s}_{m,n}(0)} .$$ \hspace{1cm} (2.43)

As a result,

$$\hat{m} = \log(1 + \hat{\mu}) - \hat{\psi}^2/2, \quad \hat{\psi} = \left[ \log(\frac{I_2/\hat{s}_{m,n}(0) + 1 + 2\hat{\mu}}{(1 + \hat{\mu})^2}) \right]^{1/2} . \quad \Box$$

The dynamics of $\hat{s}_{m,n}(t)$ under default swap measure are given by (2.31) – (2.33). The CDS option pricing is permitted under this jump-diffusion model using the formula (2.28) – (2.30) introduced in [12].

By now, the infrastructure of our extended market model, whose state variable is the mean loss rates, has already been established, which extends a similar model with forward credit spreads is introduced in [16]. The no arbitrage condition is given and the close form formula for CDS options pricing could be justified. Main work for pricing single-name credit derivatives are finished. In the next chapter, I will discuss the applications of this market model with mean loss rate on portfolio credit derivatives.
Chapter 3

Application of Market Model to CDO

3.1 Introduction of CDO

Collaterized debt obligations are the most important class of portfolio credit derivatives. A CDO is a financial instrument for the securitization of credit-risky securities related to a pool of reference entities such as bonds, loans or single-name CDSs; these securities form the asset side of the CDO. The assets are sold to a so-called special-purpose vehicle (SPV). To finance the acquisition of the assets, the SPV issues notes backed by tranches of different seniorities, which form the liability side of the structure. The tranches of the liability side are called equity, mezzanine and senior tranches. The equity tranche, which is the most junior tranche, will absorb the losses to the portfolio up to the first detachment point and then ceases to exist. Subsequent losses will then be beared by the next tranche. Losses to other senior tranches are determined similarly. This makes the CDOs attractive to different investors.

The spread of each tranche is just the premium of protection payments that are made periodically until all notional value is lost. Tranches are divided by attachment points. Take a CDO with $I$ tranches for example, the attachment points are characterized by $0 = K_0 < K_1 < \ldots < K_I \leq 1$. The value of the notional corresponding to tranche $i$ can be described below. Initially, the notional is equal to $K_i - K_{i-1}$; it is reduced whenever there is a default event
such that the cumulative loss falls in the layer $[K_{i-1}, K_i]$. In mathematical terms, $N_i(t)$, the notional of tranche $i$ at time $t$, is given by

$$N_i(t) = \begin{cases} 
K_i - K_{i-1}, & \text{for } L(t) < K_{i-1}, \\
K_i - L(t), & \text{for } L(t) \in [K_{i-1}, K_i], \\
0, & \text{for } L(t) > K_i,
\end{cases} \tag{3.1}$$

where $L(t)$ is the cumulative loss at $t$ for the portfolio in percentage. Assuming that the total number of credit names in the portfolio is $N$, $\tau_i$ is the default time of name $i$, then $L(t)$ can well be defined as:

$$L(t) = U_1 1_{\{\tau_1 \leq t\}} + \ldots + U_N 1_{\{\tau_N \leq t\}}, \tag{3.2}$$

where $U_i$ is the loss given default (LGD) of name $i$ (percentage of the total notional portfolio loss). If the recovery rate is given by $R_\tau$, the LGD of name $i$ is $U_i = (1 - R_\tau)/N$. The loss of tranche $i$ at time $t$ is:

$$L_i(t) = (L(t) - K_{i-1})1_{\{K_{i-1} \leq L(t) \leq K_i\}} + \Delta K_i 1_{\{L(t) > K_i\}},$$

where $\Delta K_i = K_i - K_{i-1}$, and can be written concisely as

$$L_i(t) = \min(\max(0, L(t) - K_{i-1}), \Delta K_i). \tag{3.3}$$

Next, we consider the pricing of the premium rate on a tranche. Let $S_i$ be the premium rate on the $i^{th}$ tranche, then the value of the fee leg of tranche $i$ is

$$PV_{fee} = S_i E^Q \left[ \sum_{j=m+1}^{n} \Delta T_j B^{-1}(T_j)(\Delta K_i - L_i(T_j)) \right]$$

$$= S_i \sum_{j=m+1}^{n} P_T_j(t) \Delta T_j (\Delta K_i - E^Q[L_i(T_j)]). \tag{3.4}$$

While the value of protection leg of tranche $i$ is

$$PV_{prot} = E^Q \left[ \sum_{j=m+1}^{n} B^{-1}(T_j)(L_i(T_j) - L_i(T_{j-1})) \right]$$

$$= \sum_{j=m+1}^{n} P_T_j(t) \left( E^Q[L_i(T_j)] - E^Q[L_i(T_{j-1})] \right). \tag{3.5}$$

No arbitrage requirement implies PVs of fee leg and protection leg are the same at time $t$, which gives the spread of tranche $i$ as

$$S_i = \frac{\sum_{j=m+1}^{n} P_T_j(t) \left( E^Q[L_i(T_j)] - E^Q[L_i(T_{j-1})] \right)}{\sum_{j=m+1}^{n} P_T_j(t) \Delta T_j (\Delta K_i - E^Q[L_i(T_j)])}. \tag{3.6}$$
3.2 Monte Carlo Method for CDO

From the last section, we understand that the key to CDO spread calculation lies in the valuation of the $L(t)$ and computation of $L_i(t)$ for $t = t_i, i = m, \ldots, n$. Now we consider the valuation of $L(t)$ by Monte Carlo simulation method. For simplicity, we assume constant recovery rate, $E^Q[R_{\tau} | T_{j-1} < \tau \leq T_j] = \bar{R}$ = constant for all names. The simulation of defaults is the focus of the algorithm.

Two set of correlations related to defaults are observed in real market. The first set is the correlation of CDS rates and hence mean loss rates, $H_j(t)$, which is the state variables for the market model. The evolution of the term structures of $H_j(t), j = m, \ldots, n$, are not independent. The second set is the correlation of default times, which is not yet prescribed in the model. It is dealt with the technique of Gaussian copula in this section, and some results based on adoption of other copula will be discussed later.

As for the mean loss rate of the $i^{th}$ name, $H_j^{(i)}(t)$, we could approximate the evolution by the Euler scheme:

$$H_j^{(i)}(t + \Delta t) = H_j^{(i)}(t) \exp \left( (\mu_j^{H} \Delta t - \frac{1}{2} \| \sigma_j^{H} \|^{2}) \Delta t + \sigma_j^{H} \Delta W^{(i)}_t \right). \quad (3.7)$$

Meanwhile, we can incorporate the correlations of the credit spreads in the evolutions by introducing correlated diffusion driving factors $\Delta W^{(i)}_t$. The hazard rate at $t$ is derived as:

$$\lambda_j^{(i)}(t) = \frac{H_j^{(i)}(t)}{(1 - (1 + \Delta t H_j^{(i)}(t)) \bar{R})}. \quad (3.8)$$

This relationship between mean loss rate $H_j^{(i)}(t)$ and hazard rate $\lambda_j^{(i)}(t)$ bridges our market model with Monte Carlo simulation of portfolio loss $L(t)$.

In order to adopt a Gaussian copula, we can take the following steps:

1. Do a Choleski decomposition of the input correlation matrix $\Sigma = AA^T$.

2. Simulate $N$ independent standard normal random variables, $\{\hat{\epsilon}_i\}_{i=1}^{N}$.

3. Transform $\{\epsilon_i\}_{i=1}^{N}$ to correlated normal random variables

$$\begin{pmatrix} \epsilon_1 \\ \vdots \\ \epsilon_N \end{pmatrix} = A \begin{pmatrix} \hat{\epsilon}_1 \\ \vdots \\ \hat{\epsilon}_N \end{pmatrix}.$$
(4) Transform the normal random variables to uniform random variables

\[(u_1, \ldots, u_N) = (N(\epsilon_1), \ldots, N(\epsilon_N))\,.

If we assume a uniform pairwise correlation such that \(\text{corr}(\epsilon_i, \epsilon_j) = \rho > 0\) for any \(i\) and \(j\), then step (1)-(3) above are simplified into the calculations of

\[\epsilon_n = \sqrt{1 - \rho} \hat{\epsilon}_n + \sqrt{\rho} \hat{\epsilon}_c,\]  

where \(\hat{\epsilon}_c\) and \(\hat{\epsilon}_n, n = 1, \ldots, N\) are independent standard normal random variables.

Now, we are ready to describe the algorithm for CDO pricing. Let \(u\) be a random variable with uniform distribution \(U(0, 1; \Sigma_u)\), where \(\Sigma_u\) stands for the correlation matrix of \(u\). To determine whether an entity is default or not within the next time interval \([T_j, T_{j+1}]\), a Poisson default standard is adopted. Given that the \(i^{th}\) entity is survival until \(T_j\), if \(u^{(i)} \leq \lambda_j^{(i)}(T_j) \Delta T_j\), this entity will default in the next time period. Otherwise, if \(u^{(i)} > \lambda_j^{(i)}(T_j) \Delta T_j\), this entity will survive continuously. Let \(N\) be the total number of credit names in the portfolio, \(T\) the maturity of the CDO, \(\Delta T\) the time interval for premium payments, \(J = T/\Delta T\) the maximal number of the premium payments, and \(M\) the number of Monte Carlo simulation paths. The algorithm for valuating loss of \(i^{th}\) tranche \(L^{(i)}(t_j)\) is developed as follows.

/* Algorithm for valuating \(L^{(i)}(t_j), j = 1, \ldots, J\) */
For \(j = 1 : J\)
\[L^{(i)}(t_j) = 0\]
end
For \(m = 1 : M\)
For \(n = 1 : N\)
\[D^{(n)}(t_0) = 1\]
end
\(N(t_0) = N\)
\(L(t_0) = 0\)
For \(j = 1 : J\)
Generate \(\{\Delta W^{(n)}\}_{n=1}^{N(t_{j-1})} \sim N(0, \Delta t_j \Sigma_H)\)
Generate \(\{u_n\}_{n=1}^{N(t_{j-1})} \sim U(0, 1; \Sigma_u)\)
\(N(t_j) = N(t_{j-1})\)
\(L(t_j) = L(t_{j-1})\)
\(l = 0\)
20
For $n = 1 : N$ repeat

If $D^{(n)}(t_{j-1}) = 1$, then

$D^{(n)}(t_j) = 1$

/* Calculate the hazard rate $\lambda^{(n)}(t_{j-1})$ */

$\lambda^{(n)}(t_{j-1}) = H^{(n)}_{j-1}(t_{j-1})/(1 + 1 + (1 + \Delta t_j H^{(n)}_{j-1}(t_{j-1})) R)$

/* Simulate default over $(t_{j-1}, t_j]$ for the $n^{th}$ name */

$l = l + 1$

If $u_l \leq \lambda^{(n)}(t_{j-1}) \Delta t_j$

$D^{(n)}(t_j) = \bar{R}$

$N(t_j) = N(t_j) - 1$

$L(t_j) = L(t_j) + (1 - D^{(n)}(t_j))/N$

end if

/* Simulate the mean loss rate $H^{(n)}(t_j)$ according to the market model */

$H^{(n)}(t_j) = H^{(n)}(t_{j-1}) \exp \left( (\mu^{H}_n(t_{j-1}) - \frac{1}{2} ||\sigma^{H}_n||^2) \Delta t_j + \sigma^{H}_n \Delta W^{(l^{(n)})}_{t_j} \right)$

end if

/* Calculate the loss of the $i^{th}$ tranche */

$L^{(i)}(t_j) = L^{(i)}(t_j) + \min(\max(0, L(t_j) - K_{i-1}), \Delta K_i)$

end

/* Average tranche loss */

For $j = 1 : J$

$L^{(i)}(t_j) = L^{(i)}(t_j)/M$

end

/* The end of the algorithm*/

The entire algorithm is rather easy to implement, and the computation time is about $J$ times more than that of the Gaussian copula method of [22].

### 3.3 Implied Base Correlation

Correlation has received a great deal of attention in recent years. A high degree of co-movement within all the companies can increase the likelihood of extreme events. Our market model for CDO pricing enable us to measure the market implied levels of correlation and to utilize them for relative pricing.
The credit risk correlation can be defined in many ways, but it generally refers to the degree of co-movement of asset values of companies. High correlation is a signal of more likelihood that assets and companies default together, while low correlation can be interpreted as more possibility of isolated defaults. The level of correlation significantly affect the value of CDO tranche. A fat-tailed loss distribution tend to have higher risk of suffering large losses than a thin-tailed loss distribution. As a result, the value of a senior tranche is lower when the loss distribution is fat-tailed. By contrast, the reverse is true for equity tranche, because high correlation increases the likelihood of avoiding any losses.

Based on the observed market spreads for a tranche, implied correlation can be calculated by finding the level of correlation that equates the theoretical spread and the market spread. This approach is analogous to the way implied volatility is derived from the options market and the type of market-implied correlation is also called the compound correlation.

Although compound correlation is well accepted as a measurement of market risk premium on CDO tranche, it has some drawbacks. Firstly, for some mezzanine tranches, there can be two levels of correlation that yields a single market spreads, which means the market-implied correlation is not well defined for these tranches. Secondly, the implied compound correlation exhibits a shape of smile. This correlation smile can cause difficulty for interpolation that is necessary for pricing a customized tranche with a non-standard attachment or detachment point.

Base correlation, which is an alternative market-implied correlation measure, does not have some of the problems of the compound correlation. Base correlation is the correlation for an equity tranche that combines all tranches up to a detachment point. For example, the 0% – 3% tranche and the 3% – 7% tranche of the DJ CDX index are combined to create a hypothetical 0% – 7% tranche. And the expected loss for this hypothetical tranche is equal to the sum of expected losses for the 0% – 3% tranche and the 3% – 7% tranche. According to this definition, we can obtain similar expression for the fee leg and protection leg of certain equity tranche. Let 0% – $K_I\%$ be the hypothetical tranche with
detachment point $K_I$, then the value of the protection leg is given by:

$$
PV_{prot} = \mathbb{E}^Q \left[ \sum_{j=m+1}^{n} B_{T_j} \left( \sum_{i=1}^{I} L^i(T_j) - \sum_{i=1}^{I} L^i(T_{j-1}) \right) \right]
$$

$$
= \sum_{j=m+1}^{n} P_{T_j}(t) \left( \sum_{i=1}^{I} (\mathbb{E}^Q[L^i(T_j)] - \mathbb{E}^Q[L^i(T_{j-1})]) \right)
$$

(3.10)

And the value of the fee leg for this tranche is:

$$
P_{V_{fee}} = \mathbb{E}^Q \left[ \sum_{j=m+1}^{n} \Delta T_j B_{T_j} \left( \sum_{i=1}^{I} S_i(\Delta K_i - L^i(T_j)) \right) \right]
$$

$$
= \sum_{j=m+1}^{n} P_{T_j}(t) \Delta T_j \left( \sum_{i=1}^{I} S_i(\Delta K_i - \mathbb{E}^Q[L^i(T_j)]) \right).
$$

(3.11)

Now, we can calculate the level of market-implied base correlation corresponds to the $0\% - K_I\%$ tranche, applying the same algorithm developed in the last section. The implied base correlation equates the market value of protection leg with fee leg for a specific equity tranche.

### 3.4 Assumptions and Initial Data Processing

To handle data, we make some reasonable assumptions and simplifications without loss of generality. The CDS rate volatility is set at the constant level of either 50% or 100%, which represents the usual range of implied swaption volatilities (see [27] and [6]). The recovery rate is taken to be $R = 40\%$, according to the industrial convention. Because of the scarcity of correlation data for CDS rates, we let the CDS rates and the Gaussian copula for default times share the same pairwise correlation. We take the number of paths to be $M = 10,000$, and the number of time stepping to be $\Delta t = 0.5$.

The standard market practice is to back out the survival probability of individual company from CDS of various maturities, under the assumption of a constant recovery rate. The survival probability, $\Lambda_j(t)$, related to the hazard rates by:

$$
\Lambda_j(t) = (1 + (T_{\eta(t)} - t)\lambda_{\eta(t)-1})^{-1} \prod_{k=\eta(t)}^{j-1} (1 + \Delta T_k \lambda_k)^{-1}.
$$

(3.12)

Taking Citigroup as the credit name for a demonstration, we see that the market quoted CDS spreads (in Table 3.1) increase with time to maturity. Therefore,
the risk-neutral hazard rate will also increase with its maturity. It is a reasonable assumption that hazard rate $\lambda(t)$ follows a piecewise linear process. Under this assumption, the mean loss rate $H_j(t)$ of Citigroup is computed.

When pricing CDO tranches, the five-years and ten-years index spread (see Table 3.2 and 3.3) are treated as input to back out the average of the hazard rate for the 125 companies in the market. According to [18], setting all credit spreads and pairwise correlations equal to their average values does not have a significant effect on the valuation of CDO tranches. Practitioners also agree with this observation. Therefore, a homogeneous market model with all companies having the same CDS rate and default probability that equal to average level is an acceptable treatment for application.

<table>
<thead>
<tr>
<th>Maturity</th>
<th>1Y</th>
<th>3Y</th>
<th>5Y</th>
<th>10Y</th>
</tr>
</thead>
<tbody>
<tr>
<td>Rate</td>
<td>0.07%</td>
<td>0.13%</td>
<td>0.19%</td>
<td>0.33%</td>
</tr>
</tbody>
</table>

Table 3.1: Citigroup CDS rates (28/7/2005, Bloomberg)

<table>
<thead>
<tr>
<th>Index</th>
</tr>
</thead>
<tbody>
<tr>
<td>0%-3% 3%-7% 7%-10% 10%-15% 15%-30%</td>
</tr>
<tr>
<td>5-year quotes 40.02 295.71 120.50 43.00 12.43 59.73</td>
</tr>
<tr>
<td>10-year quotes 58.17 632.00 301.00 154.00 49.50 81.00</td>
</tr>
</tbody>
</table>

Table 3.2: Quotes of CDX IG Tranches on 24/8/2004

<table>
<thead>
<tr>
<th>Index</th>
</tr>
</thead>
<tbody>
<tr>
<td>0%-3% 3%-6% 6%-9% 9%-12% 12%-22%</td>
</tr>
<tr>
<td>5-year quotes 24.10 127.50 54.00 32.50 18.00 37.79</td>
</tr>
<tr>
<td>10-year quotes 43.80 350.17 167.17 97.67 54.33 51.25</td>
</tr>
</tbody>
</table>

Table 3.3: Quotes of iTraxx IG Tranches on 24/8/2004

The mean loss rates for Citigroup and CDX index are shown in Figure 3.1.
On the one hand, the mean loss rate, $H_j(t)$, for Citigroup, increase with the time to maturity. As Citigroup has a rating of AAA in 2005, market quotes very small risk premium for its short term credit default swap. For long term consideration, the uncertainty of the company performance increases significantly. The curve of mean loss rate for Citigroup reflects market’s increasing risk premium for possible credit events in the future.

On the other hand, the curve of mean loss rate for CDX index exhibits a bend. The mean loss rates increase at first before arriving the maximum point at 5-year point, then decrease gradually until $t = 10$. This characteristic could be interpreted by the market model. The average of mean loss rates for all survival companies after 5-years would lower than the average of the mean loss rates for all companies comprising the underlying portfolio. The reason is, mean loss rate, $H_j(t)$, is a diffusion process under risk neutral measure, and upward moving mean loss rate is more likely to be excluded from portfolio because of the higher possibility of default. A company that survive in 5 years may have a good performance history and a relatively low hazard rate. The slope of this company’s mean loss rate is more likely downward, flat or mildly upward.

### 3.5 Results of Pricing on Correlation

As said above, compound correlation is the market-implied level of correlation for each tranche. Using the market model, we could calibrate implied compound correlations from market spreads.
From Table 3.4, we can observe the facts that for some tranches, there can be two correlations that yield the same market spread. Hence, at least for some mezzanine tranches, market implied compound correlation is not well defined. Moreover, Figure 3.2 is the implied compound correlation for tranches of iTraxx and CDX. The obvious lack of monotonicity of the implied correlation curve makes it difficult for interpolation, which is needed for pricing a customized tranche with non-standard attachment or detachment points.

The results of implied base correlation for tranches of iTraxx and CDX are demonstrated in Table 3.5, 3.6 along with Figure 3.3 and 3.4. The base correlation curves for most tranches are now monotonic, allowing a unique level
of correlation for a particular spread level. We can avoid the problem associated with compound correlation and interpolate the base correlation curve for non-standard tranches. The correlation smile disappears when we move to base correlation. Instead, the base correlation curve is upward sloping, which is often referred to as the base correlation skew.

Base correlation is not perfect also. Firstly, base correlation is less intuitive than compound correlation and is more difficult to interpret. As a result, base correlation for a certain detachment point reflects market spreads for multiple tranches. For example, if the 0% – 3% tranche becomes undervalued while the 3% – 7% tranche is overvalued, base correlation for the 0% – 7% tranche may reflect the offsetting forces and show little change. What is more, the base correlations transform the correlation smile into a skew. There is a need for more work to find a better interpretation.

<table>
<thead>
<tr>
<th>Equity Tranches</th>
<th>Base Correlation</th>
<th>Base Correlation</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Gamma = 1</td>
<td>Gamma = 0.5</td>
</tr>
<tr>
<td>iTraxx</td>
<td>0%-3%</td>
<td>0.3544</td>
</tr>
<tr>
<td></td>
<td>0%-6%</td>
<td>0.4303</td>
</tr>
<tr>
<td></td>
<td>0%-9%</td>
<td>0.4776</td>
</tr>
<tr>
<td></td>
<td>0%-12%</td>
<td>0.5185</td>
</tr>
<tr>
<td></td>
<td>0%-22%</td>
<td>0.6232</td>
</tr>
<tr>
<td>CDX</td>
<td>0%-3%</td>
<td>0.3817</td>
</tr>
<tr>
<td></td>
<td>0%-7%</td>
<td>0.4501</td>
</tr>
<tr>
<td></td>
<td>0%-10%</td>
<td>0.4784</td>
</tr>
<tr>
<td></td>
<td>0%-15%</td>
<td>0.5263</td>
</tr>
<tr>
<td></td>
<td>0%-30%</td>
<td>0.6603</td>
</tr>
</tbody>
</table>

Table 3.5: Implied Base Correlation for 10-year iTraxx and CDX
Figure 3.3: Implied Base Correlation for 10-year iTraxx and CDX

![Graphs showing implied base correlation for 10-year iTraxx and CDX.]

Table 3.6: Implied Base Correlation for 5-year iTraxx and CDX

<table>
<thead>
<tr>
<th>Equity Tranches</th>
<th>Base Correlation</th>
<th>Base Correlation</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Gamma = 1</td>
<td>Gamma = 0.5</td>
</tr>
<tr>
<td>iTraxx</td>
<td></td>
<td></td>
</tr>
<tr>
<td>0%-3%</td>
<td>0.4431</td>
<td>0.4479</td>
</tr>
<tr>
<td>0%-6%</td>
<td>0.5465</td>
<td>0.5047</td>
</tr>
<tr>
<td>0%-9%</td>
<td>0.6202</td>
<td>0.6128</td>
</tr>
<tr>
<td>0%-12%</td>
<td>0.6507</td>
<td>0.6458</td>
</tr>
<tr>
<td>0%-22%</td>
<td>0.6695</td>
<td>0.7095</td>
</tr>
<tr>
<td>CDX</td>
<td></td>
<td></td>
</tr>
<tr>
<td>0%-3%</td>
<td>0.4085</td>
<td>0.4684</td>
</tr>
<tr>
<td>0%-7%</td>
<td>0.5001</td>
<td>0.5475</td>
</tr>
<tr>
<td>0%-10%</td>
<td>0.5497</td>
<td>0.5915</td>
</tr>
<tr>
<td>0%-15%</td>
<td>0.6406</td>
<td>0.6846</td>
</tr>
<tr>
<td>0%-30%</td>
<td>0.7148</td>
<td>0.8551</td>
</tr>
</tbody>
</table>

Figure 3.4: Implied Base Correlation for 5-year iTraxx and CDX

![Graphs showing implied base correlation for 5-year iTraxx and CDX.]

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3.6 More Discussion on Market Model

Some empirical evidences suggest that asset correlations are stochastic and increase when default probabilities are high. [28] find that the correlations are higher in recessions than in expansion periods. [7] conclude that default correlation increase when default probabilities are high. To integrate this observation into our market model, a stochastic copula model could be a possible solution. While Gaussian copula is generally accepted as a standard market practice to describe the relationship between default times, a $t$-copula could provide us a distribution with more tail dependence, which means extreme events will be more likely happen together. The simulation of $t$-copula on $\nu$ degrees of freedom could be described as follow:

1. Generate $Z \sim N_d(0, \Sigma)$.
2. Generate an independent variable $W$ with a chi-squared distribution of $\nu$ degrees of freedom.
3. Set $X = \sqrt{\nu/W}Z$.
4. Return $U = (t_\nu(X_1), \ldots, t_\nu(X_d))'$, where $t_\nu$ denotes the distribution function of a standard univariate $t$ distribution.

Replacing the Gaussian copula used in CDO pricing by $t$-copula with 4 degrees of freedom, the implied base correlations are computed and illustrated in Figure 3.5. With $t$-copula, the implied base correlations for most tranches are less than those with Gaussian copula, which attributes to the larger number of defaults caused by the thick tail of $t$-copula.

![Figure 3.5: Implied Base Correlation for 10-year iTraxx and CDX with $t$-copula](image)

We can observe an interesting fact from Figure 3.3, 3.4 and 3.5. The implied base correlations, derived from mean loss rate models with different volatilities,
are getting close when the time to maturity changes from ten years to five years and when the Gaussian copula is replaced by $t$-copula. It means the default time correlation plays a more important role than the CDS rates correlation for CDO pricing.

When the degrees of freedom is larger than 30, the distribution of $t$-copula default time approximates to the one of Gaussian copula. Therefore, the stochastic copula can be implemented by choosing a state variable that represents the degrees of freedom of the $t$-copula. More work is needed in further researches.

Recall that we have derived the survival probability and hazard rate of a representative company from index spreads. Even under the assumption that all companies are homogeneous, the individual company’s hazard rate can be equal to index-implied hazard rate only for two cases: they are either perfectly independent or perfectly correlated. How to separate the information about default probability of a representative company from index spread, which also include information about market correlation would be the next challenge for us to obtain a refined pricing outcomes. One attempt is to back out simultaneously the hazard rate and correlation from index spread and, in addition, other sources of information such as spread of equity tranche. But this may be an ill-posed calibration problem. The search ends up in one of many possible solutions, depending on the initial correlation level.

In this chapter, the applications of our market model with mean loss rates on CDO pricing and implied correlation computation have been illustrated, showing the capability of our model on portfolio credit derivatives pricing and calibration. The assumptions and details of the market model setting and initial data processing have been discussed. The results of correlation pricing agree with the observations in reality and market convention. Several ways to extend our model and improve the results have also been discussed.
Chapter 4

Conclusion

In this paper, we develop an extended market model with mean loss rates based on the work of [16] for pricing single name and portfolio credit derivatives. This model integrates two jump terms into the diffusion process of mean loss rates. One is a Cox process that describes the jump to default of a specific entity, the other is a lognormal jump process with time varying coefficients that captures the jumps of CDS rates.

We have derived the no arbitrage condition of the extended market model. By changing into the default swap measure and establishing the relations between the coefficients of CDS rates and mean loss rates, the pricing formula for single name CDS options is derived.

By combining the market model with copula models of default times, we can price portfolio credit derivatives like COOs, using the method of Monte Carlo simulation. Given the market quoted CDO tranch spread, we derive the implied compound correlations as well as the implied base correlations because they are important factors to analyze a portfolio of credit risk. Compared with compound correlations, the implied base correlations avoid the problem associated with compound correlation and help us interpolate the base correlation curve for non-standard tranches.

Based on the observation that default correlations increase when default rates are high, we have discussed the possibility of adopting a $t$-copula model of default times as a potential stochastic correlation model. In addition, we also specify some details in my algorithm aiming at improving the empirical results.
Appendix A

Proof of theorems

A.1 Proof of Theorem 2.1

In [16], the no arbitrage condition of the market model under risk neutral measure was

$$
\mu^H_j(t) = \sigma^H_j(t) \sum_{k=\eta(t)}^j \frac{\Delta T_k H_k(t)}{1 + \Delta T_k H_k(t)} \left( \sigma^H_k(t) - (1 - \bar{H}_k(t)) \sigma^H_{k-1}(t) \right), \quad (A.1)
$$

where

$$
\bar{H}_j(t) = \frac{1}{\Delta T_j} \left( \frac{D_j(t)}{D_{j+1}(t)} - 1 \right).
$$

The Theorem A.1 in [11] gives the dynamics of the jump term under risk neutral measure as

$$
\begin{align*}
1 + \Delta T_j H_j(t-) & \frac{1}{\Delta T_j} \int_0^\infty e^{-\int_{t-}^{t+\Delta T_j} K_j(x,s) ds} \left( e^{-\int_{t+\Delta T_j}^{T_j+\Delta T_j} K_j(x,s) ds} - 1 \right) \nu_Q(dx,t) dt \\
\nu_Q(dx,t) dt & + \frac{1}{\Delta T_j} \int_0^\infty \left( e^{-\int_{t-}^{t+\Delta T_j} K_j(x,s) ds} - 1 \right) \mu(dx,dt),
\end{align*}
$$

(A.2)

where $\nu_Q(dx,t)$ is the intensity of the MPP under risk neutral measure, and we use $\mu(dx,dt)$ to denote the random measure assigning unit mass to each $(\tau, x)$. 

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Put these two conclusions together, we can get that $H_j(t)$ satisfies

$$
dH_j(t) = \left[ \sum_{k=n}^{j} \frac{\Delta T_k H_k(t)}{1 + \Delta T_k H_k(t)} \left( \sigma_k^H(t) - \left(1 - \frac{\bar{H}_k(t)}{H_k(t)}\sigma_{k-1}^H(t) \right) dt + dW_t \right) \right] H_j(t)\sigma_j^H(t)
$$

$$
+ \frac{1 + \Delta T_j H_j(t)}{\Delta T_j} \int_0^\infty e^{-\int_t^{T_j} K_j(x,s) ds} \left( e^{-\int_t^{T_j+\Delta T_j} K_j(x,s) ds} - 1 \right) \nu_Q(dx, t) dt
$$

$$
+ \frac{1 + \Delta T_j H_j(t)}{\Delta T_j} \int_0^\infty \left( e^{-\int_t^{T_j+\Delta T_j} K_j(x,s) ds} - 1 \right) \mu(dx, dt), \quad (A.3)
$$

where $W$ is a standard Brownian motion under risk neutral measure $Q$.

The first $dt$ term and the $dW$ terms match (2.22). For the jump terms,

$$
e^{\int_t^{T_j+\Delta T_j} K_j(x,s) ds} = \frac{\Delta T_j K_j(x,t) H_j(t)}{1 + \Delta T_j H_j(t)} + 1 = \frac{1 + \Delta T_j H_j(t)(1 + K_j(x,t))}{1 + \Delta T_j H_j(t)} ,
$$

i.e.

$$
1 + \frac{\Delta T_j H_j(t)}{\Delta T_j} \left( e^{\int_t^{T_j+\Delta T_j} K_j(x,s) ds} - 1 \right) = K_j(x,t) H_j(t). \quad (A.4)
$$

It follows that

$$
1 + \frac{\Delta T_j H_j(t)}{\Delta T_j} \int_0^\infty e^{-\int_t^{T_j} K_j(x,s) ds} \left( e^{-\int_t^{T_j+\Delta T_j} K_j(x,s) ds} - 1 \right) \nu_Q(dx, t) dt
$$

$$
= 1 + \frac{\Delta T_j H_j(t)}{\Delta T_j} \int_0^\infty e^{-\int_t^{T_j+\Delta T_j} K_j(x,s) ds} \left( 1 - e^{\int_t^{T_j+\Delta T_j} K_j(x,s) ds} \right) \nu_Q(dx, t) dt
$$

$$
= -K_j(x,t) H_j(t) \prod_{k=n}^n \frac{1 + \Delta T_k H_k(t)}{1 + \Delta T_k H_k(t)(1 + K_j(x,t))} \nu_Q(dx, t) \quad (A.6)
$$

so A.5 ensures that the intensity term matches (2.22). Finally, A.4 shows that

$$
1 + \frac{\Delta T_j H_j(t)}{\Delta T_j} \int_0^\infty \left( e^{-\int_t^{T_j+\Delta T_j} K_j(x,s) ds} - 1 \right) \mu(dx, dt) = H_j(t) L_j(t) \quad (A.7)
$$

with $L_j(t)$ as in (2.23).

### A.2 Proof of Theorem 2.2

First, we can write $\sum_{t=1}^{N(t)} K_j(X_t, \tau_i) = \int_0^t \int_{\mathbb{R}} K_j(x,s) \mu(dx, ds)$, where $\mu(dx, ds)$ is a random measure of the mark space and the time axis. We want to identify
the change in intensity associated with changing from risk neutral measure to the default swap measure. Define

\[ S(t) = \frac{\sum_{j=m}^{n-1} \Delta T_j \tilde{Z}_{j+1}(t)}{B(t)}, \]

where \( B(t) \) is the discretely compounded money market account. Recalling that

\[ \hat{f}_j(t) = \frac{1}{\Delta T_j} \left[ \frac{\tilde{Z}_j(t)}{Z_{j+1}(t)} \frac{\Lambda_j(t)}{D_j(t)} - 1 \right], \]

and

\[ \frac{\Lambda_j(t)}{D_j(t)} = 1 - \frac{\bar{R}}{1 - \bar{R}} \Delta T_{j-1} H_{j-1}(t), \]

it follows that

\[ S(t) = \prod_{j=0}^{\eta(t)-1} \frac{1}{1 + \Delta T_j f_j(T_j)} \sum_{k=m}^{n-1} \Delta T_j \prod_{j=\eta(t)}^{k} \frac{1 - \frac{\bar{R}}{1 - \bar{R}} \Delta T_{j-1} H_{j-1}(\tau -)}{(1 + \Delta T_j f_j(\tau -))(1 + \Delta T_j H_j(\tau -))(1 + K_j(X, \tau))}(1 + \Delta T_j H_j(\tau -)) \]

At a jump time of the MPP, the mark X, the percentage jump in S is:

\[ \frac{S(\tau) - S(\tau -)}{S(\tau -)} = \left( \sum_{k=m}^{n-1} \Delta T_k \prod_{j=\eta(\tau)}^{k} \frac{1 - \frac{\bar{R}}{1 - \bar{R}} \Delta T_{j-1} H_{j-1}(\tau -)}{(1 + \Delta T_j f_j(\tau -))(1 + \Delta T_j H_j(\tau -))(1 + K_j(X, \tau))}(1 + \Delta T_j H_j(\tau -)) \right) - 1, \]

which can be written as

\[ \left( \sum_{k=m}^{n-1} \Delta T_k \prod_{j=\eta(\tau)}^{k} \frac{1 - \frac{\bar{R}}{1 - \bar{R}} \Delta T_{j-1} H_{j-1}(\tau -)}{(1 + \Delta T_j f_j(\tau -))(1 + \Delta T_j H_j(\tau -))(1 + K_j(X, \tau))}(1 + \Delta T_j H_j(\tau -)) \right) - 1 \]

Multiplying numerator and denominator by \( \tilde{Z}(\eta(\tau)) \) and recalling the definition of the weight \( \tilde{\omega}_j(\tau) \), this expression simplifies to

\[ \sum_{k=m}^{n-1} \tilde{\omega}_k(\tau) \prod_{j=\eta(\tau)}^{k} \frac{(1 - \frac{\bar{R}}{1 - \bar{R}} \Delta T_{j-1} H_{j-1}(\tau -))(1 + K_{j-1}(X, \tau)))(1 + \Delta T_j H_j(\tau -))}{(1 - \frac{\bar{R}}{1 - \bar{R}} \Delta T_{j-1} H_{j-1}(\tau -))(1 + \Delta T_j H_j(\tau -))(1 + K_j(X, \tau))}(1 + \Delta T_j H_j(\tau -)) - 1 \]

(A.8)
Now write the dynamics of $S$, the drift follows from the martingale condition
\[
\frac{dS(t)}{S(t-)} = \int_{R^+} \left( \sum_{k=m}^{n-1} \tilde{\omega}_k(\tau) \prod_{j=\eta(\tau)}^k \frac{(1 - \frac{\bar{R}}{1-R} \Delta T_{j-1} H_{j-1}(\tau-)(1 + K_{j-1}(x, \tau)))}{(1 - \frac{\bar{R}}{1-R} \Delta T_{j-1} H_{j-1}(\tau-))} \right) \cdot \left( \frac{1 + \Delta T_j H_j(\tau-)}{(1 + \Delta T_j H_j(\tau-)(1 + K_j(x, \tau)))} - 1 \right) (\mu_Q(dx, t) - \nu_Q(dx, t)) dt + \ldots dW
\] (A.9)

Next define the default swap measure through the Radon-Nikodym derivative
\[
\left. \frac{dQ^D}{dQ} \right|_{\mathcal{F}_t} = S(t) \frac{B(0)}{A_{D,m,n}(0)},
\]
and invoke Girsanov’s theorem to identify the change in intensity associated to the change of measure. Girsanov’s theorem states that given a $Q$-martingale process $A(t)$ with
\[
\frac{dA(t)}{A(t-)} = \Gamma dt + \int_{R^d} (\Phi(x, t) - 1)[\mu_Q(dx, dt) - \nu_Q(dx, dt)],
\]
there is an equivalent measure $\hat{Q}$ with Radon-nikodym derivative
\[
\left. \frac{d\hat{Q}}{dQ} \right|_{\mathcal{F}_t} = A(t)
\]
such that $dW^Q = \Gamma dt + dW^{\hat{Q}}$ and the intensity of the driving jump process under $\hat{Q}$ is $\nu_{\hat{Q}}(dx, t) = \Phi(x, t)\nu_Q(dx, t)$. Therefore, we have the intensity of MPP
\[
\nu_{Q^{D'}}(dx, t) = \sum_{k=m}^{n-1} \tilde{\omega}_k(\tau) \prod_{j=\eta(\tau)}^k \frac{(1 - \frac{\bar{R}}{1-R} \Delta T_{j-1} H_{j-1}(\tau-)(1 + K_{j-1}(x, \tau)))}{(1 - \frac{\bar{R}}{1-R} \Delta T_{j-1} H_{j-1}(\tau-))} \cdot \frac{(1 + \Delta T_j H_j(\tau-))}{(1 + \Delta T_j H_j(\tau-)(1 + K_j(x, \tau)))} \nu_Q(dx, t).
\] (A.10)
Bibliography


