On Vertex Operator Algebras Associated to Jordan Algebras

by

Hongbo Zhao

A Thesis Submitted to
The Hong Kong University of Science and Technology
in Partial Fulfillment of the Requirements for
the Degree of Doctor of Philosophy
in Mathematics

12th June, Hong Kong
Authorization

I hereby declare that I am the sole author of the thesis.

I authorize the Hong Kong University of Science and Technology to lend this thesis to other institutions or individuals for the purpose of scholarly research.

I further authorize the Hong Kong University of Science and Technology to reproduce the thesis by photocopying or by other means, in total or in part, at the request of other institutions or individuals for the purpose of scholarly research.

Hongbo Zhao
On Vertex Operator Algebras Associated to Jordan Algebras

by

Hongbo Zhao

This is to certify that I have examined the above PhD thesis and have found that it is complete and satisfactory in all respects, and that any and all revisions required by the PhD qualifying examination committee have been made.

Prof. Yongchang Zhu, Supervisor

Prof. Xiaoping Wang, Acting Head of Department

Department of Mathematics

12th June 2017
Acknowledgment

First of all, I would like to express my gratitude to my supervisor, Prof. Yongchang Zhu, for his guidance during my PhD study. He is definitely a strong mathematician, as well as a good teacher. Not only did he make important contributions in his area, but he also had a panoramic view of mathematics. He was patient in explaining mathematics in a pedagogical way, although sometimes he became quite harsh when we made mistakes. I want to say, without any exaggeration, that he is the best educator I’ve ever met. I apologize that, However, as his student I’m far from the the level that he expected. What a shame!

I want to thank Prof. Jingsong Huang, Prof. Jianshu Li, and Prof. Min Yan who were involved in my PhD study and my thesis defense. I am grateful to Prof. Guowu Meng and Prof. Edmund Yik-Man Chiang, who encouraged me and helped me a lot. I also thank Prof. Chinghung Lam, who read my thesis draft carefully and made many useful comments.

Of course, I can’t forget to thank Tao Luo, who has been a good colleague of mine in the past nine years, though we were in different areas during our PhD studies. I enjoy the days with Dr. Guangze Gao, who played games and chatted with me. I think Gao is really a genius! But it is a pity that he left mathematics. I would also like to thank Diwei Li, Mingan Hu, and Xuanzong Dai, who participated in our private seminars. But I do not want to thank Diwei Li too much, as he was not so critical at my presentations.

I thank my parents Qiangshan Zhao and Xinfeng Niu, for their continuous support, although they did not want me to study mathematics. I also thank Miss Ying Ning, who accompanied me in my final year of study.

Finally, this thesis is dedicated to Shuyun Zhang, my math teacher when I was in secondary school. She frustrated me a lot at that time, and she warned me of not complying with her. Indeed, I become a mediocre person at this moment, and she seems to be correct. He–He–Ha–Ha.
Contents

Title Page i
Authorization Page ii
Signature Page iii
Acknowledgments iv
Tables of Contents vi
Abstract vii

1 Introduction 1

2 A Brief Review of Vertex Algebras 7

2.1 The Definition of Vertex Superalgebras and Vertex Operator Superalgebras 7

2.2 The Uniqueness Theorem and Some Useful Identities 13

2.3 The Reconstruction Theorem and Some Basic Examples 17

3 Vertex Operator Algebras Associated to Jordan Algebras of
| Type B | 26 |
| 3.1 The Griess Algebra of a Vertex Operator Algebra | 26 |
| 3.2 Construction of the VOAs $V_{J,1}$ and $V_{J,r}$, Where $J$ is a Type $B$ Jordan Algebra | 28 |
| 4 Correlation Functions of $V_{J,r}$ Where $J$ is of Type $B$ | 31 |
| 4.1 Diagrams, Derangements, and Some Necessary Notations | 31 |
| 4.2 Some Related Formal Power Series, and The Correlation Function of the Fields in $V_{J,1}$ | 37 |
| 4.3 Proof of Theorem 1.1 | 40 |
| 5 Simplicities of $V_{J,r}$ Where $J$ is of Type $B$ | 49 |
| 6 The Simple Quotients $\bar{V}_{J,r}$, and The Character Formula for $r = -2n, n \geq 1$, where $J$ is of type $B$ | 54 |
| 6.1 Dual-Pair Realization of Case 1 and Case 2 | 54 |
| 6.2 Dual Pair Realization of Case 3 | 58 |
| 6.3 Properties of $V_{J,r}, r \in \mathbb{Z} \neq 0$ | 61 |
| 6.4 The Character Formula of the Simple Quotient $\bar{V}_{J,r}, r = -2n, n \geq 1$ | 66 |
| 7 Construction of the VOA $V_{J,r}$, Where $J$ is of Hermitian Type | 69 |
On Vertex Operator Algebras Associated to Jordan Algebras

Hongbo Zhao
Department of Mathematics

Abstract

In this thesis, we study the VOA $V_{\mathcal{J},r}$ associated to a Hermitian type Jordan algebra $\mathcal{J}$, and mainly discuss the case when $\mathcal{J}$ is of $B$ type. The content of this thesis is divided into three parts. The first part consists of Chapter 1, Chapter 2, and Chapter 3. We give a brief account of the theory of vertex algebra. We also review the VOAs $\tilde{V}_{\mathcal{J},1}$, $V_{\mathcal{J},r}$, where $\mathcal{J}$ is a $B$ type Jordan algebra. The second part is from Chapter 4 to Chapter 6. We discuss some further properties of the VOA $V_{\mathcal{J},r}$, where $\mathcal{J}$ is a $B$ type Jordan algebra. We compute the correlation function of the fields associated to $V_{\mathcal{J},r}$, and give explicit constructions of the simple quotients $V_{\mathcal{J},r}$ when $r \in \mathbb{Z}_{\neq 0}$, using dual pair type constructions. We also calculate the character formula for the simple quotients $V_{\mathcal{J},r}$, $r = -2n$, $n \geq 1$. The third part is Chapter 7. We sketch the constructions of $V_{\mathcal{J},r}$ for all Hermitian type Jordan algebras $\mathcal{J}$. We also propose some problems for future study.
Chapter 1

Introduction

A vertex algebra is a vector space $V$, together with a family of bilinear maps labelled by $n \in \mathbb{Z}$:

$$V \times_n V \to V,$$

or equivalently, a family of $\mathbb{Z}$-labelled linear maps:

$$a \mapsto a(n) \in \text{End}(V), \text{ for all } a \in V, n \in \mathbb{Z},$$

such that certain axioms are satisfied. The concepts of vertex operator algebra (VOA) or vertex operator superalgebra (VOSA) are slight modifications of the vertex algebra. The precise definitions of these concepts will be given in Chapter 2.

Vertex operator algebra has its origins in mathematical physics. The name ‘vertex operators’ appeared in physics literatures much earlier in the late 1960s when Fubini, Veneziano and Gordan studied the dual resonance model in particle physics [FGV69]. But it was mathematicians who first studied vertex operator algebra systematically. Vertex operator algebra has a profound influence on both physics and mathematics. Not only does it lay a rigorous foundation for two dimensional conformal field theory which is extensively studied by physicists [BPZ84], but it also has intimate relationship with the representation theory of Kac-Moody algebras and finite simple groups.

We now briefly review the history of vertex operator algebra, which started from the study in representation theory of affine Lie algebras. Suppose $\mathfrak{g}$ is a simply laced finite dimensional simple Lie algebra, and $\hat{\mathfrak{g}}$ is the affine Lie algebra associated to $\mathfrak{g}$ (see for example, [Kac94]). In their paper [FK80], Kac and Frenkel constructed basic level-one irreducible representations of $\hat{\mathfrak{g}}$ using vertex operators. They gave the name ‘vertex operators’, because their expressions resemble the ‘vertex operators’ appeared in [FGV69]. In an
earlier work [LW78], Lepowsky and Wilson gave a similar ‘twisted construction’ of the basic representation for $\hat{\mathfrak{sl}}_2$, and this twisted construction was further generalized to all affine Lie algebras $\hat{\mathfrak{g}}$ by Kac, Kazhdan, Lepowsky and Wilson in [KKLW81].

A deep connection between vertex operators and finite group theory was also found later. In [FLM84], Frenkel, Lepowsky and Meurman used vertex operators to study the Conway-Norton conjecture about the monster group $\mathbb{M}$ [CN]. They constructed a $\mathbb{Z}_{\geq 0}$-graded vector space

$$V^2 = V_0^2 \oplus V_1^2 \oplus V_2^2 \cdots$$

called the ‘moonshine module’, such that $dim(V_0^2) = 1, dim(V_1^2) = 0$, the monster group $\mathbb{M}$ acts on each graded space $V_i^2$, and after multiplying $q^{-1}$, the $q$-graded character of $V^2$ equals the normalized $j$-invariant $J(q)$:

$$q^{-1}(\sum_{k \geq 0} dim(V^2)_k q^k) = J(q) \overset{\text{def}}{=} j(q) - 744 = q^{-1} + 196884q + \cdots.$$  

Here $j(q)$ is the Klein’s $j$-invariant (see for example, [Apo76]). By using vertex operators, for each $a \in V_2$, they associated a formal power series $Y(a, z)$:

$$a \mapsto Y(a, z) = \sum_k a(k)z^{-k-1},$$

so that $V_2$ becomes a commutative, but not associative algebra with the operation:

$$a \times b \overset{\text{def}}{=} a(1)b.$$  

They showed that $(V_2, \times)$ is essentially isomorphic to the Griess algebra previously defined by Griess in [Gri82]. They observed that $V$ can be viewed as the ‘affinization’ of $V_2$, just similar to the case of the affine Lie algebra that $\hat{\mathfrak{g}}$ is the affinization of $\mathfrak{g}$. This construction partially confirmed the conjecture of Conway and Norton.

Shortly afterwards in 1986, Borcherds introduced the concept of vertex algebra in his paper [Bor86], in which the axioms characterized the algebraic relations between $a(n), a \in V, n \in \mathbb{Z}$, if we are given a vertex algebra $V$. In the book [FLM89], Frenkel, Lepowsky and Meurman gave the details of the construction for the moonshine module $V^2$. They introduced the notion of vertex operator algebra, which is a vertex algebra with the Virasoro element, and satisfies some other compatibility conditions. In particular, $V^2$ has a structure of vertex operator algebra. In 1992, Borcherds finally solved the Conway-Norton conjecture in his paper [Bor92] based on the vertex operator algebra structure of $V^2$.

We now discuss the relation between vertex operator algebras and some other mathematical objects such as Lie algebras and Jordan algebras. Given a $\mathbb{Z}_{\geq 0}$ graded VOA $V = \bigoplus_{i=0}^{\infty} V_i$ with $\dim(V_0) = 1$, the weight one
subspace $V_1$ has a structure of Lie algebra, with the operation given by $[a, b] = a(0)b$, for all $a, b \in V_1$. The level $k$ VOA $V^k(g)$ associated to a simple Lie algebra $g$ gives such an example (see Section 2.5). Similarly, it is also known that if $\dim(V_0) = 1$, $\dim(V_1) = 0$, then $V_2$ has a structure of commutative (but not necessarily associative) algebra, with the operation $a \circ b = a(1)b$. For example, the moonshine VOA $V^\sharp$ mentioned before satisfies these properties. We call $V_2$ of a VOA $V$ with $\dim(V_0) = 1$, $\dim(V_1) = 0$ the Griess algebra of $V$ because of Griess’s work [Gri82], and its connection to $V^\sharp$ established in [FLM84].

In [Lam99], Lam constructed vertex algebras whose Griess algebras are simple Jordan algebras of type $A$, $B$, and $C$, and the construction for type $D$ case is partially obtained by Ashihara in [Ash11]. In [AM09], Ashihara and Miyamoto constructed for type $B$ Jordan algebra $\mathcal{J}$, a family of VOAs $V_{\mathcal{J},r}$ parameterized by a complex number $r$, such that $(V_{\mathcal{J},r})_0 = \mathbb{C}1$, $(V_{\mathcal{J},r})_1 = \{0\}$, and

$$(V_{\mathcal{J},r})_2 \cong \mathcal{J}.$$  

The VOA $V_{\mathcal{J},r}$ is further studied by Niibori and Sagaki in [NS10].

The main results in this thesis are based on two articles [Zha16a, Zha16b], which gave a further study of the VOA $V_{\mathcal{J},r}$, where $\mathcal{J}$ is a type $B$ Jordan algebra of rank $d$, $d \geq 2$.

We briefly recall the definition and some facts about Jordan algebras. A Jordan algebra $\mathcal{J}$ is a vector space, together with a bilinear map $\circ : \mathcal{J} \times \mathcal{J} \to \mathcal{J}$ satisfying

$$x \circ y = y \circ x,$$

$$(x \circ y) \circ (x \circ x) = x \circ (y \circ (x \circ x)),$$

for all $x, y \in \mathcal{J}$. A Jordan algebra $\{\mathcal{J}, \circ\}$ is commutative, which follows from the definition, but it is not necessarily associative.

Given an associative algebra $A$, it has a structure of Jordan algebra with the operation given by:

$$x \circ y = \frac{1}{2}(xy + yx), \text{ for all } x, y \in A.$$

We call $\{\mathcal{J}, \circ\}$ a special Jordan algebra if it is isomorphic to a Jordan subalgebra of $\{A, \circ\}$, for some associative algebra $A$. Otherwise we say the Jordan algebra $\{\mathcal{J}, \circ\}$ is exceptional.

The classification of finite dimensional simple Jordan algebras over $\mathbb{C}$ is well known [Alb47]. According to Jacobson’s notation in [JJ49], they are exactly the special Jordan algebras of type $A, B, C, D$, together with an exceptional one called type $E$ Jordan algebra, which is not special. We now describe type $A$ and type $B$ Jordan algebras. Let $(\mathfrak{h}, (\cdot, \cdot))$ be a finite dimensional vector
space with a non-degenerate symmetric bilinear form \((\cdot, \cdot)\), and \(\dim(\mathfrak{h}) = d\). Then \(\mathfrak{h} \otimes \mathfrak{h}\) has an associative algebra structure:

\[(a \otimes b)(u \otimes v) = (b, u)a \otimes v,\]

which induces a Jordan algebra structure on \(\mathfrak{h} \otimes \mathfrak{h}\):

\[x \circ y = \frac{1}{2}(xy + yx), \text{ for all } x, y \in \mathfrak{h} \otimes \mathfrak{h}.\]

This Jordan algebra is called the type \(A\) Jordan algebra of rank \(d\), and we set \(\tilde{L}_{a,b} \overset{\text{def.}}{=} a \otimes b \in \mathfrak{h} \otimes \mathfrak{h}\).

Let \(\mathcal{J}\) be the Jordan subalgebra of \(\mathfrak{h} \otimes \mathfrak{h}\) consists of symmetric square tensors:

\[\mathcal{J} \overset{\text{def.}}{=} \text{span}\{L_{a,b} | a, b \in \mathfrak{h}\}, \quad L_{a,b} \overset{\text{def.}}{=} a \otimes b + b \otimes a.\]

Then \(\mathcal{J} \simeq S^{2}(\mathfrak{h})\) is essentially the Jordan algebra of symmetric \(d \times d\) matrices, which is called the type \(B\) simple Jordan algebra of rank \(d\).

Through the isomorphism \((V_{\mathcal{J}}, r)_{2} \cong \mathcal{J}\), given \(L_{a,b} \in \mathcal{J}\), we have a corresponding element \(L_{a,b} \in (V_{\mathcal{J}}, r)_{2}\) and the field \(L_{a,b}(z)\) (see [AM09]). In [NS10], Niibori and Sagaki proved that \(V_{\mathcal{J},r}\) is generated by these fields. We compute the correlation function of these fields:

**Theorem 1.1** ([Zha16a]). Given \(n\) vertex operators \(L_{a_{1},b_{1}}(z_{1}), \ldots, L_{a_{n},b_{n}}(z_{n})\), and the corresponding sequence \(T = (a_{1}, b_{1}) \cdots (a_{n}, b_{n})\). Let \(DR(T)\) denote the set of permutations of sub indices \(\{1, \ldots, n\}\) without fixpoint, and \(c(\sigma)\) denote the number of disjoint cycles of an element \(\sigma \in DR(T)\). Then the corresponding correlation function is given by

\[\langle \mathbf{1}', L_{a_{1},b_{1}}(z_{1}) \cdots L_{a_{n},b_{n}}(z_{n}) \cdot 1 \rangle = \sum_{\sigma \in DR(T)} \Gamma(\sigma, T)\Gamma(\sigma; Z)r^{c(\sigma)}, \quad (1.1)\]

where the symbols are described as follows: \(\mathbf{1}'\) denotes the unique element in \((V_{\mathcal{J},r})_{0}^{*}\) such that

\[\langle \mathbf{1}', 1 \rangle = 1.\]

Assume that \(\sigma = (C_{1}) \cdots (C_{s}) = (k_{11} \cdots k_{1t_{1}}) \cdots (k_{s1} \cdots k_{s_{t_{s}}})\), then

\[\Gamma(\sigma, T) \overset{\text{def.}}{=} 2^{-s-n} \prod_{i=1}^{s} Tr(L_{a_{k_{1i}}, b_{k_{1i}}} \cdots L_{a_{k_{ti}}, b_{k_{ti}}}),\]

\[\Gamma(\sigma; Z) \overset{\text{def.}}{=} \prod_{i=1}^{n} \frac{1}{(z_{i} - z_{\sigma(i)})^{2}}.\]
where $\text{Tr}(a \otimes b)$ is the trace of $a \otimes b \in \mathfrak{h} \otimes \mathfrak{h}$ given by:

$$\text{Tr}(a \otimes b) = (a, b).$$

Another main result in [NS10] is that $V_{\mathcal{J},r}$ is simple if and only if $r \notin \mathbb{Z}$. Let $\bar{V}_{\mathcal{J},r}$ denote the simple quotient of $V_{\mathcal{J},r}$. We show that $\bar{V}_{\mathcal{J},r}$, $r \in \mathbb{Z}_{\neq 0}$ can be constructed using dual-pair type constructions. To realize all the cases, we need to consider superspaces and the corresponding orthosymplectic ‘super-groups’ (see for example, [Ser01], [LZ16], and [LZ14]). Let $W = W_0 \oplus W_1$ be an orthosymplectic superspace with $\text{sdim}(W) = (m|2n)$, and $\text{Osp}(m|2n)$ be the corresponding ‘supergroup’ which means the ‘super-Harish-Chandra pair’ $(\text{osp}(m|2n), O(m) \times \text{Sp}(2n))$ [DKW+99]. Let $\mathcal{H}(\mathfrak{h})$ be the Heisenberg VOA associated to $\mathfrak{h}$, and $\mathcal{A}(W)$ be the symplectic fermion VOSA associated to $W$ (for the definitions, see Section 2.3). Our result shows that:

**Theorem 1.2** ([Zha16b]).

1. The simple quotients $\bar{V}_{\mathcal{J},r}, r \in \mathbb{Z}_{\neq 0}$ are isomorphic to

   $$(\mathcal{H}(\mathfrak{h} \otimes W_0) \otimes \mathcal{A}(\mathfrak{h} \otimes W_1))^\text{Osp}(m|2n),$$

   where $W$ is the superspace determined by the value of $r$ as follows:

   **Case 1**, $r = m, m \geq 1$ : $\text{sdim}(W) = (m|0)$.

   **Case 2**, $r = -2n, n \geq 1$ : $\text{sdim}(W) = (0|2n)$.

   **Case 3**, $r = -2n + 1, n \geq 1$ : $\text{sdim}(W) = (1|2n)$.

2. $\bar{V}_{\mathcal{J},r}$ is generated by $(\bar{V}_{\mathcal{J},r})_2$.

3. $(\bar{V}_{\mathcal{J},r})_0 = \mathbb{C}1, (\bar{V}_{\mathcal{J},r})_1 = \{0\}$. The Griess algebra $(\bar{V}_{\mathcal{J},r})_2$ is isomorphic to the type B Jordan algebra $\mathcal{J}$, and there is a Virasoro element $\omega$ with central charge $d r$.

As by products, we reprove that $V_{\mathcal{J},r}$ is simple if and only if $r \notin \mathbb{Z}_{\neq 0}$, and we can give natural explanations to some results in [NS10].

We also compute the character formula for $\bar{V}_{\mathcal{J},r}, r \in \mathbb{Z}_{\neq 0}$ in case 2, $r = -2n, n \geq 1$. Let $\Phi_+$ be the root system of $\text{osp}(1|2n)$, $\Phi_0$ and $\Phi_1$ be even and odd roots of $\text{osp}(1|2n)$ respectively. Then $\Phi_0$ is the root system of $\text{sp}(2n)$, and the root system $\Phi$ of $\text{so}(2n + 1)$ can be realized as the sub root system of $\Phi_+$ such that $\Phi^+ \subseteq \Phi^+$. Let $\Lambda_+^0$ be the set of dominant integral weights of $\text{sp}(2n)$, $L(\lambda)$ be the simple $\text{sp}(2n)$-module with the highest weight $\lambda \in \Lambda_+^0$. 


Let $\rho_0$ be the half sum of positive roots in $\Phi_0$, $\rho_1$ be the half sum of positive roots in $\Phi_1$. Let $m^0_{\lambda_1, \ldots, \lambda_d}$ denote the dimension of the ‘multiplicity space’:

$$m^\mu_{\lambda_1, \ldots, \lambda_d} \overset{\text{def.}}{=} \dim(\Hom_{\mathfrak{sp}(2n)}(L(\mu), L(\lambda_1) \otimes \cdots \otimes L(\lambda_d))), \quad \mu, \lambda_i \in \Lambda^+_0,$$

which is related to the Clebsch-Gordan coefficients of $\mathfrak{sp}(2n)$. Our result shows that:

**Theorem 1.3 ([Zha16b]).** Let $P(q)$ be the generating function of integer partitions:

$$P(q) \overset{\text{def.}}{=} \prod_{j \geq 1} (1 - q^j)^{-1}.$$

Define the ‘branching function’ $B_\lambda(q)$:

$$B_\lambda(q) \overset{\text{def.}}{=} q^{\frac{1}{2}(\lambda + \rho_1, \lambda + \rho_1) - \frac{1}{2}(\rho_1, \rho_1)} P(q)^\alpha \prod_{\alpha \in \Phi^+} (1 - q^{(\lambda + \rho_0, \alpha)}) \quad (1.2)$$

Then:

$$\Tr_{\mathcal{V}_{\mathcal{J},r}} q^{L(0)} = \sum_{\lambda_1, \ldots, \lambda_d \in \Lambda^+_0} m^0_{\lambda_1, \ldots, \lambda_d} B_{\lambda_1}(q) \cdots B_{\lambda_d}(q).$$

Recall that the type $A$, $B$ and $C$ simple Jordan algebras are called Hermitian Jordan algebras, in which the type $B$ and type $C$ Jordan algebras can be realized as Jordan subalgebras of the type $A$ Jordan algebras by taking involutions (see for example, [McC06]). In the final part of this thesis, we construct the VOA $\mathcal{V}_{\mathcal{J},r}$ associated to a type $A$ or type $C$ Jordan algebra $\mathcal{J}$, such that $(\mathcal{V}_{\mathcal{J},r})_0 = \{0\}, (\mathcal{V}_{\mathcal{J},r})_1 = 1$, and $(\mathcal{V}_{\mathcal{J},r})_2 \cong \mathcal{J}$. We also give a uniform construction of $\mathcal{V}_{\mathcal{J},r}$ where $\mathcal{J}$ is a Hermitian Jordan algebra. A further study of these VOAs will made in the future.

This thesis is organized as follows. In Chapter 2 we provide a brief introduction to the theory of vertex algebra, and give some examples. In Chapter 3 we review the main results in [Lam99], [AM09], and [NS10]. In Chapter 4, Chapter 5 and Chapter 6, we summarize the results in [Zha16a] and [Zha16b], and give details to the proofs. In Chapter 7, we give the construction of $\mathcal{V}_{\mathcal{J},r}$ associated to a Hermitian Jordan algebra $\mathcal{J}$ of rank $d, d \geq 2$. 

6
Chapter 2

A Brief Review of Vertex Algebras

In this chapter, we give a brief introduction to the theory of vertex algebra. For our purpose, we need to deal with the super case, and introduce the notion of vertex superalgebras and vertex operator superalgebras (VOSA). Our notations and organizations mainly follow [Kac98]. Some details can be found in [Kac98], [FBZ04], and [FLM89]. We will also give some examples at the end of this chapter. All vector spaces are assumed to be over \( \mathbb{C} \).

2.1 The Definition of Vertex Superalgebras and Vertex Operator Superalgebras

We first recall some basic notations about superspaces and Lie superalgebras. More details about Lie superalgebras can be found, for example, in [Kac77].

A superspace \( V \) is a \( \mathbb{Z}/2\mathbb{Z} \)-graded vector space:

\[
V = V_0 \oplus V_1,
\]

where \( V_0, V_1 \) are called even and odd part of \( V \) respectively. A purely even (or odd) element in \( V \) is called a homogeneous element. Introduce the parity function \( p(\cdot) \) on \( V \):

\[
p(a) = \begin{cases} 
0, & a \in V_0, \\
1, & a \in V_1.
\end{cases}
\]
Then the endomorphism space $\text{End}(V)$ is also $\mathbb{Z}/2\mathbb{Z}$-graded:

$$\text{End}(V) = \text{End}(V)_0 \oplus \text{End}(V)_1.$$ 

Here

$$\text{End}(V)_\alpha \overset{\text{def.}}{=} \{x | x \in \text{End}(V), xV_\beta \subseteq V_{\alpha+\beta}, \alpha, \beta \in \mathbb{Z}/2\mathbb{Z}\},$$

and $\text{End}(V)$ has a structure of Lie superalgebra: for homogeneous elements $x, y \in \text{End}(V)$, the Lie superbracket is given by:

$$[x, y] = xy - (-1)^{p(x)p(y)}yx.$$ 

We remark that all the brackets ‘$[,]$’ in this thesis mean the Lie superbracket.

We use $V[[z, z^{-1}]]$ to denote the formal power series in $n$ variables $z_1, \cdots, z_n$ with coefficients in $V$. We also use $V((z))$ to denote the formal Laurent series with coefficients in $V$.

**Definition 2.1 ([Kac98]).** Let $V$ be a superspace, we call $a(z) \in \text{End}(V)[[z, z^{-1}]]$ a **field** (on $V$), if for all $v \in V$, we have:

$$a(z) \cdot v \in V((z)).$$

We will always write $a(z)$ as follows

$$a(z) = \sum_k a(k)z^{-k-1}$$

by convention, and we call $a(k)$ the $k$-th **mode** of $a(z)$. Then $a(z)$ is a field if and only if for any $v \in V$, there exists an integer $N$ (may depend on $v$ and $a(z)$) such that

$$a(k) \cdot v = 0, \text{ for all } k \geq N.$$ 

We say a field $a(z)$ is homogeneous, if for all $k$, $a(k)$ are purely even (or for all $k$, $a(k)$ are purely odd), and we use $p(a)$ to denote the parity of $a(k)$.

We will define the ‘$n$-th normal ordered product’ of two fields $a(z)$ and $b(z)$, which plays a key role in the theory of vertex superalgebras. Before giving the definition, we first introduce some notations. Let $F(z, w)$ be a rational function in two variables with poles only at $z = 0, w = 0$ and $z = w$. We use $t_{z,w}F(z, w)$ to denote the formal power series which is obtained by expanding $F(z, w)$ in the domain $|z| > |w|$. It may happen that for a given function $F(z, w)$, we get different formal power series when expanding it in different domains.
For example, for $F(z, w) = (z - w)^n$, introduce the following combinatorial coefficient which applies to all $m \in \mathbb{Z}, j \in \mathbb{Z}_{\geq 0}$:

$$
\binom{m}{j} \overset{\text{def.}}{=} \begin{cases} 
\frac{m(m-1)\cdots(m-j+1)}{j!}, & j \geq 1, \\
1, & j = 0.
\end{cases}
$$

When $n \geq 0$, we have:

$$
\iota_{z,w}(z - w)^n = \iota_{w,z}(z - w)^n = \sum_{k \geq 0} \binom{n}{k} (-1)^k w^k z^{n-k}.
$$

But when $n < 0$, we have:

$$
\iota_{z,w}(z - w)^n = \sum_{k \geq 0} \binom{n}{k} (-1)^k w^k z^{n-k},
$$

hence

$$
\iota_{w,z}(z - w)^n = \sum_{k \geq 0} \binom{n}{k} (-1)^{n+k} k z^{n-k} \neq \iota_{z,w}(z - w)^n
$$

in this case.

Given a field $a(z) = \sum_k a(k) z^{-k-1}$, the ‘formal residue’ is defined to be the coefficient of the term $z^{-1}$:

$$
\text{Res}_z a(z) \overset{\text{def.}}{=} a(0).
$$

We note that for a field $a(z)$, we can express its $n$-th mode $a(n)$ using the formal residue:

$$
a(n) = \text{Res}_z (z^n a(z)).
$$

The ‘$n$-th normal ordered product’ of two fields is defined as follows.

**Definition 2.2** ([Kac98]). *Given two homogeneous fields $a(w), b(w)$, we call $a(w)_{(n)} b(w)$ the $n$-th normal ordered product, or simply the ‘$n$-th’ product of $a(w), b(w)$:*

$$
a(w)_{(n)} b(w) \overset{\text{def.}}{=} \text{Res}_w [a(z) b(w) \iota_{z,w}(z - w)^n - (-1)^{p(a)p(b)} b(w) a(z) \iota_{w,z}(z - w)^n].
$$

The expansion shows that the $l$-th mode of $a(w)_{(n)} b(w)$ is given by

$$
\text{Res}_w (w^l a(w)_{(n)} b(w))
\quad = \sum_{i \geq 0} \binom{n}{i} (-1)^i (a(-i + n)b(i + l) - (-1)^{n+p(a)p(b)} b(-i + n + l)a(i)). \quad (2.1)
$$
From (2.1), it is easy to check that \( a(w)_{(n)} b(w) \) is also a field.

We also recall the notions of ‘normal ordering’ and ‘normal ordered product’, which are frequently used in literatures (for example, [Kac98]). For two modes \( a(k), b(l) \) of the homogeneous fields \( a(z), b(w) \), we define the **normal-ordering** of the product \( a(k)b(l) \) as

\[
:a(k)b(l): = \begin{cases} 
  a(k)b(l), & k < l, \\
  (-1)^{p(a)p(b)}b(l)a(k), & k \geq l.
\end{cases}
\]

We use \( :a(w)b(w): \) to denote the normal-ordered product of \( a(w), b(w) \):

\[
:a(w)b(w): \overset{\text{def}}{=} \sum_n \left( \sum_{k+l=n} :a(k)b(l): \right) w^{-n-1}.
\]  

(2.2)

By (2.1), it is checked that for all \( n \geq 0 \),

\[
a(w)_{(-n-1)} b(w) =: (\frac{\partial^n}{n!} a(w)) b(w),
\]

where \( \partial_w \) means the derivative with respect to the variable \( w \). Then we observe that the normal ordered product \( :a(w)b(w): \) is equal to the \(-1\)-th product \( a(w)_{(-1)} b(w) \) which is previously defined.

We recall the following property about the \( n \)-th product of two fields:

**Lemma 2.1** ([Kac98]). Given two fields \( a(w), b(w) \in \text{End}(V)[[w, w^{-1}]], \) and \( T \in \text{End}(V)_0 \) such that

\[
[T, a(w)] = \partial_w a(w), \quad [T, b(w)] = \partial_w b(w),
\]

then for all \( n \in \mathbb{Z} \) we have:

\[
[T, a(w)_{(n)} b(w)] = \partial_w (a(w)_{(n)} b(w)).
\]

This is proved by a direct calculation based on the following easy facts:

\[
\partial_z t_z w(z-w)^n = t_z w \partial_z (z-w)^n = -\partial_w t_z w(z-w)^n, \quad \text{for all } n \in \mathbb{Z},
\]

\[
\text{Res}_w (\partial_w c(z, w)) = 0, \quad \text{for all } c(z, w) \in \text{End}(V)[[z, z^{-1}, w, w^{-1}]].
\]

For the details see [Kac98].

The ‘locality property’ is important in defining vertex superalgebras. We now introduce the notion of ‘mutually local fields’.
Definition 2.3 ([Kac98]). Given two fields \(a(w), b(w) \in \text{End}(V)[[w, w^{-1}]]\), we say \(a(w)\) and \(b(w)\) are mutually local, or equivalently, \(a(w)\) and \(b(w)\) form a local pair, if there exists a non-negative integer \(N\) (may depend on \(a(w), b(w)\)) such that

\[
(z - w)^N[a(z), b(w)] = 0.
\]

We remark that the definition allows the case that a field \(a(z)\) is mutually local with itself.

It is checked that we have:

Lemma 2.2 ([Kac98]). If \(a(w), b(w)\) are mutually local fields, then \(\partial_w a(w)\) and \(b(w)\) are also mutually local fields.

The following lemma provides a direct way of checking the locality of two fields:

Lemma 2.3 ([Kac98]). For two fields \(a(w)\) and \(b(w)\), if there exists a non-negative integer \(N\) such that:

\[
[a(z), b(w)] = \sum_{k=0}^{N} c_k(w)(\iota_{z,w}(z - w)^{-k} - \iota_{w,z}(z - w)^{-k}),
\]

(2.3)

where \(c_k(w) \in \text{End}(V)[[w, w^{-1}]]\), then

\[
(z - w)^{N+1}[a(z), b(w)] = 0.
\]

In particular \(a(w), b(w)\) are mutually local.

The proof of this lemma follows from the facts that

\[
\iota_{z,w}(z - w)^n - \iota_{w,z}(z - w)^n = 0, \text{ for all } n \geq 0,
\]

and

\[
(z - w)^n \iota_{z,w}(z - w)^k = \iota_{z,w}(z - w)^{n+k}, \text{ for all } n \geq 0.
\]

Therefore we have

\[
(z - w)^{N+1}[a(z), b(w)] = \sum_{k=0}^{N} c_k(w)(\iota_{z,w}(z - w)^{N-k} - \iota_{w,z}(z - w)^{N-k}) = 0.
\]
We explain some notations which are frequently used in literatures (for example, [Kac98]), but are not used in this thesis. The following ‘operator product expansion’

\[ a(z)b(w) \sim \sum_{k=0}^{N} c_k(w)(z-w)^{-k-1} \]

essentially means the identity (2.3). The ‘formal delta function’ \( \delta(z-w) \) means the formal power series

\[ \iota_{z,w}(z-w)^{-1} - \iota_{w,z}(z-w)^{-1} \]

The name ‘formal delta function’ is given, because we can check the following identity

\[ \text{Res}_z[a(z)\delta(z-w)] = a(w), \text{ for all } a(z) \in \text{End}(V)[[z,z^{-1}]], \]

which is similar to the property of ‘Dirac delta function’ appeared in analysis.

The following Theorem called Dong’s lemma about locality is very useful in the theory of vertex superalgebras.

**Theorem 2.1** (Dong’s Lemma, [Kac98]). If \( a(z), b(z), c(z) \) are three mutually local fields, then for all \( n \in \mathbb{Z} \), \( a(z)_{(n)}b(z) \) and \( c(z) \) are mutually local.

For the proof, see for example, [Kac98].

Now we can give the definition of vertex superalgebra.

**Definition 2.4** ([Kac98]). A **vertex superalgebra** is a quadruple \( \{V, T, 1, Y(\cdot, z)\} \), where \( V \) is a superspace, \( T \in \text{End}(V)_{\bar{0}} \) is a linear map called translation, \( 1 \in V_{\bar{0}} \) is a distinguished vector called vacuum, \( \text{Id}_V \in \text{End}(V) \) is the identity map over \( V \), and \( Y(\cdot, z) : a \mapsto Y(a, z) \) is a parity-preserving linear map from \( V \) to \( \text{End}(V)[[z,z^{-1}]] \), such that:

1. \([T, Y(a, z)] = \partial_z Y(a, z)\).
2. \(T \cdot 1 = 0, \ Y(a, z) \cdot 1 \in V[[z]], \ Y(a, z) \cdot 1|_{z=0} = a, \ Y(1, z) = \text{Id}_V.\)
3. For all \( a, b \in V \), \( Y(a, z) \) and \( Y(b, w) \) are mutually local fields.
If $V_{1} = \{0\}$, we simply call $V$ a vertex algebra.

A vertex operator superalgebra (VOSA) is a vertex superalgebra $V$, which contains an even element $\omega \in V_{0}$ called the Virasoro element, satisfying the compatibility conditions below: set

$$L(n) \overset{\text{def.}}{=} \omega(n+1).$$

Then the compatibility conditions for $\omega$ and $V$ are:

1. the modes $L(n)$ satisfy the Virasoro relation:

$$[L(m), L(n)] = (m-n)L(m+n) + \delta_{m+n,0} \frac{m^3 - m}{12} C \cdot Id_{V},$$

and

$$T = L(-1)$$

holds. Here $C \in \mathbb{C}$ is a constant called the central charge of $\omega$ (or $V$).

2. $V$ is a direct sum of $L(0)$-eigenspaces with half-integer eigenvalues bounded from below:

$$V = \bigoplus_{k \in \frac{1}{2} \mathbb{Z}} V_{k}, \quad V_{k} \overset{\text{def.}}{=} \{ v | L(0)v = kv, v \in V \},$$

there exists $N$ s.t., $V_{k} = \{0\}$, for all $k < N$.

And each eigenspace is finite dimensional:

$$\dim(V_{k}) < +\infty.$$ 

If $V_{1} = \{0\}$ and the eigenvalues of $L(0)$ are in $\mathbb{Z}$, we call $V$ a vertex operator algebra (VOA).

It is not easy to give non-trivial examples of VOAs and VOSAs at this moment. We will provide some examples after establishing the uniqueness theorem and the reconstruction theorem in next two sections.

### 2.2 The Uniqueness Theorem and Some Useful Identities

The following ‘uniqueness theorem’ is very useful in establishing the reconstruction theorem and some other identities. The locality property plays a key role in the proof.
Theorem 2.2 (Uniqueness theorem, [Kac98]). Given a superspace $V$, a ‘vacuum element’ $1 \in V_0$, and a parity preserving linear map $Y(\cdot, z) : V \to \text{End}(V)[[z, z^{-1}]]$ such that

1. For all $a, b \in V$, $Y(a, z)$ and $Y(b, w)$ are mutually local fields.

2. $Y(a, z) \cdot 1 \in V[[z]]$, $Y(a, z) \cdot 1|_{z=0} = a$.

Assume that $A(z)$ is a field on $V$ which is mutually local with $Y(b, z)$, for all $b \in V$, and there exists $a \in V$ such that

$$A(z) \cdot 1 = Y(a, z) \cdot 1.$$ 

Then we have

$$A(z) = Y(a, z).$$

In particular, the theorem holds when $V$ is a vertex superalgebra.

Proof of Theorem 2.2. We may assume that $Y(a, z)$ and $A(z)$ are both homogeneous. Take arbitrary homogeneous element $b \in V$. Because $Y(b, z)$ is mutually local with $Y(a, z)$ and $A(z)$, there exists a non-negative integer $N$ such that

$$(z - w)^N[A(z), Y(b, w)] = (z - w)^N[Y(a, z), Y(b, w)] = 0.$$ 

Therefore we have

$$(-1)^{p(a)p(b)}(z - w)^N A(z) Y(b, w) \cdot 1$$

$$= (z - w)^N Y(b, w) A(z) \cdot 1$$

$$= (z - w)^N Y(b, w) Y(a, z) \cdot 1$$

$$= (-1)^{p(a)p(b)}(z - w)^N Y(a, z) Y(b, w) \cdot 1.$$ 

Take $w = 0$ and use $Y(b, z) \cdot 1|_{w=0} = b$, we have

$$z^N A(z) b = z^N Y(a, z) b.$$ 

Our choice of $b \in V$ is arbitrary, hence we have

$$z^N A(z) = z^N Y(a, z),$$

14
which implies

\[ A(z) = Y(a, z). \]

We can prove some identities using the uniqueness theorem.

**Corollary 2.1** ([Kac98]). Given a vertex superalgebra \( V \), for all \( a \in V \), we have

\[ Y(Ta, z) = \partial_z Y(a, z). \]  \hspace{1cm} (2.4)

**Proof.** Take \( A(z) = \partial_z Y(a, z) \). Because \([T, Y(a, z)] = \partial_z Y(a, z)\) and \( T \cdot 1 = 0 \), it follows that

\[ A(z) \cdot 1 = [T, Y(a, z)] \cdot 1 = Y(Ta, z) \cdot 1. \]

Then we conclude the proof using Lemma 2.2 and Theorem 2.2.

**Proposition 2.1** ([Kac98]). Given a vertex superalgebra \( V \), for all \( a, b \in V \), we have:

\[ Y(a(n)b, w) = Y(a, w)(n)Y(b, w). \]

Equivalently, for homogeneous elements \( a, b \in V \), we have:

\[ (a(n)b)(l) = \sum_{i \geq 0} \binom{n}{i} (-1)^i (a(-i + n)b(i + l) - (-1)^{n+p(a)p(b)}b(-i + n + l)a(i)). \]  \hspace{1cm} (2.5)

We sketch the proof of Proposition 2.1. A direct computation shows that the field

\[ X(w) \overset{\text{def}}{=} Y(a, w)(n)Y(b, w) \]

satisfies

\[ X(w) \cdot 1 \in V[[w]], \quad X(w) \cdot 1 \big|_{w=0} = a(n)b. \]

It follows that

\[ X(w) \cdot 1 = Y(a(n)b, w) \cdot 1. \]

Therefore by Theorem 2.1 and Theorem 2.2, Proposition 2.1 holds. For the details see for example, [Kac98].
We note that when $m \geq 0$,
\[
z^m = (z-w+w)^m = \sum_{j \geq 0} \binom{m}{j} (z-w)^j w^{m-j}
\]
\[= t_{z,w} \sum_{j \geq 0} \binom{m}{j} (z-w)^j w^{m-j}
\]
\[= t_{w,z} \sum_{j \geq 0} \binom{m}{j} (z-w)^j w^{m-j}.
\]

And when $m < 0$, for any integer $N \geq 0$, we have
\[
z^m - \sum_{j \geq 0} \binom{m}{j} (z-w)^j w^{m-j} \in (z-w)^{N+m+2} F(z,w), F(z,w) \in \mathbb{C}[z, w, z^{-1}, w^{-1}].
\]

By Proposition 2.1, we have:

**Corollary 2.2 ([Kac98]).**
\[
[a(m), Y(b,z)] = \sum_{j \geq 0} \binom{m}{j} Y(a(j)b, z) z^{m-j}.
\]

Equivalently:
\[
[a(m), b(n)] = \sum_{j \geq 0} \binom{m}{j} (a(j)b)(m + n - j). \quad (2.6)
\]

For the details see [Kac98]. In many cases the computation of the right hand side in (2.6) is easier than the left hand side. The identities (2.4), (2.5) and (2.6) will be frequently used throughout this thesis. An application of these identities will appear in Example 1, Section 2.3.

We also have:

**Proposition 2.2** (Borcherds’s identity, [Kac98]). Let
\[
F(z,w) \overset{\text{def.}}{=} z^m(z-w)^n w^l, m,n,l \in \mathbb{Z}.
\]

Then for $a, b$ homogeneous, we have:
\[
\text{Res}_z (Y(a,z)Y(b,w) t_{z,w} F(z,w) - (-1)^{p(a)p(b)} Y(b,w)Y(a,z) t_{w,z} F(z,w))
\]
\[= \text{Res}_{z-w} (Y(Y(a, z-w)b, w) t_{z-w,w} F(z,w)).
\]
Equivalently we have:

\[
\sum_{j \geq 0} (a(n+j)b)(m+k-j) = \sum_{j \geq 0} (-1)^j \binom{n}{j} a(m+n-j)b(k+j) - (-1)^{p(a)p(b)} \sum_{j \geq 0} (-1)^{j+n} \binom{n}{j} b(n+k-j)a(m+j). \tag{2.7}
\]

For the proof, see [Kac98].

Finally, we have another identity whose proof can also be found in [Kac98]:

**Proposition 2.3.** For homogeneous elements \(a, b \in V\) we have:

\[
Y(a, z)b = (-1)^{p(a)p(b)} e^{zT} Y(b, -z)a,
\]

or equivalently,

\[
a(m)b = \sum_{k \geq 0} (-1)^{p(a)p(b)+m+k} \frac{T^k}{k!} b(m+k)a. \tag{2.8}
\]

We remark that in Borcherds’s paper [Bor86], he used different but equivalent conditions defining vertex algebra, where (2.7) and (2.8) appeared as axioms.

## 2.3 The Reconstruction Theorem and Some Basic Examples

The following reconstruction theorem tells that, we can construct vertex superalgebras from a family of mutually local fields.

**Theorem 2.3** (Reconstruction theorem, [Kac98]). Given a superspace \(V\), a ‘vacuum vector’ \(1 \in V_0\), a ‘translation’ \(T \in \text{End}(V)_0\), and a set of vectors \(V_S \subset V\) with \(1 \in V_S\). Assume that for all \(a \in V\), we can associate a corresponding field \(a(z)\). Set

\[
V_S(z) \overset{\text{def}}{=} \{a(z) | a \in V_S\}.
\]

If the following conditions hold:
1. \[ [T, a(z)] = \partial_z a(z), a(z) \cdot 1 \in V[z], a(z) \cdot 1 \big|_{z=0} = a, \quad T \cdot 1 = 0, \text{ for all } a(z) \in V_S(z). \]

2. The field associated to 1 is \( \text{Id}|_V \).

3. \( \text{span}\{ a_i (j_1) \cdots a_i (j_l) \cdot 1 | a_i \in V_S \} = V \).

4. Any pair of fields in \( V_S(z) \) are mutually local.

Then we have a unique vertex superalgebra structure on \( V \), satisfying the following property: Let \( I = i_1 \cdots i_l \) and \( J = j_1 \cdots j_l \) denote the multi-indices.

For \( a = \sum c_{I,J} a_{i_1}(j_1) \cdots a_{i_l}(j_l) \cdot 1 \in V \), we have:

\[
Y(a, z) = \sum c_{I,J} a_{i_1}(z)(j_1) \cdots a_{i_l}(z)(j_l) \cdot \text{Id}|_V.
\]

We simply say the local fields \( V_S(z) \) generate the vertex superalgebra \( V \).

**Proof of Theorem 2.3.** Later on we use capitalized letters \( I \) and \( J \) to denote the corresponding multi-indices \( i_1 \cdots i_l \) and \( j_1 \cdots j_l \). Because \( \text{span}\{ a_i (j_1) \cdots a_i (j_l) \cdot 1 | a_i \in V_S \} = V \).

For an element \( a \in V \), suppose

\[
a = \sum c_{I,J} a_{i_1}(j_1) \cdots a_{i_l}(j_l) \cdot 1
\]

for some \( a_{i_1}, \cdots, a_{i_l} \in V_S \). Define a map \( Y_1(\cdot, z) : a \mapsto Y_1(a, z) \) by:

\[
Y_1(a, z) \overset{\text{def}}{=} \sum c_{I,J} a_{i_1}(z)(j_1) \cdots a_{i_l}(z)(j_l) \cdot \text{Id}|_V.
\]

A direct calculation shows that the axiom (1) and (2) in Definition 2.4 are satisfied by \( Y_1(a, z) \) defined above. By Dong’s lemma (Theorem 2.1), the locality property (axiom (3) in Definition 2.4) is also satisfied by \( Y_1(a, z) \), for all \( a \in V \). It is obvious that \( Y_1(\cdot, z) \) is defined for all \( a \in V \).

It remains to show that if we choose a different expression of \( a \), we will get a field which is equal to \( Y_1(a, z) \). Suppose

\[
a = \sum c'_{I',J'} a'_{i_1}(j'_1) \cdots a'_{i_l}(j'_l) \cdot 1
\]

for another group of elements \( a'_{i_1}, \cdots, a'_{i_l} \in V_S \). We define the following field:

\[
Y_2(a, z) \overset{\text{def}}{=} \sum c'_{I',J'} a'_{i_1}(z)(j'_1) \cdots a'_{i_l}(z)(j'_l) \cdot \text{Id}|_V.
\]
It is already shown that fields $Y_1(a, z)$ satisfy the conditions in Theorem 2.2. We note that

$$Y_2(a, z) \cdot 1|_{z=0} = Y_1(a, z) \cdot 1|_{z=0} = a,$$
and $Y_2(a, z)$ is mutually local with all $Y_1(b, z)$ for all $b \in V$, by Dong’s Lemma (Theorem 2.1). By Theorem 2.2, we have

$$Y_1(a, z) = Y_2(a, z).$$

Hence we conclude the proof of Theorem 2.3.

The proof of Theorem 2.3 also tells us how to construct vertex superalgebras from mutually local fields. We now give some examples. These examples will play an important role in our constructions. We need another lemma which is obtained by a direct calculation using (2.4) and (2.6):

**Lemma 2.4** ([Kac98]). Given a vertex superalgebra $V$, if there is an element $\omega \in V$ and a complex number $C$ such that

$$\omega(0)\omega = T\omega, \quad \omega(1)\omega = 2\omega, \quad \omega(2)\omega = 0, \quad \omega(3)\omega = \frac{C}{2} \cdot Id|_V.$$

Then $L(n) \triangleq \omega(n + 1)$ satisfy the Virasoro relation:

$$[L(m), L(n)] = (m - n)L(m + n) + \delta_{m+n,0} \frac{m^3 - m}{12} C \cdot Id|_V.$$

**Example 1.** Let $\mathfrak{h}$ be a finite dimensional vector space with a symmetric non-degenerate bilinear form $(\cdot, \cdot)$, $\dim(\mathfrak{h}) = d$. Define the following Lie algebra associated to $(\mathfrak{h}, (\cdot, \cdot))$:

$$\hat{\mathfrak{h}} = \mathfrak{h} \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}c,$$

with the Lie bracket:

$$[a(m), b(n)] = m(a, b)\delta_{m+n,0}c, \quad [x, c] = 0, \quad \text{for all } x \in \hat{\mathfrak{h}}.$$

Here $a(m) = a \otimes t^m \in \mathfrak{h} \otimes \mathbb{C}[t, t^{-1}]$, and $c$ is the central element. Take the following Lie algebra decomposition:

$$\hat{\mathfrak{h}}_- \triangleq \mathfrak{h} \otimes \mathbb{C}t^{-1}[t^{-1}],$$

$$\hat{\mathfrak{h}}_+ \triangleq \mathfrak{h} \otimes \mathbb{C}[t] \bigoplus \mathbb{C}c,$$

$$\hat{\mathfrak{h}} = \hat{\mathfrak{h}}_- \bigoplus \hat{\mathfrak{h}}_+.$$

Define the following one dimensional $\hat{\mathfrak{h}}_+$-module $\mathbb{C}1$:

$$x \cdot 1 = 0, \quad c \cdot 1 = 1, \quad \text{for all } x \in \mathfrak{h} \otimes \mathbb{C}[t].$$
Let $U(\mathfrak{g})$ denote the universal enveloping algebra of a Lie algebra $\mathfrak{g}$. Then we have an induced $\hat{\mathfrak{h}}$-module:

$$U(\hat{\mathfrak{h}}) \otimes _{U(\hat{\mathfrak{h}})} \otimes \mathbb{C}.$$

Note that $\hat{\mathfrak{h}}$ is commutative. Let $S(\hat{\mathfrak{h}})$ denote the symmetric algebra associated with $\hat{\mathfrak{h}}$, then by PBW theorem, the induced module $U(\hat{\mathfrak{h}}) \otimes _{U(\hat{\mathfrak{h}})} \otimes \mathbb{C}$ is isomorphic to $S(\hat{\mathfrak{h}}) \cdot 1$ as vector spaces.

For each $a \in \mathfrak{h}$, we associate a corresponding formal power series $a(z)$:

$$a(z) = \sum_k a(k)z^{-k-1}.$$

The commutation relation implies that

$$[a(z), b(w)] = \sum_{i, j \in \mathbb{Z}} [a(i), b(j)]z^{-i-1}w^{-j-1}
= \sum_{i, j \in \mathbb{Z}} i\delta_{i+j,0}(a, b)c z^{-i-1}w^{-j-1}
= \sum_{i \in \mathbb{Z}} i(a, b)c z^{-i-1}w^{i-1}
= c[t_{z,w}(z - w)^{-2} - t_{w,z}(z - w)^{-2}].$$

By Lemma 2.3, $a(w), b(w)$ form a local pair:

$$(z - w)^2[a(z), b(w)] = 0.$$

Introduce an operator $T$ which acts on $\hat{\mathfrak{h}}$ by

$$[T, a(n)] = -na(n - 1), \quad [T, c] = 0,$$

and we extend $T$ to an element in $End(S(\hat{\mathfrak{h}}) \cdot 1)$ by setting $T \cdot 1 = 0$. Then it is clear that the $\hat{\mathfrak{h}}$-module $S(\hat{\mathfrak{h}}) \cdot 1$, the set of fields $\{a(z) | a \in \mathfrak{h}\}$, and the elements $T, 1$ satisfy the conditions in Theorem 2.3. Therefore we conclude that $S(\hat{\mathfrak{h}}) \cdot 1$ has a structure of vertex algebra, and for each element

$$a = \sum c_{I,J}a_i(j_1) \cdots a_i(j_l) \cdot 1, \quad a_i \in \mathfrak{h},$$

the corresponding field is given by:

$$a \mapsto Y(a, z) = \sum c_{I,J}a_i(z)(j_1) \cdots a_i(z)(j_l) \cdot Id_{V}.$$

Let $\{e_i\}$ be an orthonormal basis of $\mathfrak{h}$. Define

$$\omega = \frac{1}{2} \sum_i e_i(-1)e_i(-1) \cdot 1.$$
By (2.6) and (2.5), it is checked that
\[
\omega(0)e_i = \sum_j e_j(-2)e_j(1)e_i(-1) \cdot 1 = Te_i,
\]
\[
\omega(1)e_i = \sum_j e_j(-1)e_j(1)e_i(-1) \cdot 1 = e_i.
\]

By (2.6),
\[
[\omega(0), e_i(-n)] = ne_i(-n-1),
\]
which implies that \(\omega(0) = T\). By (2.6) and (2.5) again, we deduce that
\[
[\omega(1), e_i(-n)] = \left(\frac{1}{0}\right)(e_i(-2) \cdot 1)(-n+1) + \left(\frac{1}{1}\right)e_i(-n) = ne_i(-n).
\]

Therefore \(\omega(1)\) acts semisimply on \(V = S(\hat{h} \cdot 1)\):
\[
V = \bigoplus_{k \in \mathbb{Z}_{\geq 0}} V_k, \quad V_k = \{v | v \in V, \omega(1)v = kv\}.
\]

More precisely,
\[
V_k = \text{span}\{e_i_1(-j_1) \cdots e_i_s(-j_s) \cdot 1 | j_i \geq 1, \sum j_i = k\},
\]
and we check that \(\text{dim}(V_k) < +\infty\), for all \(k \geq 0\). We say an element \(v \in V_k\) is of degree \(k\), denoted by
\[
\text{deg}(v) = k.
\]

We also check that
\[
\omega(1)\omega = 2\omega
\]
\[
\omega(2)\omega = \frac{1}{2} \sum_{i,j} e_j(0)e_j(1)e_i(-1)e_i(-1) \cdot 1 = 0,
\]
\[
\omega(3)\omega = \frac{1}{4} \sum_{i,j} e_j(1)e_j(1)e_i(-1)e_i(-1) \cdot 1 = \frac{d}{2},
\]
using (2.6) and (2.5). By Lemma 2.4, \(L(n) = \omega(n + 1)\) satisfy the Virasoro relation. The central charge is equal to \(d\).

We’ve checked all the compatibility conditions for \(\omega\), hence we conclude that \(S(\hat{h} \cdot 1)\) is a vertex operator algebra. The VOA \(S(\hat{h} \cdot 1)\) is called the Heisenberg VOA associated to \(h\). We denote it by \(H(h)\).

**Example 2.** Let \(W\) be a finite dimensional symplectic space with the symplectic form \(\langle \cdot, \cdot \rangle\), \(\text{dim}(W) = 2n\). Define the corresponding Lie superalgebra:
\[
\hat{W} = W \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}c,
\]
with parities:
\[ p(x) = 1, \quad p(c) = 0, \quad \text{for all } x \in W \otimes \mathbb{C}[t, t^{-1}]. \]

The Lie superbracket is given by:
\[ [a(m), b(n)] = m(a, b)\delta_{m+n,0}, \quad [x, c] = 0, \quad \text{for all } x \in \hat{W}. \]

Here \( a(m) = a \otimes t^m \in W \otimes \mathbb{C}[t, t^{-1}] \), and \( c \) is the central element. Take the following Lie superalgebra decomposition:
\[
\hat{W}_- \overset{\text{def.}}{=} W \otimes \mathbb{C}t^{-1}[t^{-1}], \quad \hat{W}_+ \overset{\text{def.}}{=} W \otimes \mathbb{C}[t] \bigoplus \mathbb{C}c, \quad \hat{W} = \hat{W}_- \bigoplus \hat{W}_+.
\]

Define the following one dimensional \( \hat{W}_+ \)-module \( \mathbb{C}1 \):
\[ x \cdot 1 = 0, \quad c \cdot 1 = 1, \quad \text{for all } x \in W \otimes \mathbb{C}[t], \]

Then we have an induced \( \hat{W} \)-module:
\[ U(\hat{W}) \otimes_{U(\hat{W}_+)} \mathbb{C}1. \]

Note that \( \hat{W}_- \) is supercommutative, by the super version of PBW theorem, this induced module is isomorphic to \( \wedge(\hat{W}_-) \cdot 1 \) as superspaces.

For each \( a \in W \), we associate a corresponding formal power series \( a(z) \):
\[ a(z) = \sum_k a(k)z^{-k-1}. \]

By the computations similar to Example 1, it is checked that:
\[ [a(z), b(w)] = c[i_{z,w}(z-w)^{-2} - i_{w,z}(z-w)^{-2}]. \]

Therefore \( a(w), b(w) \) are mutually local fields by Lemma 2.3:
\[ (z - w)^2[a(z), b(w)] = 0. \]

Introduce an operator \( T \in End(\wedge(\hat{W}_-) \cdot 1) \), which acts on \( W \) by
\[ [T, a(n)] = -na(n-1), \quad [T, c] = 0, \]

and we extend \( T \) to an element in \( End(\wedge(\hat{W}_-) \cdot 1) \) by setting \( T \cdot 1 = 0 \). Then by applying Theorem 2.3 again, we conclude that \( \wedge(\hat{W}_-) \cdot 1 \) has a structure of vertex superalgebra, and for each element
\[ a = \sum c_{I,J}a_i(j_1) \cdots a_i(j_l) \cdot 1, \quad a_i \in W, \]
the corresponding field is given by:

\[ a \mapsto Y(a, z) = \sum c_{I,J} a_i(z)_{(j_1)} \cdots a_i(z)_{(j_l)} \cdot Id_V. \]

Let \( \{\psi_i, \psi_i^*\} \) be a symplectic basis of \( W \) such that

\[ \langle \psi_i^*, \psi_j \rangle = \delta_{i,j}, \langle \psi_i^*, \psi_j^* \rangle = \langle \psi_i, \psi_j \rangle = 0. \]

Define

\[ \omega = \frac{1}{2} \sum_i \psi_i(-1)\psi_i^*(-1) \cdot 1 - \frac{1}{2} \sum_i \psi_i^*(-1)\psi_i(-1) \cdot 1. \]

By the computations similar to Example 1, it is checked that \( \omega(0) = T, \omega(1) \) acts semisimply on \( V = \bigwedge (\hat{W}_-) \cdot 1: \)

\[ V = \bigoplus_{k \geq 0} V_k, V_k = \{v | v \in V, \omega(1)v = kv\}. \]

More precisely,

\[ V_k = \text{span}\{\psi_i(-j_1) \cdots \psi_i(-j_s)\psi_{p_1}(-q_1) \cdots \psi_{p_t}(-q_t) \cdot 1 | j_l, q_m \geq 1, \sum j_l + \sum q_m = k\}, \]

and we check that \( \text{dim}(V_k) < +\infty \), for all \( k \geq 0 \). We also compute that

\[ \omega(1)\omega = 2\omega, \omega(2)\omega = 0, \omega(3)\omega = -n. \]

By Lemma 2.4, \( L(n) = \omega(n + 1) \) satisfy the Virasoro relation. The central charge is equal to \(-2n\). Therefore \( \bigwedge (\hat{W}_-) \cdot 1 \) is a vertex operator superalgebra. The VOSA \( \bigwedge (\hat{W}_-) \cdot 1 \) is called the symplectic fermion VOSA associated to \( W \) in the literatures \[\text{[Kau95]}\]. We denote it by \( \mathcal{A}(W) \).

**Example 3.** Let \( \mathfrak{g} \) be a finite dimensional simple Lie algebra, \( \langle \cdot, \cdot \rangle \) be the corresponding normalized Killing form, and \( \hat{\mathfrak{g}} \) be its corresponding affine Lie algebra:

\[ \hat{\mathfrak{g}} \overset{\text{def.}}{=} \mathfrak{g} \otimes \mathbb{C}[t, t^{-1}] \bigoplus \mathbb{C}c. \]

The Lie bracket is given by:

\[ [x(m), y(n)] = [x, y](m + n) + m(x, y)\delta_{m+n,0}c, [z, c] = 0, \]

for all \( x(m) \overset{\text{def.}}{=} x \otimes t^m \in \mathfrak{g} \otimes \mathbb{C}[t, t^{-1}], z \in \hat{\mathfrak{g}}. \)

Let \( \mathfrak{h} \) be the Cartan subalgebra of \( \mathfrak{g} \), and \( M(k\Lambda_0) \) be the Verma module of \( \hat{\mathfrak{g}} \) with the highest weight \( k\Lambda_0 \). Here \( \Lambda_0 \) is the weight such that

\[ \langle \Lambda_0, c \rangle = 1, \langle \Lambda_0, h \rangle = 0, \text{ for all } h \in \mathfrak{h}. \]
Let $1$ be the highest weight vector in $M(k\Lambda_0)$. For each $x = x(-1)1$, we associate the corresponding field:

$$x(z) \overset{\text{def}}{=} \sum_k x(k)z^{-k-1}.$$  

We compute that

$$[x(z), y(w)] = \sum_{i,j \in \mathbb{Z}} [x, y](i + j)z^{-i-1}w^{-j-1}$$

$$+ \sum_{i,j \in \mathbb{Z}} i\delta_{i+j,0}(x, y)cz^{-i-1}w^{-j-1}$$

$$= [x, y](w)(i_z w(z - w)^{-1} - \iota_{w,z}(z - w)^{-1})$$

$$+ (x, y)c(t_z w(z - w)^{-2} - \iota_{w,z}(z - w)^{-2}).$$

Therefore $x(w), y(w)$ are mutually local by Lemma 2.3:

$$(z - w)^2[x(z), y(w)] = 0.$$  

Define the operator $T$ by:

$$T \cdot 1 = 0, [T, x(n)] = -nx(n - 1).$$

Then $M(k\Lambda_0)$ has a structure of vertex algebra, and for each element

$$a = \sum c_{I,J}a_i(j_1)\cdots a_i(j_l) \cdot 1, a_i \in \mathfrak{g},$$

the corresponding field is given by:

$$a \mapsto Y(a, z) = \sum c_{I,J}a_i(z)(j_1)\cdots a_i(z)(j_l) \cdot 1.$$  

The vertex algebra $M(k\Lambda_0)$ is called the **universal vertex algebra associated to** $\mathfrak{g}$ **of level** $k$ (see for example, [Kac98]). We denote it by $V^k(\mathfrak{g})$.

Let $h^\vee$ be the dual Coxeter number of $\mathfrak{g}$ ([Kac94]), and $l = \dim(\mathfrak{g})$. When $h + k^\vee \neq 0$, let $\{a_i\}$ be an orthonormal basis of $\mathfrak{g}$ with respect to the normalized Killing form. Define

$$\omega = \frac{1}{2(k + h^\vee)} \sum_i a_i(-1)a_i(-1) \cdot 1.$$  

It is computed that

$$\omega(0) = T, \omega(1)\omega = 2\omega, \omega(2)\omega = 0, \omega(3)\omega = \frac{kl}{2(k + h^\vee)},$$

and

$$V = \bigoplus_{k \geq 0} V_k, \quad V_k = \{v \mid v \in V, \omega(1)v = kv\}.$$
We also observe that
\[ V_k = \text{span}\{a_{i_1}(-j_1) \cdots a_{i_s}(-j_s) \cdot 1| j_i \geq 1, \sum j_i = k\}, \]
and \( \dim(V_k) < +\infty \), for all \( k \geq 0 \). In this situation, \( V^k(g) \) has a structure of vertex operator algebra with the central charge \( \frac{kl}{k+h} \). The construction of the Virasoro element \( \omega \), and its corresponding field \( Y'(\omega, z) \) are called **Sugawara construction** in the literatures. The details of Sugawara construction, and the corresponding computations can be found, for example, in [Kac98].
Chapter 3

Vertex Operator Algebras

Associated to Jordan Algebras of Type $B$

In this chapter, we first recall the notion of Griess algebra. Then we review the construction of the VOA $\overline{V}_J$ given by Lam in [Lam99], and the VOA $V_{J,r}$ constructed by Ashihara and Miyamoto in [AM09], where $J$ is a type $B$ Jordan algebra. From this chapter to Chapter 6, all Jordan algebras $J$ are type $B$ Jordan algebras.

3.1 The Griess Algebra of a Vertex Operator Algebra

From now on, we assume that $V$ is a vertex operator algebra with a $\mathbb{Z}_{\geq 0}$-gradation given by $\omega(1)$:

$$V = \bigoplus_{k \geq 0} V_k, \quad V_k \overset{\text{def.}}{=} \{ v \in V \mid \omega(1)v = kv \},$$

and $V_0 = \mathbb{C}1$.

We recall the following result, which tells the relation between $V$ and Lie algebras:
Proposition 3.1 ([Bor86]). Let $TV$ denote the following subspace of $V$:

$$TV \overset{\text{def.}}{=} \text{span}\{Ta \mid a \in V\}.$$  

Then $V/TV$ has a structure of Lie algebra with the Lie bracket:

$$[a, b] \overset{\text{def.}}{=} a(0)b.$$  

This can be checked by using identities (2.6), (2.5), and (2.4) in Section 2.2.

We now further assume that $V_1 = \{0\}$. In this situation, $a(2)b = 0$ for all $a, b \in V_2$, because $a(2)b \in V_1 = \{0\}$, and $a(3)b \in V_0$ is a constant multiple of 1. By (2.8), and

$$\deg(a(n)b) = \deg(a) - 1 - n + \deg(b),$$

we check that for all $a, b \in V_2$,

$$a(1)b = b(1)a - Tb(2)a + \frac{T^2}{2}b(3)a = b(1)a.$$  

We also check that

$$a(3)b = b(3)a$$

using (2.8). Therefore we have the following definition:

**Definition 3.1.** Assume that $V_0 = \mathbb{C}1, V_1 = \{0\}$. Define the bilinear operation ‘$\circ$’ on $V_2$:

$$a \circ b \overset{\text{def.}}{=} a(1)b, \text{ for all } a, b \in V_2,$$

and the bilinear form ‘$\langle \cdot, \cdot \rangle$’ on $V_2$:

$$\langle a, b \rangle \overset{\text{def.}}{=} a(3)b, \text{ for all } a, b \in V_2.$$  

Then $V_2$ is a commutative (but not necessarily associative) algebra with the operation ‘$\circ$’, and the symmetric bilinear form ‘$\langle \cdot, \cdot \rangle$’. We call $V_2$ the **Griess algebra** of $V$.

It is already mentioned in the introduction that Jordan algebras provide the example of commutative (but not necessarily associative) algebras. In the remaining part of this thesis, we focus on the case when the Griess algebra $V_2$ is isomorphic to a Jordan algebra $J$. This can be viewed as an analogue of the VOA $V^k(g)$. 

27
3.2 Construction of the VOAs $\tilde{V}_{\mathcal{J},1}$ and $V_{\mathcal{J},r}$; Where $\mathcal{J}$ is a Type $B$ Jordan Algebra

We first briefly review the VOA constructed by Lam in [Lam99], which is denoted by $\overline{V}_{\mathcal{J},1}$ in this thesis. We assume that $\mathfrak{h}$ is a finite dimensional vector space with a symmetric non-degenerate bilinear form $(\cdot, \cdot)$, dim$(\mathfrak{h}) = d$, $d \geq 2$. Let $\mathcal{J}$ denote the type $B$ simple Jordan algebra of rank $d$, which is realized as symmetric tensors in $\mathfrak{h} \otimes \mathfrak{h}$.

Recall the Heisenberg VOA $\mathcal{H}(\mathfrak{h})$ which is introduced as Example 1 in Section 2.3. We check that the map $\theta : a \mapsto -a$, for all $a \in \mathfrak{h}$
induces an automorphism of the VOA $\mathcal{H}(\mathfrak{h})$. Define $\overline{V}_{\mathcal{J},1}$ as the fixpoint sub-VOA of $\mathcal{H}(\mathfrak{h})$:

$$\overline{V}_{\mathcal{J},1} \overset{\text{def.}}{=} \mathcal{H}(\mathfrak{h})^{\theta}.$$ 

It follows that $(\overline{V}_{\mathcal{J},1})_0 = \mathbb{C}1$, $(\overline{V}_{\mathcal{J},1})_1 = \{0\}$, and

$$(\overline{V}_{\mathcal{J},1})_2 = \text{span}\{a(-1)b | a, b \in \mathfrak{h}\}.$$ 

The Virasoro element of $\overline{V}_{\mathcal{J},1}$ is the same as the Virasoro element $\omega$ of the VOA $\mathcal{H}(\mathfrak{h})$, and the central charge is equal to $d$. Moreover,

$$(\overline{V}_{\mathcal{J},1})_2 \cong \mathcal{J}$$

as Griess algebras, and $\overline{V}_{\mathcal{J},1}$ is generated by $(\overline{V}_{\mathcal{J},1})_2$. For the details, see [Lam99].

We briefly review the construction of $V_{\mathcal{J},r}$ given by Ashihara and Miyamoto in [AM09]. We will also review the main results in [NS10] about the properties of $V_{\mathcal{J},r}$. Recall that in Example 1, Section 2.3, we have a Lie algebra $\hat{\mathfrak{h}}$ associated to $\mathfrak{h}$:

$$\hat{\mathfrak{h}} = \mathfrak{h} \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}c.$$ 

The Lie bracket over $\hat{\mathfrak{h}}$ is:

$$[a(m), b(n)] = m(a, b)\delta_{m+n,0}c, \ [x, c] = 0, \text{ for all } x \in \hat{\mathfrak{h}}.$$ 

Consider the localized associative algebra $U(\hat{\mathfrak{h}})[c^{-1}]$ which is obtained by inverting the central element $c$ in $U(\mathfrak{h})$. It is checked that $[U(\hat{\mathfrak{h}}), U(\hat{\mathfrak{h}})] \subseteq cU(\mathfrak{h})$. Therefore we can define a ‘new’ Lie bracket $[x, y]_{\text{new}}$ on $U(\hat{\mathfrak{h}})$:

$$[x, y]_{\text{new}} \overset{\text{def.}}{=} \frac{1}{c} [x, y], \text{ for all } x, y \in U(\hat{\mathfrak{h}}),$$
because $c^{-1}$ cancels with the $c$ appeared in $[x, y]$, and $\frac{1}{c}[x, y]$ is still in $U(\hat{\mathfrak{h}})$.

We consider the subspace $\mathcal{L}$ of $U(\hat{\mathfrak{h}})$ consists of ‘quadratic’ elements:
\[
\mathcal{L} \overset{\text{def.}}{=} \text{span}\{a(m)b(n) | a, b \in \mathfrak{h}, m, n \in \mathbb{Z}\}.
\]

Recall the definition of ‘normal ordering’ given in Section 2.1. Define
\[
L_{a,b}(m,n) \overset{\text{def.}}{=} \frac{1}{2} : a(m)b(n) :,
\]

and the function
\[
1_m = \begin{cases} 
1, & m \geq 0, \\
0, & m < 0.
\end{cases}
\]

We check that
\[
\frac{1}{2} a(m)b(n) = L_{a,b}(m,n) + \frac{1}{2} m\delta_{m+n,0}(a,b)1_{m-n} c,
\]

and
\[
\mathcal{L} = \text{span}\{L_{a,b}(m,n) | a, b \in \mathfrak{h}, m, n \in \mathbb{Z}\} \oplus \mathbb{C}c.
\]

By a direct computation, we see that for $L_{a,b}(m,n), L_{u,v}(k,l) \in \mathcal{L}$,
\[
[L_{a,b}(m,n), L_{u,v}(k,l)]_{\text{new}} = \\
= \frac{1}{2} n\delta_{n+k,0}(b,u)L_{u,v}(m,l) + \frac{1}{2} m\delta_{m+k,0}(a,u)L_{b,v}(n,l) \\
+ \frac{1}{2} n\delta_{n+l,0}(b,v)L_{u,a}(k,m) + \frac{1}{2} m\delta_{m+l,0}(a,v)L_{u,b}(k,n) \\
+ \frac{mnc}{4} \delta_{m+k,0}\delta_{m+l,0}(b,u)(a,v)1_m + \frac{mnc}{4} \delta_{m+k,0}\delta_{m+l,0}(a,u)(b,v)1_n \\
- \frac{mnc}{4} \delta_{m+k,0}\delta_{m+l,0}(b,v)(a,u)1_{-m} - \frac{mnc}{4} \delta_{m+k,0}\delta_{m+l,0}(a,v)(b,u)1_{-n}. \quad (3.1)
\]

We remark that the above commutation relation for the Lie algebra $\mathcal{L}$ is similar to the commutation relation for the Lie algebra $\hat{\mathfrak{gl}}$ appeared in [KR96], where $\hat{\mathfrak{gl}}$ is the central extension of the Lie algebra $\tilde{\mathfrak{gl}}$ consists of infinite matrices with finitely many non-zero entries. In both cases, the central extensions arise because of the normal orderings.

We now consider the Lie algebra $\mathcal{L}$ with the ‘new’ Lie bracket $[\cdot, \cdot]_{\text{new}}$, and we drop the word ‘new’ for convenience. Define
\[
\mathfrak{B}_+ \overset{\text{def.}}{=} \text{span}\{L_{a,b}(m,n) | n \geq 0 \text{ or } m \geq 0\},
\]
\[
\mathcal{L}_- \overset{\text{def.}}{=} \text{span}\{L_{a,b}(m,n) | m, n < 0\}, \quad \mathcal{L}_+ \overset{\text{def.}}{=} \mathfrak{B}_+ \bigoplus \mathbb{C}c.
\]

Then we have a decomposition of $\mathcal{L}$:
\[
\mathcal{L} = \mathcal{L}_- \bigoplus \mathcal{L}_+ = \mathcal{L}_- \bigoplus \mathfrak{B}_+ \bigoplus \mathbb{C}c.
\]
Define a 1-dimensional $L_+$-module $C_1$:
\[
x_1 = 0, \quad \text{for all } x \in \mathfrak{B}_+, \quad c_1 = r1.
\]
Then by induction from $U(L_+)$ to $U(L)$, we have a $U(L)$-module $M_r$:
\[
M_r \overset{\text{def}}{=} U(L) \otimes_{U(L_+)} C_1 \cong U(L_{-1})
\]
\[
= \text{span}\{ L_{a_1,b_1}(-m_1,-n_1) \cdots L_{a_k,b_k}(-m_k,-n_k) \cdot 1 | \\
m_i, n_i \in \mathbb{Z}_{\geq 1}, a_i, b_i \in \mathfrak{h} \}. 
\]
(3.2)
Because $c$ acts as $r$ on $M_r$, we can take $c = r$ in (3.1).

For $a, b \in \mathfrak{h}$, define the operators $L_{a,b}(l)$ and the fields $L_{a,b}(z)$:
\[
L_{a,b}(l) \overset{\text{def}}{=} \sum_{k \in \mathbb{Z}} L_{a,b}(-k + l - 1, k), \quad L_{a,b}(z) \overset{\text{def}}{=} \sum_{l \in \mathbb{Z}} L_{a,b}(l) z^{-l-1}.
\]
It is proved in [AM09] that these fields are mutually local.

By Theorem 2.3, these mutually local fields generate a vertex algebra:
\[
V_{\mathcal{J},r} = \text{span}\{ L_{a_1,b_1}(m_1) \cdots L_{a_k,b_k}(m_k) \cdot 1 | m_i \in \mathbb{Z}, a_i, b_i \in \mathfrak{h} \}.
\]
It is also shown in [AM09] that $V_{\mathcal{J},r}$ is a VOA. Let $\{e_1, \cdots, e_k\}$ be an orthonormal basis of $\mathfrak{h}$, then the Virasoro element is given by
\[
\omega = \sum_k L_{e_k,e_k},
\]
and the central charge is equal to $dr$. $\omega(1)$ acts semisimply on $V_{\mathcal{J},r}$:
\[
V_{\mathcal{J},r} = \bigoplus_{k \in \mathbb{Z}_{\geq 0}} (V_{\mathcal{J},r})_k, \quad (V_{\mathcal{J},r})_k = \text{span}\{ v | \omega(1)v = kv \},
\]
and $(V_{\mathcal{J},r})_0 = \mathbb{C}1, (V_{\mathcal{J},r})_1 = \{0\}, (V_{\mathcal{J},r})_2 \simeq \mathcal{J}$ as Griess algebras. For the details, see [AM09].

The following is one of the main results in [NS10].

**Theorem 3.1 ([NS10])**. $M_r = V_{\mathcal{J},r}$ holds. Equivalently, $V = V_{\mathcal{J},r}$ is generated by $V_2$.

This fact will be used later in Section 6.3.

We compare the VOAs $\bar{V}_{\mathcal{J},1}$ and $V_{\mathcal{J},r}$. We observe that $\bar{V}_{\mathcal{J},1}$ and $V_{\mathcal{J},r}$ have isomorphic Griess algebras
\[
(\bar{V}_{\mathcal{J},1})_2 \simeq (V_{\mathcal{J},r})_2 \simeq \mathcal{J},
\]
and they are both generated by the degree two subspaces $V_2$. The difference is that their central charges are different. The analogues between these two VOAs will provide some ideas for the proof of Theorem 1.1. The precise relation will be clear in Chapter 6.
Chapter 4

Correlation Functions of $V_{\mathcal{J},r}$

Where $\mathcal{J}$ is of Type $B$

Let $\mathcal{J}$ denote the type $B$ Jordan algebra of rank $d$. In this chapter, we prove Theorem 1.1, which gives the correlation function of the fields $L_{a_1, b_1}(z_1), \ldots, L_{a_n, b_n}(z_n)$ associated to the VOA $V_{\mathcal{J},r}$. The formula for the correlation function of the fields associated to a Virasoro vertex algebra, was previously shown in [HT12]. Our approach is different, and the correlation function of the fields associated to a Virasoro vertex algebra computed in [HT12], can be obtained as a corollary of Theorem 1.1.

4.1 Diagrams, Derangements, and Some Necessary Notations

In this section, we introduce some notations. We recall the VOA $V_{\mathcal{J},r}$ described in Section 3.2. Let $L_{a_1, b_1}(z_1), \ldots, L_{a_n, b_n}(z_n)$ denote $n$ fields associated to the VOA $V_{\mathcal{J},r}$, then we have a corresponding sequence

$$T = (a_1, b_1) \cdots (a_n, b_n).$$

Let $DR(T)$ denote the set of permutations of the $n$-element set $\{1, \ldots, n\}$ without fixpoint, which is called the set of derangements in some literatures [FS09]. We view $DR(T)$ as a subset of the permutation group $S_n$. For an element $\sigma \in DR(T)$, we decompose it as disjoint cycles

$$\sigma = (C_1) \cdots (C_s),$$
and we use $\sigma(i)$ to denote the image of $i$ under the action of $\sigma$. We also use the symbol
\[ c(\sigma) \overset{\text{def.}}{=} s \]
to denote the number of disjoint cycles in $\sigma$. We remark that the case $n = 0$ is also allowed, and we call the corresponding sequence $T$ the empty sequence, denoted by
\[ T = \emptyset. \]

Given the sequence $T = (a_1, b_1) \cdots (a_n, b_n)$, a sign on $T$ is a function $\epsilon : \{a_1, b_1, \cdots, a_n, b_n\} \to \{+, -\}$. If $\epsilon(a_i) = \epsilon_i$, $\epsilon(b_i) = \delta_i$, we also write
\[ T^\epsilon = (a_1^{\epsilon_1}, b_1^{\delta_1}) \cdots (a_n^{\epsilon_n}, b_n^{\delta_n}). \]

We call $T^\epsilon$ a signed sequence, and $(a_i^{\epsilon_i}, b_i^{\delta_i})$ the $i$-th pair of $T^\epsilon$.

We will use capital letters $A, B$ to denote certain parts in a sequence or a signed sequence $T^\epsilon$. For example, for $T^\epsilon = (a_1^{\epsilon_1}, b_1^{\delta_1}) \cdots (a_n^{\epsilon_n}, b_n^{\delta_n})$, $1 \leq k \leq n - 1$, we also write
\[ T^\epsilon = A(a_k^{\epsilon_k}, b_k^{\delta_k})(a_{k+1}^{\epsilon_{k+1}}, b_{k+1}^{\delta_{k+1}})B, \]
where
\[ A = (a_1^{\epsilon_1}, b_1^{\delta_1}) \cdots (a_{k-1}^{\epsilon_{k-1}}, b_{k-1}^{\delta_{k-1}}), \quad B = (a_{k+2}^{\epsilon_{k+2}}, b_{k+2}^{\delta_{k+2}}) \cdots (a_n^{\epsilon_n}, b_n^{\delta_n}). \]

We remark that there is only one signed sequence associated to the empty sequence $T = \emptyset$, which is called the empty signed sequence by convention. We denote it as $T^\epsilon = \emptyset$.

We will introduce a notation called the diagram over a sequence $T = (a_1, b_1) \cdots (a_n, b_n)$, denoted by $D(T)$. It will be shown that there is a surjective map from $D(T)$ to $DR(T)$. We will also introduce the notation of diagram over $T$ compatible with the sign $\epsilon$, denoted by $D(T^\epsilon)$. It will also be shown that two obvious operations of diagrams turn out to have correspondences with the commutation relations, which will be used to prove Theorem 1.1.

A diagram over the sequence $T = (a_1, b_1) \cdots (a_n, b_n)$ is a graph, with the vertex set $V = \{a_1, b_1, \cdots, a_n, b_n\}$, and the edge set $E$ consisting of unordered pairs $\{u, v\}$, $u, v \in V$ satisfying:

1. $|E| = n$.
2. $\{a_i, b_i\}, \{a_i, a_j\}, \{b_i, b_j\} \notin E$, for all $i = 1, \cdots, n$.
3. Any two edges have no common vertex.
Let $D(T)$ denote the set of all diagrams over $T$. We observe that such a graph always has $2n$ vertices and $n$ disjoint edges.

A diagram over the signed sequence $T^\epsilon = (a_1^\epsilon_1, b_1^\epsilon_1)(a_2^\epsilon_2, b_2^\epsilon_2)\cdots(a_n^\epsilon_n, b_n^\epsilon_n)$ is a diagram over $T$ compatible with the sign $\epsilon$ in the following sense:

(4). For all $\{u, v\} \in E$, if $u = a_i$ or $b_i$, $v = a_j$ or $b_j$, $i < j$, then $\epsilon(u) = +$ and $\epsilon(v) = -$.

Let $D(T^\epsilon)$ denote the set of diagrams over $T$ which are compatible with $\epsilon$. It may happen that for some $\epsilon$, $D(T^\epsilon) = \emptyset$. For example, if $T^\epsilon = (a_1^+, b_1^+)(a_2^-, b_2^-)(a_3^-, b_3^-)(a_4^+, b_4^+)$, then $D(T^\epsilon) = \emptyset$.

For the empty sequence $T = \emptyset$, or the empty signed sequence $T^\epsilon = \emptyset$, there is only one diagram $D$ in $D(\emptyset)$, which is called the empty diagram, denoted by $D = \emptyset$. We note that the set $D(\emptyset)$ is non-empty, because it contains the empty diagram $D = \emptyset$ by our convention.

Given a diagram in $D(T^\epsilon)$, we can get a diagram in $D(T)$ by forgetting the sign $\epsilon$. Conversely, for a diagram $D \in D(T)$, there is a unique way of assigning a sign $\epsilon$ to $T$ satisfying the compatibility condition. Therefore we have

$$D(T) = \coprod_{\epsilon} D(T^\epsilon).$$  \hspace{1cm} (4.1)

It will be helpful illustrating $D(T)$ or $D(T^\epsilon)$ using graphs. We give some examples and non-examples. Let $T = (a_1, b_1)(a_2, b_2)(a_3, b_3)(a_4, b_4)$. Then the following is a diagram over $T$:

\[
\begin{array}{c}
(a_1 & b_1) & (a_2 & b_2) & (a_3 & b_3) & (a_4 & b_4)
\end{array}
\]

There is a unique way of adding the sign:

\[
\begin{array}{c}
(a_1^+ & b_1^+) & (a_2^- & b_2^-) & (a_3^- & b_3^-) & (a_4^+ & b_4^+)
\end{array}
\]

But the followings are not diagrams over $T$:

\[
\begin{array}{c}
(a_1 & b_1) & (a_2 & b_2) & (a_3 & b_3) & (a_4 & b_4)
\end{array}
\]
because they violate the conditions (1), (2), and (3) in the definition respectively. The following graph

\[
\begin{align*}
(a_1^+ b_1^-) & \quad (a_2^+ b_2^-) & \quad (a_3^+ b_3^-) & \quad (a_4^- b_4^-) \\
(a_1^- b_1^+) & \quad (a_2^- b_2^+) & \quad (a_3^- b_3^+) & \quad (a_4^+ b_4^+)
\end{align*}
\]

is a diagram over \( T \), but it is not compatible with the sign \( \epsilon = (++)(+)(-+)(--). \)

We will establish a map from \( D(T) \) to \( DR(T) \) in two steps.

First, for a diagram \( D = \{V, E\} \in D(T) \), we can obtain a new graph by identifying two vertices \( a_i, b_i \) in the same pair \( (a_i, b_i) \), for all \( i = 1, \ldots, n \). The defining condition of \( D(T) \) implies that this graph is always a disjoint unions of cycle graphs.

For example, let \( T = (a_1, b_1) \cdots (a_6, b_6) \), and \( D \in D(T) \) which is a diagram graphically represented by

\[
\begin{align*}
(a_1) & \quad (a_2) & \quad (a_3) & \quad (a_4) & \quad (a_5) & \quad (a_6) \\
(b_1) & \quad (b_2) & \quad (b_3) & \quad (b_4) & \quad (b_5) & \quad (b_6)
\end{align*}
\]

The corresponding graph is given by

\[
\begin{array}{c}
1 \quad 2 \\
3 \quad 5
\end{array}
\]

Next, we add an orientation on each cycle graph to get a directed graph. For a cycle with vertices labelled by \( i_1, \ldots, i_t \), in which \( i_1 \) be the smallest number, when \( t \geq 3 \), there are two possibilities adding the orientation. We choose the edge \( \{a_{i_1}, b_j\} \in E \) (or \( \{a_{i_1}, a_j\} \in E \)), and take the orientation in the direction from \( i_1 \) to \( j \).
For example, for diagram (4.2), the corresponding directed graph is

\[
\begin{array}{c}
4 & 6 \\
\downarrow & \downarrow \\
1 & 2 & 3 & 5,
\end{array}
\]

where \(i_1 = 3, j = 5\), in the second cycle graph. We remark that these graphs are essentially the same as what is called ‘Virasoro graphs’ in [HT12].

For every such directed graph, we can get a corresponding element in \(DR(T)\) in an obvious way. For the above example, \((12)(3564)\) is the corresponding derangement. We use \(\sigma_D\) to denote the derangement corresponding to a diagram \(D\).

We have the following combinatorial lemma:

**Lemma 4.1.** Given a sequence \(T = (a_1, b_1) \cdots (a_n, b_n), n \geq 2\), the map defined above:

\[D \mapsto \sigma_D.\]

is surjective. Moreover, for \(\sigma = (C_1) \cdots (C_s) \in DR(T)\), there are exactly \(2^{n-s}\) diagrams \(D \in D(T)\) such that \(\sigma_D = \sigma\).

We also consider some operations on the diagram. Suppose

\[T = (a_1, b_1) \cdots (a_n, b_n) = A(a_i, b_i)(a_{i+1}, b_{i+1})B,\]

\(D = (V, E) \in D(T)\), and \(e = \{a_i, b_{i+1}\} \in E\). Then we can form a new diagram \(D_1 = (V_1, E_1)\) by deleting \(e\), where \(V_1 = V - \{a_i\} - \{b_{i+1}\}, E_1 = E - e\), and \(D_1\) is a diagram over the new sequence

\[T_1 = A(a_{i+1}, b_i)B.\]

This process is described graphically by:

\[
\begin{array}{c}
\cdots (a_{i+1} b_i) \\
D_1
\end{array}
\quad \cdots = \quad \begin{array}{c}
\cdots (a_i b_i) (a_{i+1} b_{i+1}) \\
D
\end{array}
\quad - \quad \begin{array}{c}
a_i \\
b_{i+1}
\end{array}
\quad \begin{array}{c}
e
\end{array}.
\]

We simply write \(D_1 = D - e\), or equivalently \(D = D_1 + e\).

Similarly, if \(e_1 = \{a_i, a_{i+1}\}, e_2 = \{b_i, b_{i+1}\} \in E\), we can form a new diagram \(D_2 = (V_2, E_2)\) by deleting both \(e_1, e_2\), where \(V_2 = V - \{a_i\} - \{a_{i+1}\} - \{b_i\} - \{b_{i+1}\}, E_2 = E - e_1 - e_2\), and \(D_2\) is a diagram over the new sequence

\[T_2 = AB.\]
This process is described graphically as

\[ D_2 = \cdots - a_i e_1 - b_i e_2. \]

We also write \( D_2 = D - e_1 - e_2 \) or \( D = D_2 + e_1 + e_2 \). Note that \( D_2 = \emptyset \) may happen.

Later on, we use capital letters \( Z, W \) to denote the set of variables in the formal power series \( L_{a_1, b_1}(z_1, w_1), \cdots, L_{a_n, b_n}(z_n, w_n) \):

\[ Z = \{ z_1, \cdots, z_n \}, \quad W = \{ w_1, \cdots, w_n \}. \]

We also need to introduce some notations which relate the graphical data to formal power series.

Given a signed sequence \( T^\epsilon = (a_1^\epsilon, b_1^\epsilon) \cdots (a_n^\epsilon, b_n^\epsilon) \), we use \( P(T^\epsilon; Z, W) \) to denote the following product of formal power series associated to \( T^\epsilon \):

\[ P(T^\epsilon; Z, W) \overset{\text{def.}}{=} L_{a_1^\epsilon, b_1^\epsilon}(z_1, w_1) \cdots L_{a_n^\epsilon, b_n^\epsilon}(z_n, w_n). \]

It is obvious that for a signed sequence \( AB \),

\[ P(AB; Z, W) = P(A; Z, W)P(B; Z, W). \]

For a diagram \( D = (V, E) \) over \( T = (a_1, b_1) \cdots (a_n, b_n) \), and for an edge \( e = \{a_i, b_j\} \in E \), define

\[ K(e; Z, W) \overset{\text{def.}}{=} \frac{1}{(z_i - w_j)^2}, \quad Q(e; Z, W) \overset{\text{def.}}{=} \frac{1}{2}(a_i, b_j). \]

Similarly for an edge \( e = \{a_i, a_j\} \), define

\[ K(e; Z, W) \overset{\text{def.}}{=} \frac{1}{(z_i - z_j)^2}, \quad Q(e; Z, W) \overset{\text{def.}}{=} \frac{1}{2}(a_i, a_j). \]

and for \( e = \{b_i, b_j\} \):

\[ K(e; Z, W) \overset{\text{def.}}{=} \frac{1}{(w_i - w_j)^2}, \quad Q(e; Z, W) \overset{\text{def.}}{=} \frac{1}{2}(b_i, b_j). \]

We also define the functions \( \Gamma(D), R(D; Z, W) \):

\[ \Gamma(D) \overset{\text{def.}}{=} \prod_{(u, v) \in E} (u, v), \quad R(D; Z, W) \overset{\text{def.}}{=} r^{e(\sigma_D)} \prod_{e \in E} Q(e; Z, W). \]
We observe that $R(D; Z, W)$ can also be written as

$$R(D; Z, W) = \Gamma(D)_c e^{(c_D)} \prod_{e \in E} K(e; Z, W), \quad (4.3)$$

through a direct computation. For the empty diagram $D = \emptyset$, we define

$$R(\emptyset; Z, W) = 1$$

by convention.

We also recall the notations in Theorem 1.1, Chapter 1. Assume that $\sigma = (C_1) \cdots (C_s) = (k_{11} \cdots k_{1t_1}) \cdots (k_{s1} \cdots k_{st_s})$ is an element in $D R(T)$. We define

$$\Gamma(\sigma, T) \overset{\text{def.}}{=} 2^{-s-n} \prod_{i=1}^{s} Tr(L_{a_{k_{i1}}}, b_{k_{i1}} \cdots L_{a_{k_{it_i}}, b_{k_{it_i}}});$$

$$\Gamma(\sigma; Z) \overset{\text{def.}}{=} \prod_{i=1}^{n} \frac{1}{(z_i - z_{\sigma(i)})^2},$$

where $Tr(a \otimes b)$ is the trace of $a \otimes b \in \mathfrak{h} \otimes \mathfrak{h}$ given by:

$$Tr(a \otimes b) = (a, b).$$

### 4.2 Some Related Formal Power Series, and The Correlation Function of the Fields in $\bar{V}_{J,1}$

Given a VOA $V$ such that $V$ is $\mathbb{Z}_{\geq 0}$-graded

$$V = \bigoplus_{k \in \mathbb{Z}_{\geq 0}} V_k,$$

let $V^*$ denote the graded dual of $V$:

$$V^* \overset{\text{def.}}{=} \bigoplus_{k \in \mathbb{Z}_{\geq 0}} (V_k)^*.$$

Suppose $V_0 = \mathbb{C}1$. Let $1'$ denote the unique element in $(V_0)^*$ satisfying $\langle 1', 1 \rangle = 1$. Then the correlation function of the fields $Y(a_1, z_1), \cdots, Y(a_n, z_n)$, $a_i \in V$, is defined by

$$\langle 1', Y(a_1, z_1) \cdots Y(a_n, z_n) 1 \rangle,$$
which is a formal power series in \(\mathbb{C}[[z_1, z_1^{-1} \cdots z_n, z_n^{-1}]]\). It is shown in [ELM89] that this formal power series converges to a rational function in the domain \(|z_1| > \cdots > |z_n|\).

Recall the VOA \(\tilde{V}_{J,1}\) which is described in Section 3.2. We now give the correlation function of the fields \(Y(a_1(-1)b_1, z_1), \ldots, Y(a_n(-1)b_n, z_n)\) associated to the VOA \(\tilde{V}_{J,1}\), where \(a_i, b_i \in \mathfrak{h}\). Because \(\tilde{V}_{J,1}\) is realized as a sub VOA of the Heisenberg VOA \(\mathcal{H}(\mathfrak{h})\), we can compute the correlation function in the VOA \(\mathcal{H}(\mathfrak{h})\). With the notations given in Section 4.1, the result is given by the following proposition, which is a corollary of the well known ‘Wick theorem’ appeared in physics literatures [PS95].

**Proposition 4.1.** Given \(n\) fields \(Y(\frac{1}{2}a_1(-1)b_1, z_1), \ldots, Y(\frac{1}{2}a_n(-1)b_n, z_n)\) associated to the VOA \(\tilde{V}_{J,1}\), where \(a_i, b_i \in \mathfrak{h}\). Let \(T = (a_1, b_1) \cdots (a_n, b_n)\) denote the corresponding sequence. The correlation function of these fields is given by:

\[
\langle 1', Y(\frac{1}{2}a_1(-1)b_1, z_1) \cdots Y(\frac{1}{2}a_n(-1)b_n, z_n)1 \rangle
\]

\[
= \sum_{D \in D(T)} \Gamma(D) \prod_{e \in E} K(e; Z, Z)
\]

\[
= \sum_{\sigma \in DR(T)} \Gamma(\sigma, T)\Gamma(\sigma; Z)
\]

We omit the proof of this proposition. This formula can be derived using the ‘Wick theorem’ in [Kac98].

We note that (4.4) is identical to (1.1) when \(r = 1\). The ‘Wick theorem’ in [Kac98] can not be applied directly to the proof of Theorem 1.1, although there are some similarities between the VOAs \(V_{J,r}\) and \(\tilde{V}_{J,1}\).

But we can still solve this problem by comparing \(V_{J,r}\) and \(\tilde{V}_{J,1}\). Recall the Heisenberg VOA \(\mathcal{H}(\mathfrak{h})\) defined in Section 2.3, and the ‘normal ordered product’ introduced in Chapter 2. By Proposition 2.1 and (2.2), for \(a, b \in \mathfrak{h}\), we have

\[
Y(a(-1)b, z)
\]

\[
= Y(a, z)(-1)Y(b, z)
\]

\[
= : a(z)b(z) : 
\]

\[
= a_-(z)b_+(z) + b_-(z)a_+(z) + a_-(z)b_-(z) + a_+(z)b_+(z),
\]

where

\[
Y(a, z) = a(z) = a_-(z) + a_+(z),
\]

38
Motivated by the proof of ‘Wick theorem’ in [Kac98], we introduce some two-variable formal power series $L_{a,b}^{-+}(z,w)$, $L_{a,b}^{+-}(z,w)$, $L_{a,b}^{++}(z,w)$, and $L_{a,b}(z,w)$, whose coefficients are in $\mathcal{L}$:

$$
L_{a,b}(z,w) \overset{\text{def.}}{=} \sum_{m,n \in \mathbb{Z}} L_{a,b}(m,n) z^{-m-1} w^{-n-1},
$$

$$
L_{a,b}^{-+}(z,w) \overset{\text{def.}}{=} \sum_{m,n<0} L_{a,b}(m,n) z^{-m-1} w^{-n-1},
$$

$$
L_{a,b}^{+-}(z,w) \overset{\text{def.}}{=} \sum_{m<0,n \geq 0} L_{a,b}(m,n) z^{-m-1} w^{-n-1},
$$

$$
L_{a,b}^{++}(z,w) \overset{\text{def.}}{=} \sum_{m,n \geq 0} L_{a,b}(m,n) z^{-m-1} w^{-n-1}.
$$

It is easy to see that

$$
L_{a,b}^{-+}(z,w) = L_{b,a}^{-+}(w,z), L_{a,b}^{+-}(z,w) = L_{b,a}^{+-}(w,z), L_{a,b}^{++}(z,w) = L_{b,a}^{++}(w,z),
$$

$$
L_{a,b}(z,w) = L_{a,b}^{-+}(z,w) + L_{a,b}^{+-}(z,w) + L_{a,b}^{++}(z,w) + L_{a,b}(z,w),
$$

(4.5)

and

$$
L_{a,b}(z) = L_{a,b}(z,z)
$$
as fields on $V_{\mathcal{J},r}$.

We have the following closed formulas for the commutation relations of these formal power series, which will be used later.

**Proposition 4.2.** As formal power series with coefficients taking value in $\text{End}(V_{\mathcal{J},r})$, we have

$$
[L_{a,b}^{++}(x,y), L_{a,v}^{-+}(z,w)] = \frac{1}{2}(b,v)_{t_{y,w}(y - w)} L_{a,v}^{++}(z,x) + \frac{1}{2}(a,v)_{t_{x,w}(x - w)} L_{u,b}^{+-}(z,y)
$$

$$
+ \frac{1}{2}(a,u)_{t_{z,x}(x - z)} L_{u,b}^{+-}(w,y) + \frac{1}{2}(b,u)_{t_{y,z}(y - z)} L_{v,a}^{++}(w,x)
$$

$$
+ \frac{1}{4} r(a,u)(b,v)_{t_{x,z}(x - z)} L_{v,a}^{++}(w,y) + \frac{1}{4} r(a,v)(b,u)_{t_{x,w}(x - w)} L_{u,b}^{+-}(z,y - z)^{-2},
$$

(4.6)

$$
[L_{a,b}^{++}(x,y), L_{a,v}^{-+}(z,w)]
$$

39
\[
\frac{1}{2} (a, u) L_{b,w}^{++}(y, w) t_{x,z}(x - z)^{-2} + \frac{1}{2} (b, u) L_{a,v}^{++}(x, w) t_{y,z}(y - z)^{-2}, \quad (4.7)
\]

\[
[L_{a,b}^{++}(x, y), L_{u,v}^{++}(z, w)] = 0.
\]

**Proof.** We only prove (4.6), and the other one is obtained in a similar way. A direct computation in the Heisenberg VOA \(\mathcal{H}(\mathfrak{h})\) shows that

\[
[a_+(z), b_-(w)] = (a, b) t_{z,w} \frac{1}{(z - w)^2}. \quad (4.8)
\]

We view \(a_\pm(x) b_\pm(y)\) as a formal power series with coefficients in \(\mathcal{L}\). Then we have

\[
L_{a,b}^{++}(z, w) = \frac{1}{2} a_+(z) b_+(w), \quad L_{a,b}^{--}(z, w) = \frac{1}{2} a_-(z) b_+(w),
\]

\[
L_{a,b}^{-+}(z, w) = \frac{1}{2} b_-(w) a_+(z), \quad L_{a,b}^{+-}(z, w) = \frac{1}{2} a_-(z) b_-(w).
\]

Hence (4.6) follows from the following computation by taking \(c = r\):

\[
[a_+(x) b_+(y), u_-(z) v_-(w)]_{\text{new}} = \frac{1}{c} [a_+(x) b_+(y), u_-(z) v_-(w)]
\]

\[
= (b, v) t_{y,w} (y - w)^{-2} u_-(z) a_+(x) + (a, v) t_{x,w} (x - w)^{-2} u_-(z) b_+(y)
+ (a, u) t_{x,z} (x - z)^{-2} b_+(y) v_-(w) + (b, u) t_{y,z} (y - z)^{-2} a_+(x) v_-(w)
\]

\[
= (b, v) t_{y,w} (y - w)^{-2} u_-(z) a_+(x) + (a, v) t_{x,w} (x - w)^{-2} u_-(z) b_+(y)
+ (a, u) t_{x,z} (x - z)^{-2} v_-(w) b_+(y) + (b, u) t_{y,z} (y - z)^{-2} v_-(w) a_+(x)
+ (a, v) (b, u) c t_{x,z} (x - z)^{-2} t_{y,w} (y - w)^{-2}
+ (a, v) (b, u) c t_{x,w} (x - w)^{-2} t_{y,z} (y - z)^{-2}.
\]

### 4.3 Proof of Theorem 1.1

In this section, we finish the proof of Theorem 1.1. To compute the correlation function

\[
\langle 1^{'}, L_{a_1,b_1}(z_1) \cdots L_{a_n,b_n}(z_n) \rangle,
\]

we need to compute the following ‘correlation function’ of formal power series:

\[
\langle 1^{'}, L_{a_1,b_1}(z_1, w_1) \cdots L_{a_n,b_n}(z_n, w_n) \rangle.
\]

By (4.5), we have

\[
\langle 1^{'}, L_{a_1,b_1}(z_1, w_1) \cdots L_{a_n,b_n}(z_n, w_n) \rangle = \sum_{c} \langle 1^{'}, L_{a_1,b_1}^{c, \delta_1}(z_1, w_1) \cdots L_{a_n,b_n}^{c, \delta_n}(z_n, w_n) \rangle
\]

40
Lemma 4.2. Given a sign \( \langle \epsilon \rangle \) we need the following lemma, which shows that for some sign \( \epsilon \) to compute each term in the summation of the right hand side.

where \( \epsilon = (\epsilon_1, \delta_1) \cdots (\epsilon_n, \delta_n) \) runs over all \( 4^n \) possible signs over \( T \). We need to compute each term in the summation of the right hand side.

We need the following lemma, which shows that for some sign \( \epsilon \), we have \( \langle 1', P(T^c; Z, W) \cdot 1 \rangle = 0 \).

**Lemma 4.2.** Given a sign \( \epsilon = (\epsilon_1, \delta_1) \cdots (\epsilon_n, \delta_n) \). If \( (\epsilon_1, \delta_1) = (--) \), \((+-)\), or \((-+)\), then

\[ \langle 1', P(T^c; Z, W) \cdot 1 \rangle = 0. \]

**Proof of Lemma 4.2.** Suppose \( x = L_{a_1, b_1}(k_1, l_1) \cdots L_{a_n, b_n}(k_n, l_n) \), it is enough to show that if \( k_1 < 0 \), or \( l_1 < 0 \), then

\[ \langle 1', x \cdot 1 \rangle = 0. \quad (4.10) \]

We first prove this for the case \( k_1, l_1 < 0 \). If \( L_{a_2, b_2}(k_2, l_2) \cdots L_{a_n, b_n}(k_n, l_n) \cdot 1 = 0 \), then \((4.10)\) obviously holds. Otherwise \( L_{a_2, b_2}(k_2, l_2) \cdots L_{a_n, b_n}(k_n, l_n) \cdot 1 \neq 0 \), then \( \deg(L_{a_2, b_2}(k_2, l_2) \cdots L_{a_n, b_n}(k_n, l_n) \cdot 1) \geq 0 \). It is clear that

\[ \deg(x \cdot 1) = \deg(L_{a_1, b_1}(k_1, l_1)L_{a_2, b_2}(k_2, l_2) \cdots L_{a_n, b_n}(k_n, l_n) \cdot 1) \geq 2, \]

hence

\[ \langle 1', x \cdot 1 \rangle = 0. \]

We now prove the case \( k_1 < 0, l_1 \geq 0 \). From the structure of \( V_{j,r} \), we can assume that \( k_i, l_i < 0 \) for all \( i \geq 2 \). When \( n = 1 \), \((4.10)\) trivially holds. We now prove the general case by induction on \( n \). Because

\[ \langle 1', x \cdot 1 \rangle = \langle 1', L_{a_1, b_1}(k_1, l_1)L_{a_2, b_2}(k_2, l_2) \cdots L_{a_n, b_n}(k_n, l_n) \cdot 1 \rangle \]

\[ = \langle 1', L_{a_2, b_2}(k_2, l_2)L_{a_1, b_1}(k_1, l_1) \cdots L_{a_n, b_n}(k_n, l_n) \cdot 1 \rangle \]

\[ + \langle 1', [L_{a_2, b_2}(k_2, l_2), L_{a_1, b_1}(k_1, l_1)] \cdots L_{a_n, b_n}(k_n, l_n) \cdot 1 \rangle, \]

we conclude that \((4.10)\) still holds by the induction hypothesis and the commutation relation \((3.1)\).

The following key lemma tells us that, for each sign \( \epsilon \), how to compute \( \langle 1', P(T^c; Z, W) \cdot 1 \rangle \). It turns out that we can compute \( \langle 1', P(T^c; Z, W) \cdot 1 \rangle \) using the functions \( R(D; Z, W) \) defined in Section 4.1, where \( D \) are diagrams in \( D(T^c) \).

**Lemma 4.3.** Given a signed sequence \( T^c = (a_1^{\epsilon_1}, b_1^{\delta_1}) \cdots (a_n^{\epsilon_n}, b_n^{\delta_n}) \).
Case I. If $D(T) = \emptyset$, then
\[
\langle 1', P(T^e; Z, W) \cdot 1 \rangle = 0. \tag{4.11}
\]

Case II. If $D(T) \neq \emptyset$, then
\[
\langle 1', P(T^e; Z, W) \cdot 1 \rangle = \sum_{D \in D(T^e)} R(D; Z, W). \tag{4.12}
\]

Proof of Lemma 4.3. We now prove the lemma using induction. Let $T^e(n, k)$ denote the set of signed sequences $T^e$ which have $n$ pairs, and in which the leftmost $k$ pairs have the form $(a^+, b^+)$, and the $(k + 1)$-th pair is not of the form $(a^+, b^+)$, or $k = n$. For example, let
\[
T^e = (a_1^+, b_1^+)(a_2^+, b_2^+)(a_3^+, b_3^+)(a_4^+, b_4^-)(a_5^-, b_5^-)(a_6^-, b_6^-),
\]
then we have $T^e \in T^e(6, 2)$.

We consider the extreme case when $k = 0$. It is obvious that in this case, $(\epsilon_1, \delta_1) = (--)$, $(+-)$, or $(--)$, and
\[
D(T^e) = \emptyset.
\]
Hence this extreme case is in Case I of Lemma 4.3. By Lemma 4.2, (4.11) holds.

We consider another extreme case when $n = 0$. In this case $T^e = \emptyset$, $D(T^e) = \{\emptyset\}$, and this extreme case is in Case II of Lemma 4.3. We have
\[
\langle 1', P(\emptyset; Z, W) \cdot 1 \rangle = \sum_{D \in D(\emptyset)} R(D; Z, W) = 1
\]
by our convention. Therefore (4.12) holds.

It’s obvious that the set of signed sequences over $T$ is a disjoint union of $T^e(n, k), 0 \leq k \leq n$. We prove Lemma 4.3 by induction on $n + k$. Because we have verified the extreme cases $k = 0$ and $n = 0$, therefore it is enough to show that the correctness for $T^e \in T^e(n, k - 1) \cup T^e(n - 1, k) \cup T^e(n - 1, k - 1) \cup T^e(n - 2, k - 1)$ will imply the correctness for $T^e \in T^e(n, k)$. We can also assume that $k \geq 1$.

For $T^e \in T^e(n, k)$, suppose $T^e = A(a_k^+, b_k^+)B$, then there are two cases:

Case 1 The signed sequence $T^e = A(a_k^+, b_k^+)(a_{k+1}^-, b_{k+1}^+)B$, or $A(a_k^+, b_k^+)(a_{k+1}^+, b_{k+1}^-)B$.

Case 2 The signed sequence $T^e = A(a_k^+, b_k^+)(a_{k+1}^-, b_{k+1}^-)B$. 

42
Case 1: We only need to consider the subcase \(T^e = A(a^+_k, b^+_k)(a^-_{k+1}, b^+_{k+1})B\), and the other subcase \(T^e = A(a^+_k, b^+_k)(a^+_{k+1}, b^-_{k+1})B\) is dealt with in the same way. Using commutation relation (4.7), we see that

\[
\langle 1', P(T^e; Z, W) \cdot 1 \rangle
= \langle 1', P(A; Z, W) L_{a_{k+1}, b_{k+1}}^+ (z_{k+1}, w_{k+1}) L_{a_k, b_k}^+ (z_k, w_k) P(B; Z, W) \cdot 1 \rangle
+ \langle 1, P(A; Z, W) L_{a_{k+1}, b_{k+1}}^+ (z_{k+1}, w_{k+1}) L_{a_k, b_k}^+ (z_k, w_k) P(B; Z, W) \cdot 1 \rangle
+ \frac{1}{2} (b_k, a_{k+1})(w_k - z_{k+1})^{-2} \langle 1', P(A; Z, W) L_{a_k, b_k}^+ (z_k, w_{k+1}) P(B; Z, W) \cdot 1 \rangle
+ \frac{1}{2} (a_k, a_{k+1})(z_k - z_{k+1})^{-2} \langle 1', P(A; Z, W) L_{a_k, b_k}^+ (w_k, w_{k+1}) P(B; Z, W) \cdot 1 \rangle.
\]

(4.13)

Let

\[
T_1^{e_1} = A(a^-_{k+1}, b^+_k)(a^+_{k+1}, b^+_k)B, \quad T_2^{e_2} = A(a^+_k, b^+_{k+1})B, \quad T_3^{e_3} = A(b^+_k, b^-_{k+1})B.
\]

Observe that \(D(T^e)\) can be decomposed into three disjoint subsets \(D(T_1^{e_1}), D(T_2^{e_2}) + e_1, D(T_3^{e_3}) + e_2\), where \(e_1 = \{b^+_k, a^-_{k+1}\}\), \(e_2 = \{a^+_k, a^-_{k+1}\}\):

1. There is no edge connecting \((a^+_k, b^+_k)\) and \((a^-_{k+1}, b^+_{k+1})\):

\[
\cdots \quad (a^+_k \quad b^+_k) \quad (a^-_{k+1} \quad b^+_{k+1}) \quad \cdots \quad \cdots.
\]

2. There is one edge connecting \((a^+_k, b^+_k)\) and \((a^-_{k+1}, b^+_{k+1})\), and there are two subcases:

\[
\cdots (a^+_k \quad b^+_k) \quad (a^-_{k+1} \quad b^+_{k+1}) \quad \cdots, \quad (a^+_k \quad b^+_k) \quad (a^-_{k+1} \quad b^+_{k+1}) \quad \cdots \quad \cdots.
\]

By our notation of diagram operations defined in Section 4.1, we have:

\[
D(T^e) = (D(T_1^{e_1})) \coprod (D(T_2^{e_2}) + e_1) \coprod (D(T_3^{e_3}) + e_2).
\]

(4.14)

We first consider Case I when \(D(T^e) = \emptyset\). In this situation, \(D(T_i^{e_i}) = \emptyset\), for all \(i = 1, 2, 3\). By the induction hypothesis, if \(D(T_i^{e_i}) = \emptyset\), then

\[
\langle 1', P(T_i^{e_i}; Z, W) \cdot 1 \rangle = 0.
\]

(4.15)
Hence
\[
\langle 1', P(T^e; Z, W) \cdot 1 \rangle
= \langle 1', P(T_1^e; Z, W) \cdot 1 \rangle
+ Q(e_1; Z, W)\langle 1', P(T_2^e; Z, W) \cdot 1 \rangle
+ Q(e_2; Z, W)\langle 1', P(T_3^e; Z, W) \cdot 1 \rangle
= 0,
\]
and we prove (4.11) for Case I in Lemma 4.3.

Next, we consider Case II when \( D(T^e) \neq \emptyset \). We observe that \( T_1^e \in T^e(n, k - 1), T_2^e, T_3^e \in T^e(n - 1, k) \). We also note that \( D(T_i^e) = \emptyset \), for some \( i = 1, 2, 3 \) may happen. Rewrite (4.13), by the induction hypothesis and (4.14), (4.15), we have:
\[
\langle 1', P(T^e; Z, W) \cdot 1 \rangle
= \langle 1', P(T_1^e; Z, W) \cdot 1 \rangle
+ Q(e_1; Z, W)\langle 1', P(T_2^e; Z, W) \cdot 1 \rangle
+ Q(e_2; Z, W)\langle 1', P(T_3^e; Z, W) \cdot 1 \rangle
= \sum_{D \in D(T_1^e)} R(D; Z, W)
+ \sum_{D \in D(T_2^e)} R(D; Z, W)
+ \sum_{D \in D(T_3^e)} R(D; Z, W)
= \sum_{D \in D(T^e)} R(D; Z, W).
\]
The sixth and the seventh line in the computation is by observing that, if \( D_1 = D - e \), then \( c(\sigma_D) = c(\sigma_{D_1}) \), and we have
\[
R(D; Z, W) = Q(e; Z, W)R(D_1; Z, W).
\] (4.16)

Hence we finish the proof of Lemma 4.3 for Case 1.

**Case 2:** The arguments here are similar, but there are more subcases. Formula (4.16) tells us that
\[
\langle 1, P(T^e; Z, W) \cdot 1 \rangle
= \langle 1, P(A; Z, W) L_{\bar{a}_{k+1}, \bar{b}_{k+1}}(z_{k+1}, w_{k+1}) L_{a_{k+1}, b_{k+1}}^+(z_k, w_k) P(B; Z, W) \cdot 1 \rangle
+ \langle 1, P(A; Z, W) [L_{\bar{a}_{k+1}^-, \bar{b}_{k+1}^+}(z_k, w_k), L_{a_{k+1}^-, b_{k+1}^+}(z_{k+1}, w_{k+1})] P(B; Z, W) \cdot 1 \rangle
= \langle 1, P(A; Z, W) L_{\bar{a}_{k+1}, \bar{b}_{k+1}}^-(z_{k+1}, w_{k+1}) L_{a_{k+1}, b_{k+1}}^+(z_k, w_k) P(B; Z, W) \cdot 1 \rangle
+ \frac{1}{2}(a_k, b_{k+1})(z_k - w_{k+1})^{-2}\langle 1, P(A; Z, W) L_{\bar{a}_{k+1}^-}^-(z_{k+1}, w_k) P(B; Z, W) \cdot 1 \rangle
+ \frac{1}{2}(a_k, a_{k+1})(z_k - z_{k+1})^{-2}\langle 1, P(A; Z, W) L_{ \bar{b}_{k+1}^+, b_k}^-(w_{k+1}, w_k) P(B; Z, W) \cdot 1 \rangle
\]
\[
+ \frac{1}{2} (b_k, b_{k+1})(w_k - w_{k+1})^{-2}(1, P(A; Z, W) L_{a_k+1, a_k}^{-} (z_{k+1}, z_k) P(B; Z, W) \cdot 1) \\
+ \frac{1}{2} (b_k, a_{k+1})(w_k - z_{k+1})^{-2}(1, P(A; Z, W) L_{b_{k+1}, a_k}^{-} (w_k, z_k) P(B; Z, W) \cdot 1) \\
+ \frac{r}{4} (a_k, a_{k+1})(z_k - z_{k+1})^{-2}(b_k, b_{k+1})(w_k - w_{k+1})^{-2}(1, P(AB; Z, W) \cdot 1) \\
+ \frac{r}{4} (a_k, b_{k+1})(z_k - w_{k+1})^{-2}(a_{k+1}, b_k)(w_k - z_{k+1})^{-2}(1, P(AB; Z, W) \cdot 1).
\]

(4.17)

We note that \( D(T^c) \) can also be decomposed into three disjoint subsets:

1. There is no edge connecting \((a_k^+, b_k^-)\) and \((a_{k+1}^-, b_{k+1}^-)\):

2. There is exactly one edge connecting \((a_k^+, b_k^-)\) and \((a_{k+1}^-, b_{k+1}^-)\), and there are four subcases:

3. There are exactly two edges connecting \((a_k^+, b_k^+)\) and \((a_{k+1}^-, b_{k+1}^-)\), and there are two subcases

Let

\[
T_{1}^{c1} = A(a_{k+1}^- b_{k+1}^-)(a_k^+, b_k^+)B, \quad T_{3}^{c3} = AB, \\
T_{21}^{c21} = A(a_{k+1}^- b_k^+)B, \quad T_{22}^{c22} = A(b_{k+1}^-, b_k^+)B, \\
T_{23}^{c23} = A(a_{k+1}^-, a_k^+)B, \quad T_{24}^{c24} = A(b_{k+1}^-, a_k^+)B,
\]
Then we also have a set decomposition:

\[ D(T^e) = (D(T_{11}^e)) \bigcup (D(T_{21}^e) + e_1) \bigcup (D(T_{22}^e) + e_2) \bigcup (D(T_{23}^e) + e_3) \bigcup (D(T_{24}^e) + e_4) \bigcup (D(T_{3}^e) + e_2 + e_3) \bigcup (D(T_{3}^e) + e_1 + e_4). \]  

(4.18)

We note that \( T_{11}^e \in T^e(n, k - 1) \), \( T_{21}^e \in T^e(n - 1, k - 1) \), and \( T_{3}^e \in T^e(n - 2, k - 1) \).

We first consider Case I, when \( D(T^e) = \emptyset \). In this situation, the sets \( D(T_{11}^e), D(T_{21}^e), D(T_{3}^e) \) are all empty. By (4.15), (4.17), and the induction hypothesis, we have

\[
\langle 1', P(T^e; Z, W) \cdot 1 \rangle = \langle 1', P(T_{11}^e; Z, W) \cdot 1 \rangle \\
= Q(e_1; Z, W) \langle 1', P(T_{21}^e; Z, W) \cdot 1 \rangle + Q(e_2; Z, W) \langle 1', P(T_{22}^e; Z, W) \cdot 1 \rangle \\
+ Q(e_3; Z, W) \langle 1', P(T_{23}^e; Z, W) \cdot 1 \rangle + Q(e_4; Z, W) \langle 1', P(T_{24}^e; Z, W) \cdot 1 \rangle \\
+ (rQ(e_2; Z, W)Q(e_3; Z, W) + rQ(e_1; Z, W)Q(e_4; Z, W)) \langle 1', P(T_{3}^e; Z, W) \cdot 1 \rangle = 0.
\]

Next, we consider Case II when \( D(T^e) \neq \emptyset \). We note that some of the sets \( D(T_{11}^e), D(T_{21}^e), D(T_{3}^e) \) may be empty. By (4.15), (4.17), (4.18), and the induction hypothesis,

\[
\langle 1', P(T^e; Z, W) \cdot 1 \rangle = \langle 1', P(T_{11}^e; Z, W) \cdot 1 \rangle \\
= Q(e_1; Z, W) \langle 1', P(T_{21}^e; Z, W) \cdot 1 \rangle + Q(e_2; Z, W) \langle 1', P(T_{22}^e; Z, W) \cdot 1 \rangle \\
+ Q(e_3; Z, W) \langle 1', P(T_{23}^e; Z, W) \cdot 1 \rangle + Q(e_4; Z, W) \langle 1', P(T_{24}^e; Z, W) \cdot 1 \rangle \\
+ (rQ(e_2; Z, W)Q(e_3; Z, W) + rQ(e_1; Z, W)Q(e_4; Z, W)) \langle 1', P(T_{3}^e; Z, W) \cdot 1 \rangle = \sum_{D \in D(T_{11}^e)} R(D; Z, W) \\
+ \sum_{D \in D(T_{21}^e)} R(D; Z, W) + Q(e_1; Z, W) \sum_{D \in D(T_{22}^e)} R(D; Z, W) \\
+ \sum_{D \in D(T_{23}^e)} R(D; Z, W) + Q(e_2; Z, W) \sum_{D \in D(T_{24}^e)} R(D; Z, W) \\
+ (rQ(e_2; Z, W)Q(e_3; Z, W) + rQ(e_1; Z, W)Q(e_4; Z, W)) \sum_{D \in D(T_{3}^e)} R(D; Z, W). \]

46
\[
\sum_{D \in D(T'_1)} R(D; Z, W) + \sum_{D \in D(T'_{21}) + e_1} R(D; Z, W) + \sum_{D \in D(T'_{22}) + e_2} R(D; Z, W)
\]
\[
+ \sum_{D \in D(T'_{33}) + e_3} R(D; Z, W) + \sum_{D \in D(T'_{34}) + e_4} R(D; Z, W)
\]
\[
+ \sum_{D \in D(T'_{35}) + e_2 + e_3} R(D; Z, W) + \sum_{D \in D(T'_{4})} R(D; Z, W)
\]
\[
= \sum_{D \in D(T')} R(D; Z, W).
\]

Here we also use (4.16), and note that for \(D_2 = D - e_1 - e_2\), we have \(c(D_2) = c(D) - 1\), and:
\[
R(D; Z, W) = rQ(e_1; Z, W)Q(e_2; Z, W)R(D_2; Z, W). 
\]

Hence we finish the proof of Lemma 4.3.

Using (4.1), (4.9), and Lemma 4.3, we see that
\[
\langle 1', P(T; Z, W) \cdot 1 \rangle = \sum_{\epsilon} \sum_{D \in D(T')} R(D; Z, W) = \sum_{D \in D(T)} R(D; Z, W).
\]

Hence we have:

**Proposition 4.3.**
\[
\langle 1', P(T; Z, W) \cdot 1 \rangle = \sum_{D \in D(T)} R(D; Z, W). 
\]

Theorem 1.1 is a corollary of Proposition 4.3 by letting \(z_1 = w_1, \ldots, z_n = w_n\). We note that for \(D \in DR(T)\), we have
\[
\prod_{e \in E_D} K(e, Z, Z) = \Gamma(\sigma_D; Z). 
\]

We also note that for all \(\sigma \in DR(T)\),
\[
\Gamma(\sigma; T) = \sum_{D \in D(T), s.t. \sigma_D = \sigma} \Gamma(D) 
\]
by a direct computation.

By (4.20), we have
\[
\langle 1', L_{a_1, b_1}(z_1) \cdots L_{a_n, b_n}(z_n) \cdot 1 \rangle = \sum_{\epsilon} \langle 1', P(T; Z, Z) \cdot 1 \rangle
\]
\[
= \sum_{D \in D(T)} R(D; Z, Z)
\]
\[
= \sum_{D \in D(T)} (\Gamma(D)r^{e(\sigma_D)} \prod_{e \in E_D} K(e; Z, Z))
\]

47
\begin{align*}
&= \sum_{D \in D(T)} (\Gamma(D)\Gamma(\sigma_D; Z)\Gamma(\sigma)) \\
&= \sum_{\sigma \in DR(T)} \left( \sum_{D \in D(T), \sigma_D = \sigma} \Gamma(D)\Gamma(\sigma; Z)\Gamma(\sigma) \right) \\
&= \sum_{\sigma \in DR(T)} \Gamma(\sigma; T)\Gamma(\sigma; Z)\Gamma(\sigma).
\end{align*}

Here we use (4.3), (4.21) and (4.22). Hence we get (1.1), and Theorem 1.1 is proved.
A main result in [NS10] is that $V_{J,r}$ is simple if and only if $r \notin \mathbb{Z}$. In this chapter, we reprove the 'if' part using a different method, and our method also applies to the cases when $J$ is of type $A$ or $C$.

Let $\bar{V}_{J,r}$ denote the simple quotient of $V_{J,r}$. Our main observation is that the Lie algebra $L$ is closely related to the infinite rank symplectic Lie algebra $C_\infty$ (see for example, Chap. 7 of [Kac94]). The Lie algebra $C_\infty$ is a Lie subalgebra of $L$. An ideal $\mathcal{I} \subseteq L$ acts as 0 on $V_{J,r}$, and $C_\infty \oplus \mathbb{C} c$ is isomorphic to $L/\mathcal{I}$, where $c$ is the central element in $L$. We also observe that $V_{J,r}$ is a generalized Verma module for $C_\infty$ [KR93]. Therefore we can reprove $V_{J,r} = \bar{V}_{J,r}$ if $r \notin \mathbb{Z}$, using the irreducibility criteria for the generalized Verma module.

It is easy to see that

$$W_N \overset{\text{def.}}{=} \text{span}\{a(i) | a \in \mathfrak{h}, 1 \leq |i| \leq N\}$$

is a $2dN$-dimensional symplectic space. The symplectic form is given by:

$$\langle a(m), b(n) \rangle = [a(m), b(n)]_{\text{new}} = m(a, b) \delta_{m+n,0}.$$ 

Let $k$ be an integer such that $1 \leq k \leq dN$. Then $k = (i-1)N + j$ for some $i, j$, with $1 \leq i \leq d$ and $1 \leq j \leq N$. We set

$$v_k = \frac{1}{j} e_i(j), \quad v_{-k} = e_i(-j),$$

where $\{e_i\}$ is an orthonormal basis of $\mathfrak{h}$. It is easy to check that $\{v_k | 1 \leq |k| \leq dN\}$ is a symplectic basis of $W_N$ such that $\langle v_k, v_l \rangle = \delta_{k+l,0}$, for all $k > 0$. 

49
We need the following lemma:

**Lemma 5.1.**

\[
\text{span}\left\{ \frac{1}{2}(v_kv_l + v_lv_k) \mid 1 \leq |k|, |l| \leq dN \right\}
\]

is a Lie algebra isomorphic to \(\mathfrak{sp}(2dN)\).

**Proof:** It is easy to see that the adjoint action of \(x \in \text{span}\left\{ \frac{1}{2}(v_kv_l + v_lv_k) \mid 1 \leq |k|, |l| \leq dN \right\}\) on \(W_N\)

\[
x \cdot v \overset{\text{def.}}{=} [x, v]_{\text{new}}
\]
preserves the symplectic form on \(W_N\), that is,

\[
\langle x \cdot u, v \rangle + \langle u, x \cdot v \rangle = 0,
\]
for all \(u, v \in W_N\).

Therefore

\[
\text{span}\left\{ \frac{1}{2}(v_kv_l + v_lv_k) \mid 1 \leq |k|, |l| \leq dN \right\} \subseteq \mathfrak{sp}(2dN).
\]

We conclude the proof of Lemma 5.1 by counting the dimension.

For convenience, we set

\[
g^{(N)} \overset{\text{def.}}{=} \mathfrak{sp}(2dN) \simeq \text{span}\left\{ \frac{1}{2}(v_kv_l + v_lv_k) \mid 1 \leq |k|, |l| \leq dN \right\}.
\]

We now analyze the root space decomposition of \(g^{(N)}\). Note that:

\[
g^{(N)} = g_+^{(N)} \bigoplus h^{(N)} \bigoplus g_-^{(N)},
\]

where

\[
g_+^{(N)} = \text{span}\left\{ \frac{1}{2}(v_kv_l + v_lv_k) \mid k + l > 0 \right\} = \text{span}\{v_kv_l \mid k + l > 0\},
\]

\[
h^{(N)} = \text{span}\left\{ \frac{1}{2}(v_kv_l + v_lv_k) \mid k + l = 0 \right\} = \text{span}\{v_-kv_k + \frac{c}{2} \mid k > 0\},
\]

\[
g_-^{(N)} = \text{span}\left\{ \frac{1}{2}(v_kv_l + v_lv_k) \mid k + l < 0 \right\} = \text{span}\{v_kv_l \mid k + l < 0\}.
\]

Consider the elements \(\epsilon_k \in (h^{(N)})^*, k = 1, \cdots, dN\) such that

\[
\epsilon_l(v_-kv_k + \frac{c}{2}) = -\delta_{k,l}.
\]

The positive and negative roots with respect to the triangular decomposition are:

\[
\Phi_+^{(N)} = \{+\epsilon_i + \epsilon_j \mid i \leq j\} \cup \{-\epsilon_i + \epsilon_j \mid i < j\},
\]

\[
\Phi_-^{(N)} = \{-\epsilon_i - \epsilon_j \mid i \leq j\} \cup \{+\epsilon_i - \epsilon_j \mid i < j\}.
\]
The corresponding simple roots are:
\[ \Delta^{(N)} = \{2\epsilon_1\} \cup \{-\epsilon_i + \epsilon_{i+1} | 1 \leq i < dN\}. \]

The half sum of positive roots is:
\[ \rho^{(N)} = \frac{1}{2} \sum_{\alpha \in \Phi_+^{(N)}} \alpha = \sum_{1 \leq i \leq dN} i\epsilon_i. \]

We recall the notion of ‘generalized Verma module of scalar type’ for \( \mathfrak{g}^{(N)} \) (for notations and conventions, see for example, Chap. 9 of [Hum08]). Define
\[ n^{(N)}_{-} \overset{\text{def.}}{=} \text{span}\{v_kv_l | k, l < 0\}, \]
\[ t^{(N)} \overset{\text{def.}}{=} \text{span}\{v_kv_l + \frac{c}{2}\delta_{k+l,0} | k < 0, l > 0\}, \]
\[ u^{(N)} \overset{\text{def.}}{=} \text{span}\{v_kv_l | k, l > 0\}, \]
\[ p^{(N)} \overset{\text{def.}}{=} t^{(N)} \oplus u^{(N)}. \]

Then we have decompositions:
\[ \mathfrak{g}^{(N)} = p^{(N)} \oplus n^{(N)}_{-} = t^{(N)} \oplus u^{(N)} \oplus n^{(N)}_{-}. \]

We also define the following set \( \Phi_I^{(N)} \):
\[ \Phi_I^{(N)} \overset{\text{def.}}{=} \{-\epsilon_i + \epsilon_j | i < j\} \cup \{\epsilon_i - \epsilon_j | i < j\}. \]

Note that \( t^{(N)} \) is spanned by \( \mathfrak{h}^{(N)} \), and the root spaces \((\mathfrak{g}^{(N)})_{\alpha}\), where \( \alpha \in \Phi_I^{(N)} \).

Define the 1-dimensional \( p^{(N)} \)-module of weight \( \lambda^{(N)} \in (\mathfrak{h}^{(N)})^* \) spanned by the element 1 as follows:
\[ x \cdot 1 = 0, \quad h \cdot 1 = \lambda^{(N)}(h) \cdot 1, \quad \text{for all } h \in \mathfrak{h}^{(N)}, \quad x \in (\mathfrak{g}^{(N)})_{\alpha}, \quad \alpha \in \Phi_I^{(N)} \cup \Phi_+^{(N)}. \]

The generalized Verma module \( M_I(\lambda^{(N)}) \) is defined as follows:
\[ M_I(\lambda^{(N)}) \overset{\text{def.}}{=} U(\mathfrak{g}^{(N)}) \otimes_{U(\mathfrak{g}^{(N)})} \mathbb{C} \cdot 1 \simeq U(n^{(N)}_{-}) \cdot 1. \]

It is known that \( M_I(\lambda^{(N)}) \) is a ‘generalized Verma module of scalar type’ (see for example, [Hum08]).

By Theorem 3.1 and (3.2),
\[ V_{\mathcal{J},r} \simeq U(\mathcal{L}) \otimes_{U(\mathcal{L}_+)} \mathbb{C}1 \]
as \( \mathcal{L} \)-modules, which are isomorphic to \( U(\mathcal{L}_-)1 \) as vector spaces. We want to show that \( V_{\mathcal{J},r} \) is a generalized Verma module for \( C_{\infty} \) ([KR93]). Observe that
the union of $\mathfrak{g}^{(N)}$ is isomorphic to the infinite rank symplectic Lie algebra $C_\infty$.

$$C_\infty = \cup_{k \geq 1} \mathfrak{g}^{(k)}.$$  

Define

$$\mathcal{I} \overset{\text{def.}}{=} \text{span}\{L_{a,b}(m,n)\mid a, b \in \mathfrak{h}, mn = 0\}.$$  

Then $\mathcal{I}$ is an ideal of $\mathcal{L}$, $\mathcal{I}$ acts as 0 on $V_{J,r}$. We also have:

$$\mathcal{L} = C_\infty \bigoplus \mathcal{I} \bigoplus C_c. \quad (5.2)$$  

Note that there are increasing exhaustive filtrations:

$$\{0\} = \mathfrak{n}_-^{(0)} \subseteq \mathfrak{n}_-^{(1)} \subseteq \cdots \subseteq \mathcal{L}_-, \quad \{0\} = \mathfrak{p}_-^{(0)} \subseteq \mathfrak{p}_-^{(1)} \subseteq \cdots \subseteq C_\infty \cap \mathcal{L}_+.$$  

(5.3)

It is easy to compute that:

$$(v_{-k}v_k + \frac{C}{2}) \cdot 1 = \frac{r}{2} \cdot 1, \quad v_kv_l \cdot 1 = 0,$$

for all $v_kv_l \in (\mathfrak{g}^{(N)})_{\alpha}, \alpha \in \Phi^{(N)} \cup \Phi^{(N)}_+$. Let

$$\lambda^{(N)} = -\frac{r}{2} \sum_{i=1, \ldots, dN} \epsilon_i.$$  

By comparing (3.2), (5.1), and use (5.3), we have an embedding of $\mathfrak{g}^{(N)}_-$ module:

$$M_I(\lambda^{(N)}) \hookrightarrow V_{J,r}.$$  

We also have an exhaustive filtration:

$$\{0\} \subseteq M_I(\lambda^{(1)}) \subseteq \cdots \subseteq M_I(\lambda^{(N)}) \subseteq \cdots \subseteq V_{J,r}.$$  

(5.4)

Therefore we conclude that $V_{J,r}$ is a ‘generalized Verma module of scalar type’ for $C_\infty$.

**Proof of the irreducibility of $V_{J,r}$ when $r \notin \mathbb{Z}$.** We first need the following lemma, whose proof can be found in [NS10]:

**Lemma 5.2 (NS10).** All proper VOA ideals of $V_{J,r}$ are also proper $\mathcal{L}$-submodules of $V_{J,r}$; hence $V_{J,r}$ is simple if and only if $M_r = V_{J,r}$ is simple as a $\mathcal{L}$-module.

From (5.2), it is easy to see that $\mathcal{L}$-submodules of $V_{J,r}$ are also in 1-1 correspondence with the $C_\infty$ submodules of $V_{J,r}$. Therefore the simplicity of the VOA $V_{J,r}$, is reduced to the simplicity of $V_{J,r}$ as a $C_\infty$-module.

We need another lemma about the simplicity of a finite dimensional generalized Verma module of scalar type, which can be found in [Hum08], Section 9.12:
Lemma 5.3 ([Hum08]). If $\lambda^{(N)}$ is a dominant integral weight for $g^{(N)}$, and

$$\langle \lambda^{(N)} + \rho^{(N)}, \beta^\vee \rangle \notin \mathbb{Z}_{>0}, \text{ for all } \beta \in \Phi^{(N)}_+ - \Phi^{(N)}_I,$$

then $M_I(\lambda^{(N)})$ is an irreducible $g^{(N)}$-module.

It is easy to compute that

$$\beta^\vee = \begin{cases} -v_k v_k - v_l v_l - c, & k \neq l, \\ -v_k v_k - \frac{c}{2}, & k = l, \end{cases}$$

for $\beta = \epsilon_k + \epsilon_l \in \Phi^{(N)}_+ - \Phi^{(N)}_I = \{\epsilon_i + \epsilon_j | 1 \leq i \leq j \leq dN\}$. Therefore

$$\langle \lambda^{(N)} + \rho^{(N)}, \beta^\vee \rangle = \begin{cases} -r + k + l, & k \neq l, \\ -\frac{r}{2} + k, & k = l. \end{cases}$$

It’s obvious that when $r \notin \mathbb{Z}$,

$$\langle \lambda^{(N)} + \rho^{(N)}, \beta^\vee \rangle \notin \mathbb{Z}_{>0}, \text{ for all } \beta \in \Phi^{(N)}_+ - \Phi^{(N)}_I.$$

Hence by Lemma 5.3, it is shown that $M_I(\lambda^{(N)})$ is irreducible as a $g^{(N)}$-module when $r \notin \mathbb{Z}$.

We now conclude the proof by contradiction. Suppose the contrary that $V_{J,r}$ is not simple when $r \notin \mathbb{Z}$, then it has a proper $C_\infty$ submodule $M$. Note that the filtration (5.4) is exhaustive, we deduce that $M \cap M_I(\lambda^{(N)})$ is a proper $g^{(N)}$-submodule of $M_I(\lambda^{(N)})$, for some $N$. This contradicts to the result that $M_I(\lambda^{(N)})$ is irreducible for all $N$ when $r \notin \mathbb{Z}$. Hence we conclude the proof.
Chapter 6

The Simple Quotients $\tilde{V}_{J,r}$, and
The Character Formula for

$r = -2n, n \geq 1$, where $J$ is of type $B$

In Chapter 5, we’ve proved that $V_{J,r}$ is irreducible when $r \notin \mathbb{Z}$. In this chapter, we construct the VOA $\tilde{V}_{J,r}$, $r \in \mathbb{Z}_{\neq 0}$, using the dual-pair type construction. As a corollary, $V_{J,r}$ is reducible when $r \in \mathbb{Z}_{\neq 0}$, and $\tilde{V}_{J,r}$ is the corresponding simple quotient. We also compute the character formula for $\tilde{V}_{J,r}, r = -2n, n \geq 1$. Here we exclude the trivial commutative case when $r = 0$.

6.1 Dual-Pair Realization of Case 1 and Case 2

Recall that in Chapter 1, Theorem 1.2, we divide the construction of $\tilde{V}_{J,r}$ into three cases according to the value of $r$. In this section, we give details about the constructions for Case 1 and Case 2 in Theorem 1.2.
As in Chapter 1, let $h$ denote a finite dimensional vector space with the symmetric non-degenerate bilinear form $(\cdot, \cdot)$, $\dim(h) = d$, $d \geq 2$, $V_m$ denote a $m$-dimensional vector space with the symmetric non-degenerate bilinear form $(\cdot, \cdot)$, and $W_n$ denote a $2n$-dimensional symplectic space with the symplectic form $\langle \cdot, \cdot \rangle$. We also assume that $\{e_i\}$ is an orthonormal basis of $h$. Our construction of the VOAs $\bar{V}_{J,r}$, $r \in \mathbb{Z} \neq 0$ for these two cases are given as follows.

**Construction of Case 1, $r = m \geq 1$.** Recall the Heisenberg VOA constructed as Example 1 in Section 2.3. We note that $h \otimes V_m$ is a $dm$-dimensional vector space with a non-degenerate symmetric bilinear form. Then we have the corresponding Lie algebra $\widehat{h \otimes V_m}$, and the corresponding Heisenberg VOA $\mathcal{H}(h \otimes V_m)$. The orthogonal group $O(dm)$ acts on $\widehat{h \otimes V_m}$ and $\mathcal{H}(h \otimes V_m)$ as automorphisms. The subgroup $O(m)$ acts on the component $V_m$, therefore we construct $\bar{V}_{J,m}$ as the fixpoint sub VOA of $\mathcal{H}(h \otimes V_m)$:

$$\bar{V}_{J,m} \overset{\text{def.}}{=} \mathcal{H}(h \otimes V_m)^{O(m)}.$$

We describe $\mathcal{H}(h \otimes V_m)^{O(m)}$ more explicitly using the invariant theory of $O(m)$. Recall that given a Lie algebra $g$, we have the corresponding standard filtration of the enveloping algebra $U(g)$:

$$\mathbb{C} = U^{(0)}(g) \subseteq \cdots U^{(n)}(g) \subseteq \cdots U(g).$$

We note that $O(m)$ also acts on $U(\widehat{h \otimes V_m})$ as automorphism. We consider the ‘degree two’ $O(m)$-invariants, and take $c = 1$. Let $(c - 1)$ denote the two sided ideal in $U(\widehat{h \otimes V_m})$ generated by $c - 1$. Define

$$\mathcal{L}_m \overset{\text{def.}}{=} U^{(2)}(\widehat{h \otimes V_m})^{O(m)}/((c - 1) \cap U^{(2)}(\widehat{h \otimes V_m})).$$

We will show that $\mathcal{L}_m$ is a Lie algebra.

Recall the definition of ‘normal ordering’ in Section 2.1. Let $f_1, \cdots, f_m$ be an orthonormal basis of $V_m$. Set

$$L_{a,b}^m(k, l) \overset{\text{def.}}{=} \frac{1}{2} \sum_{i=1, \cdots, m} : (a \otimes f_i)(k)(b \otimes f_i)(l) :.$$

By the invariant theory of $O(m)$ [GW09], we have

$$\mathcal{L}_m = \text{span}\{L_{a,b}^m(k, l) | a, b \in h, k, l \in \mathbb{Z}\}.$$

A direct computation shows that

$$[L_{a,b}^m(s, t), L_{a,v}^m(k, l)] = \frac{1}{2} t \delta_{t+k,0} (b, u) L_{a,v}^m(s, l) + \frac{1}{2} s \delta_{s+k,0} (a, u) L_{b,v}^m(t, l).$$
\[
\frac{1}{2} \delta_{t,t,0}(b,v)L_{a,b}^{m}(k,s) + \frac{1}{2} s \delta_{s+t,0}(a,v)L_{a,b}^{m}(k,t) \\
+ \frac{mst}{4} \delta_{t+k,0}(b,u)(a,v)\mathbb{I}_s + \frac{mst}{4} \delta_{s+k,0}(a,u)(b,v)\mathbb{I}_t \\
- \frac{mst}{4} \delta_{s+k,0}(b,v)(a,u)\mathbb{I}_{-s} - \frac{mst}{4} \delta_{t+k,0}(a,v)(b,u)\mathbb{I}_{-t}.
\] (6.1)

Hence \( \mathcal{L}_m \) is a Lie algebra. Moreover, the associative algebra \( U(\mathfrak{h} \otimes \mathbb{V}_m)^{O(m)}/(c-1) \) is generated by \( \mathcal{L}_m \), by the invariant theory of \( O(m) \):

\[
U(\mathfrak{h} \otimes \mathbb{V}_m)^{O(m)}/(c-1) = \text{span}\{L_{a_1,b_1}(m_1,n_1) \cdots L_{a_k,b_k}(m_k,n_k) | a_i, b_i \in \mathfrak{h}, m_i, n_i \in \mathbb{Z}\}.
\]

We remark that the computation of (6.1) is very similar to (3.1), because

\[
[(a \otimes f_i)(k)(b \otimes f_i)(l), (a \otimes f_j)(k)(b \otimes f_j)(l)] = 0, \text{ for all } i \neq j.
\]

By the structure of the Heisenberg VOA \( \mathcal{H}(\mathfrak{h} \otimes \mathbb{V}_m) \), we have the following description of \( \mathbf{V}_{\mathcal{J},m} \) through the invariant theory of \( O(m) \) [GW09]:

\[
\mathbf{V}_{\mathcal{J},m} = \mathcal{H}(\mathfrak{h} \otimes \mathbb{V}_m)^{O(m)} \simeq (S((\mathfrak{h} \otimes \mathbb{V}_m)_{-}))^{1}^{O(m)} \\
= \text{span}\{L_{a_1,b_1}(-m_1,-n_1) \cdots L_{a_k,b_k}(-m_k,-n_k) \cdot 1 | a_i, b_i \in \mathfrak{h}, m_i, n_i \geq 1\}.
\]

The Virasoro element \( \omega \) is given by

\[
\omega = \sum_{k=1, \ldots, d} L_{e_k,e_k}^{m}.
\]

It is computed that

\[
\omega(3)\omega = \frac{dm}{2} \omega.
\]

Therefore the central charge is equal to \( dm \).

It is obvious that when \( m = 1 \), \( O(1) = \{ \pm 1 \} \). Hence our construction of the VOA \( \mathbf{V}_{\mathcal{J},r} \) when \( r = m = 1 \), is exactly the same as the VOA \( \mathbf{V}_{\mathcal{J},1} \) constructed by Lam in [Lam99], which is previously described in Section 3.2.

**Construction of Case 2, \( r = -2k, n \geq 1 \).** Recall the symplectic fermion VOSA constructed as Example 2 in Section 2.3. We note that \( \mathfrak{h} \otimes \mathbb{W}_n \) is a \( 2dn \)-dimensional symplectic space. Then we have the corresponding Lie superalgebra \( \mathfrak{h} \otimes \mathbb{W}_n \), and the corresponding symplectic fermion VOSA \( \mathcal{A}(\mathfrak{h} \otimes \mathbb{W}_n) \). The symplectic group \( \text{Sp}(2dn) \) acts on \( \mathfrak{h} \otimes \mathbb{W}_n \) and \( \mathcal{A}(\mathfrak{h} \otimes \mathbb{W}_n) \) as automorphisms. The subgroup \( \text{Sp}(2n) \) acts on the component \( \mathbb{W}_n \), therefore we construct \( \mathbf{V}_{\mathcal{J},-2n} \) as the fixpoint sub VOSA of \( \mathcal{A}(\mathfrak{h} \otimes \mathbb{W}_n) \):

\[
\mathbf{V}_{\mathcal{J},-2n} \overset{\text{def}}{=} \mathcal{A}(\mathfrak{h} \otimes \mathbb{W}_n)^{\text{Sp}(2n)}.
\]
We also describe $\tilde{V}_{\mathcal{J},-2n}$ explicitly using the invariant theory of $Sp(2n)$. We use $(c-1)$ to denote the two sided ideal in $U(\hat{\mathfrak{h}} \otimes W_n)$ generated by $c-1$. Define

$$L_{-2n} \overset{\text{def.}}{=} U^{(2)}(\hat{\mathfrak{h}} \otimes W_n)^{Sp(2n)}/((c-1) \cap U^{(2)}(\hat{\mathfrak{h}} \otimes W_n)).$$

We will show that $L_{-2n}$ is a Lie algebra. Let $\psi_1, \cdots, \psi_n, \psi_1^*, \cdots, \psi_n^*$ be a symplectic basis of $W_n$ such that

$$\langle \psi_i^*, \psi_j \rangle = \delta_{i,j}, \langle \psi_i^*, \psi_j^* \rangle = \langle \psi_i, \psi_j \rangle = 0.$$

We recall the ‘normal ordering’ in the super case:

$$: a(m)b(n) := \begin{cases} (-1)^{p(a)p(b)} b(n)a(m), m \geq n, \\ a(m)b(n), m < n, \end{cases}$$

where $p(\cdot)$ is the parity function. Let

$$L_{-2n}^{-a,b}(k, l) \overset{\text{def.}}{=} \frac{1}{2} \sum_{j=1, \cdots, n} : (a \otimes \psi_j)(k)(b \otimes \psi_j^*)(l) : - \frac{1}{2} \sum_{j=1, \cdots, n} : (a \otimes \psi_j^*)(k)(b \otimes \psi_j)(l) :.$$

By the invariant theory of $Sp(2n)$ [GW09], we have:

$$L_{-2n} = \text{span}\{L_{-2n}^{-a,b}(k, l) | a, b \in \mathfrak{h}, k, l \in \mathbb{Z}\}.$$

A direct computation shows that:

$$[L_{-2n}^{-a,b}(s, t), L_{-2n}^{-u,v}(k, l)]$$

$$= \frac{1}{2} t \delta_{t+k,0}(b, u)L_{-2n}^{-a,v}(s, l) + \frac{1}{2} s \delta_{s+k,0}(a, u)L_{-2n}^{-2n}(t, l)$$

$$+ \frac{1}{2} t \delta_{t+l,0}(b, v)L_{-2n}^{-a,u}(k, s) + \frac{1}{2} s \delta_{s+t,0}(a, v)L_{-2n}^{-2n}(k, t)$$

$$- \frac{nst}{2} \delta_{s+k,0}\delta_{s+l,0}(b, u)(a, v)1_s - \frac{nst}{2} \delta_{s+k,0}\delta_{t+l,0}(a, u)(b, v)1_t$$

$$+ \frac{nst}{2} \delta_{s+k,0}\delta_{t+l,0}(b, v)(a, u)1_s - \frac{nst}{2} \delta_{s+k,0}\delta_{s+l,0}(a, v)(b, u)1_t. \quad (6.2)$$

We note that $L_{-2n}$ is even, therefore we’ve checked that $L_{-2n}$ is a Lie algebra. Moreover, the associative algebra $U(\hat{\mathfrak{h}} \otimes W_{-2n})^{Sp(2n)}/(c-1)$ is generated by $L_{-2n}$, by the invariant theory of $Sp(2n)$:

$$U(\hat{\mathfrak{h}} \otimes W_{-2n})^{O(m)}/(c-1)$$

$$= \text{span}\{L_{-2n}^{-2n}(m_1, n_1) \cdots L_{-2n}^{-2n}(m_k, n_k) | a_i, b_i \in \mathfrak{h}, m_i, n_i \in \mathbb{Z}\}.$$

We recall the structure of the symplectic fermion VOA given as Example 2 in section 2.3. By the invariant theory of $Sp(2n)$ [GW09], the fixpoint Sub VOA $\tilde{V}_{\mathcal{J},-2n}$ is explicitly described by:

$$\tilde{V}_{\mathcal{J},-2n} = A(\hat{\mathfrak{h}} \otimes W_n)^{Sp(2n)} \simeq (\hat{\mathfrak{h}} \otimes W_n)_- \cdot 1)^{Sp(2n)}.$$
\[\mathcal{V}_{\mathcal{J}, -2n} = \text{span}\{L_{a_1, b_1}^{-2n}(-m_1, -n_1) \cdots L_{a_k, b_k}^{-2n}(-m_k, -n_k) \cdot 1 \mid a_i, b_i \in \mathfrak{h}, m_i, n_i \geq 1\}.\]

We also note that \(\mathcal{V}_{\mathcal{J}, -2n}\) is actually a VOA, because all the elements are even. The Virasoro element is:

\[\omega = \sum_{k=1, \ldots, d} L_{e_k, e_k}^{-2n}.\]

It is computed that

\[\omega(3)\omega = -dn\omega,\]

Hence the central charge is equal to \(-2dn\).

We now compare (3.1), (6.1), and (6.2). The main observation of our construction is that we can unify (6.1) and (6.2) when \(r = m \geq 1\) or \(r = -2n, n \geq 1\):

\[\begin{aligned}
&[L_{a,b}^{-2n}(s,t), L_{u,v}^{-2n}(k,l)]_{\text{new}} \\
= &\frac{1}{2} \delta_{t+k,0}(b,u)L_{a,v}^{-2n}(s,l) + \frac{1}{2} s \delta_{s+k,0}(a,u)L_{b,v}^{-2n}(t,l) \\
&+ \frac{1}{2} t \delta_{t+l,0}(b,v)L_{u,a}^{-2n}(k,s) + \frac{1}{2} s \delta_{s+l,0}(a,v)L_{u,b}^{-2n}(k,t) \\
&+ \frac{rst}{4} \delta_{t+k,0}(b,u)(a,v)\mathbb{1}_s + \frac{rst}{4} \delta_{s+k,0}(a,u)(b,v)\mathbb{1}_t \\
&- \frac{rst}{4} \delta_{s+k,0}(b,v)(a,u)\mathbb{1}_{-s} - \frac{rst}{4} \delta_{t+k,0}(a,v)(b,u)\mathbb{1}_{-t}. \\
\end{aligned}\]

Moreover, the commutation relations (3.1) and (6.3) for the Lie algebras \(\mathcal{L}\) and \(\mathcal{L}_r\) are similar if we take \(c = r\) in (3.1). We will use this fact in Section 6.3. It’s easy to see that \(\mathcal{V}_{\mathcal{J}, r}\) in Case 1 and Case 2 can also be uniformly written as:

\[\mathcal{V}_{\mathcal{J}, r} = \text{span}\{L_{a_1, b_1}^r(-m_1, -n_1) \cdots L_{a_k, b_k}^r(-m_k, -n_k) \cdot 1 \mid a_i, b_i \in \mathfrak{h}, m_i, n_i \geq 1\}.\]

(6.4)

### 6.2 Dual Pair Realization of Case 3

In this section, we give details about the construction for Case 3, the VOA \(\mathcal{V}_{\mathcal{J}, r}\) when \(r = -2n + 1\). This case is slightly different from Case 1 and Case 2, because we need to consider an orthosymplectic superspace \(W\), the corresponding orthosymplectic Lie algebra \(osp(1|2n)\), and the corresponding orthosymplectic ‘supergroup’ \(Osp(1|2n)\), all of which act on \(W\).

Recall that a superspace \(W\) is a \(\mathbb{Z}/2\mathbb{Z}\)-graded space with \(W = W_0 \oplus W_1\), where \(W_0\) and \(W_1\) are called even and odd part of \(W\) respectively. A superspace \(W\)
is called orthosymplectic if $W$ has a supersymmetric bilinear form $(\cdot, \cdot)$, such that $(\cdot, \cdot)$ restricts to $W_0$ is non-degenerate symmetric, to $W_1$ is symplectic, and $W_0$, $W_1$ are orthogonal to each other:

$$(u, v) = 0, \text{ for all } u \in W_0, v \in W_1.$$  

For our purpose we set $W_0 = V_m$, $W_1 = W_n$.

We say the ‘superdimension’ of $W$ is $(m|2n)$, and we write:

$$\text{sdim}(W) = (m|2n).$$

Given an orthosymplectic super space $W$ with $\text{sdim}(W) = (m|2n)$, we have the corresponding Lie superalgebra $\mathfrak{osp}(m|2n)$, and the corresponding ‘super-group’ $Osp(m|2n)$. For general theory about $\mathfrak{osp}(m|2n)$ and $Osp(m|2n)$, see for example, [Kac77], [Ser01], and [DKW+99]. The orthosymplectic ‘super-group’ $Osp(m|2n)$ here means the ‘super Harich-Chandra pair’ $(\mathfrak{osp}(m|2n), O(m) \times Sp(2n))$ (see for example, [DKW+99]), where $O(m) \times Sp(2n)$ acts on $\mathfrak{osp}(m|2n)$ through the adjoint action:

$$g \cdot x \overset{\text{def.}}{=} gxg^{-1}.\text{We say } Osp(m|2n) \text{ ‘acts’ on a superspace } M, \text{ which means that } M \text{ is a } (\mathfrak{osp}(m|2n), O(m) \times Sp(2n))-\text{module such that:}$$

$$g(xv) = (g \cdot x)(gv), \text{ for all } g \in O(m) \times Sp(2n), x \in \mathfrak{osp}(m|2n), v \in M.$$  

Let $M^{\mathfrak{osp}(m|2n)}$ denote the subspace of $M$ which is killed by all elements in $\mathfrak{osp}(m|2n)$:

$$M^{\mathfrak{osp}(m|2n)} \overset{\text{def.}}{=} \{m \mid m \in M, xm = 0, \text{ for all } x \in \mathfrak{osp}(m|2n)\}.$$  

It’s easy to see that $O(m) \times Sp(2n)$ acts on $M^{\mathfrak{osp}(m|2n)}$. We define:

$$M^{Osp(m|2n)} \overset{\text{def.}}{=} (M^{\mathfrak{osp}(m|2n)})O(m)\times Sp(2n).$$

**Construction of Case 3,** $r = -2n + 1, n \geq 1$. We now focus on the special case $\text{sdim}(W) = (1|2n)$. We simply say ‘a superspace $W$’, and we omit the adjective ‘orthosymplectic’ for convenience. Observe that $\mathfrak{h} \otimes W$ is a superspace with the supersymmetric bilinear form:

$$(a \otimes u, b \otimes v) = (a, b)(u, v), \text{ for all } a \otimes u, b \otimes v \in \mathfrak{h} \otimes W.$$  

The even and odd parts are given by

$$(\mathfrak{h} \otimes W)_0 = \mathfrak{h} \otimes V_1 \simeq \mathfrak{h}, \quad (\mathfrak{h} \otimes W)_1 = \mathfrak{h} \otimes W_n.$$
It is easy to see that we have a corresponding super Lie algebra \( \hat{\mathfrak{h}} \otimes W \):
\[
\hat{\mathfrak{h}} \otimes W = (\mathfrak{h} \otimes W) \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}c,
\]
with the super Lie bracket given by:
\[
[a(m), b(n)] = m(a, b)\delta_{m+n,0}c, \quad [x, c] = 0, \text{ for all } x \in \hat{\mathfrak{h}} \otimes W.
\]
Here \( a(m) = at^m \). It is also easy to check that \( \hat{\mathfrak{h}} \otimes W \) is a super-commutative Lie subalgebra. The corresponding VOSA is essentially isomorphic to a tensor product VOSA:
\[
\mathcal{H}(\mathfrak{h} \otimes V_1) \otimes \mathcal{A}(\mathfrak{h} \otimes W_n) \simeq \mathcal{H}(\mathfrak{h}) \otimes \mathcal{A}(\mathfrak{h} \otimes W_n).
\]

It’s obvious that \( Osp(d|2\text{nd}) \) acts on \( \hat{\mathfrak{h}} \otimes W \) and \( \mathcal{H}(\mathfrak{h}) \otimes \mathcal{A}(\mathfrak{h} \otimes W_n) \). \( Osp(1|2n) \) acts on the component \( W \), therefore we define the sub VOSA \( \hat{\mathcal{V}}_{\mathcal{J}, -2n+1} \) as:
\[
\hat{\mathcal{V}}_{\mathcal{J}, -2n+1} \overset{\text{def.}}{=} (\mathcal{H}(\mathfrak{h}) \otimes \mathcal{A}(\mathfrak{h} \otimes W_n))^{Osp(1|2n)}.
\]

There is also an invariant theory for orthosymplectic supergroups. Set
\[
L_{-2n+1}^{a,b}(k, l) \overset{\text{def.}}{=} L_{a,b}(k, l) + L_{a,b}^{-2n}(k, l).
\]
Define
\[
\mathcal{L}_{-2n+1} \overset{\text{def.}}{=} U(2)(\hat{\mathfrak{h}} \otimes W)^{Osp(1|2n)} / ((c - 1) \cap U(2)(\hat{\mathfrak{h}} \otimes W)).
\]
By the invariant theory of \( Osp(1|2n) \) (see for example, [Ser01], [LZ16], and [LZ14]), we have:
\[
\mathcal{L}_{-2n+1} = \text{span}\{L_{a,b}^{-2n+1}(k, l), 1| a, b \in \mathfrak{h}, k, l \in \mathbb{Z}\}.
\]
Note that our observation at the end of Section 6.1 also applies here, and \( U(\hat{\mathfrak{h}} \otimes W)^{Osp(1|2n)} \) is generated by \( \mathcal{L}_{-2n+1} \). The formulas (6.3) and (6.4) still hold in this case. The Virasoro element is given by:
\[
\omega = \sum_k L_{-2n+1}^{e_k, e_k}
\]
and the central charge is equal to \( dr = -(2n - 1)d \). We also observe that \( \mathcal{L}_{-2n+1} \) is a Lie algebra, and \( \hat{\mathcal{V}}_{\mathcal{J}, -2n+1} \) is a VOA, because they are all even.

Our construction of \( \hat{\mathcal{V}}_{\mathcal{J}, r} \) in all three cases can be unified using the approach of this section, and (6.3) gives the commutation relation for all three cases.
We note that Case 1 (Case 2, Case 3, resp.) corresponds to the construction using superspace \( W \) with superdimension \((m|0), (0|2n), (1|2n)\), resp.)

We remark that these dual-pair constructions are analogues to dual pairs \((O(m), C_{\infty}),(Sp(2n), C_{\infty}), (Osp(1|2n), C_{\infty})\) studied by W. Wang in [Wan99a, Wan99b]. For each finite dimensional simple module for \(O(m), Sp(2n), Osp(1|2n)\), there is a corresponding \(C_{\infty}\)-module, which is also the corresponding \(\bar{V}_{J,r}\)-module, where \(r\) is the level with respect to each case.

6.3 Properties of \(\bar{V}_{J,r}, r \in \mathbb{Z}_{\neq 0}\)

In this section, we finish the proof of Theorem 1.2. We fix the following notation:

\[
\bar{V}_{J,r} = \text{span}\{L_{a,b}^{r}(-m_{1}, -n_{1}) \cdots L_{a_{k},b_{k}}^{r}(-m_{k}, -n_{k}) \cdot 1 | a_{i}, b_{i} \in \mathfrak{h}, m_{i}, n_{i} \geq 1\}.
\]

The formula (6.3) will be important for our computations.

**Proof of (3) in Theorem 1.2.** The central charges have already been computed in Section 6.2. Now we check the isomorphism between the Griess algebra \(V_{2}\) and the Jordan algebra \(J\). First, \(V_{0} = \mathbb{C}1, V_{1} = \{0\}\) is obvious, and

\[
V_{2} = \text{span}\{L_{a,b}^{r}(-1, -1) \cdot 1 | a, b \in \mathfrak{h}\}
\]

is also clear. By (6.3), it is computed that

\[
L_{a,b}^{r}(1)L_{u,v}^{r} = L_{a,b}^{r}(-1, 1)L_{u,v}^{r}(-1, -1) \cdot 1 + L_{a,b}^{r}(1, -1)L_{u,v}^{r}(1, -1) \cdot 1
\]

\[
= (b, u)L_{a,v}^{r} + (b, v)L_{a,u}^{r} + (a, u)L_{b,v}^{r} + (a, v)L_{b,u}^{r}.
\]

Therefore

\[
L_{a,b}^{r} \mapsto L_{a,b}, V_{2} \rightarrow J
\]

gives the isomorphism.

**Proof of (2) in Theorem 1.2.** The proof is essentially similar to the proof of Proposition 3.1 in [NS10]. It is enough to prove this for \(d = 2\). We may assume that \(a, b\) form an orthonormal basis of \(\mathfrak{h}\) such that:

\[
(a, a) = (b, b) = 1, (a, b) = 0.
\]

Let \(\bar{M}_{r}\) denote the VOA which is generated by \(V_{2}\):

\[
\bar{M}_{r} = \text{span}\{L_{a_{1},b_{1}}^{r}(l_{1}) \cdots L_{a_{k},b_{k}}^{r}(l_{k}) \cdot 1 | a_{i}, b_{i} \in \mathfrak{h}, l_{i} \in \mathbb{Z}\}.
\]

We need to show \(V = \bar{M}_{r}\). Because \(\bar{M}_{r} \subseteq V\) is obvious, it is enough to prove the converse.
We prove it by induction on the ‘length’ of the elements in $V$. For

$$L_{a_1,b_1}^r(-m_1,-n_1) \cdots L_{a_k,b_k}^r(-m_k,-n_k) \cdot 1 \in V,$$

we call it is of ‘length $k$’. We use $P(k)$ to denote the subspace of $V$ spanned by elements of length less or equal to $k$. We have a filtration:

$$\mathbb{C} \cdot 1 = P(0) \subseteq \cdots P(k) \subseteq \cdots V.$$

When $k = 0$, $\mathbb{C} \cdot 1 \in \bar{M}_r$ obviously holds.

Suppose $P(k) \in \bar{M}_r$ already holds. Let:

$$x \overset{\text{def.}}{=} L_{a_1,b_1}^r(-m_1,-n_1) \cdots L_{a_k,b_k}^r(-m_k,-n_k).$$

We want to show

$$L_{a,a}^r(k,l)x \cdot 1, L_{a,b}^r(k,l)x \cdot 1 \in \bar{M}_r. \quad (6.5)$$

Lemma 6.1.

$$L_{a,b}^r(-1,-1)x \cdot 1 \in \bar{M}_r.$$

Proof of Lemma 6.1. Let $y \overset{\text{def.}}{=} \sum_{k \neq -1} L_{a,b}^r(-k-2,k)$. We check that

$$L_{a,b}^r(-1) = L_{a,b}^r(-1,-1) + \sum_{k \neq -1} L_{a,b}^r(-k-2,k) = L_{a,b}^r(-1,-1) + y,$$

and

$$L_{a,b}^r(m,n) \cdot 1 = 0, \ [L_{a,b}^r(m,n),x] \cdot 1 \in P(k), \text{ if } m \geq 0, \text{ or } n \geq 0$$

hold. We also check that $L_{a,b}^r(-1)x \cdot 1 \in \bar{M}_r$, and:

$$L_{a,b}^r(-1)x \cdot 1 = L_{a,b}^r(-1,-1)x \cdot 1 + yx \cdot 1$$

$$= L_{a,b}^r(-1,-1)x \cdot 1 + [y,x] \cdot 1.$$

Therefore

$$L_{a,b}^r(-1,-1)x \cdot 1 = L_{a,b}^r(-1)x \cdot 1 - [y,x] \cdot 1 \in \bar{M}_r.$$

Hence we conclude the proof of Lemma 6.1.

For the remaining part, we divide (6.5) into two cases:

Case 1. $L_{a,b}^r(-m,-n)x \cdot 1 \in P(k+1) \subseteq \bar{M}_r$, for all $m,n \geq 1$.

Case 2. $L_{a,a}^r(-m,-n)x \cdot 1 \in P(k+1) \subseteq \bar{M}_r$, for all $m,n \geq 1$. 

62
Proof of Case 1. Take two elements $L_{a,a}^r, L_{b,b}^r$. A direct computation shows:

\[
[L_{a,a}^r(0), L_{a,b}(-m, -n)] = mL_{a,b}^r(-m - 1, -n),
\]

\[
[L_{b,b}^r(0), L_{a,b}(-m, -n)] = nL_{a,b}^r(-m, -n - 1).
\]

Therefore we have:

\[
\frac{L_{a,a}^r(0)^{m-1}L_{b,b}^r(0)^{n-1}}{(m-1)!(n-1)!} L_{a,b}^r(-1, -1) x \cdot 1
\]

\[
=L_{a,b}(-m, -n) x \cdot 1 + L_{a,b}^r(-1, -1) \frac{L_{a,a}^r(0)^{m-1}L_{b,b}^r(0)^{n-1}}{(m-1)!(n-1)!} x \cdot 1 \in \bar{M}_r
\]

by the induction hypothesis. Note that

\[
L_{a,b}^r(-1, -1) \frac{L_{a,a}^r(0)^{m-1}L_{b,b}^r(0)^{n-1}}{(m-1)!(n-1)!} x \cdot 1
\]

\[
=L_{a,b}(-1, -1)[\frac{L_{a,a}^r(0)^{m-1}L_{b,b}^r(0)^{n-1}}{(m-1)!(n-1)!}, x] \cdot 1 \in \bar{M}_r.
\]

By Lemma 6.1, we have

\[
L_{a,b}^r(-m, -n) x \cdot 1 \in \bar{M}_r.
\]

Hence we conclude the proof of this case.

Proof of Case 2. First, it is shown that

\[
(L_{a,b}^r(-m, -n) \cdot 1)(k) x \cdot 1
\]

\[
=[\frac{L_{a,a}^r(0)^{m-1}L_{b,b}^r(0)^{n-1}}{(m-1)!(n-1)!}, L_{a,b}^r(k)] x \cdot 1 \in \bar{M}_r.
\]

From the proof of Case 1, we have

\[
L_{b,a}^r(1)L_{a,b}^r(-m, -n) x \cdot 1 \in \bar{M}_r,
\]

\[
(L_{b,a}^r(-2, -1) \cdot 1)(2)L_{a,b}^r(-m, -n) x \cdot 1 \in \bar{M}_r,
\]

\[
L_{a,b}^r(-m, -n)(L_{b,a}^r(-1, -1) \cdot 1)(1) x \cdot 1 \in \bar{M}_r,
\]

and

\[
L_{a,b}^r(-m, -n)(L_{b,a}^r(-2, -1) \cdot 1)(2) x \cdot 1 \in \bar{M}_r.
\]

Then we deduce that

\[
[(L_{b,a}^r(-1, -1) \cdot 1)(1), L_{a,b}^r(-m, -n)] x \cdot 1 \in \bar{M}_r,
\]

\[
[(L_{b,a}^r(-2, -1) \cdot 1)(2), L_{a,b}^r(-m, -n)] x \cdot 1 \in \bar{M}_r.
\]

A direct computation shows that

\[
[(L_{b,a}^r(-1, -1) \cdot 1)(1), L_{a,b}^r(-m, -n)] = mL_{b,b}^r(-m, -n) \cdot 1 + nL_{a,a}^r(-m, -n),
\]

63
\[(L_{r,a}(-2, -1) \cdot 1)(2), L_{a,b}(-m, -n)]
\[= m(m - 1)L_{b,b}(-m, -n) \cdot 1 - n(n + 1)L_{a,a}(-m, -n).\]

Then we have:
\[mL_{r,b,b}(-m, -n)x \cdot 1 + nL_{r,a,a}(-m, -n)x \cdot 1 = [(L_{b,a}(-1, -1)x \cdot 1)(1), L_{r,b}(-m, -n)]x \cdot 1 \in \bar{M}_r,
\[m(m - 1)L_{r,b,b}(-m, -n) \cdot 1 - n(n + 1)L_{r,a,a}(-m, -n)x \cdot 1 = [(L_{r,a}(-2, -1) \cdot 1)(2), L_{r,b}(-m, -n)]x \cdot 1 \in \bar{M}_r.\]

Hence we have
\[L_{r,b,b}(-m, -n)x \cdot 1, L_{r,a,a}(-m, -n)x \cdot 1 \in \bar{M}_r,
\]
and we’ve proved Case 1 and Case 2 for (6.5). By induction on \(k\), \(P(k) \subseteq \bar{M}_r\), for all \(k \geq 1\), therefore \(V \subseteq \bar{M}_r\), and we conclude the proof.

**Remark.** We note that (3) in Theorem 1.2 is also satisfied by \(V_{J,r}\) (see \[AM09\]), and (2) in Theorem 1.2 also holds for \(V_{J,r}\) with the assumption \(d \geq 2\) (see \[NS10\]). But (2) fails if \(d = 1\) for both \(V_{J,r}\) and \(\bar{V}_{J,r}\). Let \(\mathcal{H}(\mathfrak{h})^+\) be the fixpoint sub-VOA of \(\mathcal{H}(\mathfrak{h})\) under the action of \(-1\) on \(\mathfrak{h}\). It is shown in \[DN99\] that when \(d = 1\), \(\mathcal{H}(\mathfrak{h})^+\) can be generated by the Virasoro element \(\omega\) and another degree 4 element \(J\). This suggests that from the view of Griess algebra, we should exclude the case \(d = 1\).

**Proof of (1) in Theorem 1.2.** This is done by establishing the relation between \(\mathcal{L}\) and \(\mathcal{L}_r\). By (6.3), it is obvious that the following map
\[U(\mathcal{L})/(c - r) \rightarrow U(\mathcal{L}_r)/(c - 1), \quad L_{a,b}(m, n) \mapsto L_{a,b}^r(m, n) \quad (6.6)\]
is an associative algebra homomorphism. Here \((c - r)\) and \((c - 1)\) means the corresponding two sided ideals generated by \(c - r\) and \(c - 1\) respectively. Note that \(L_{a,b}(k), L_{a,b}^r(k)\) are in certain completion of \(U(\mathcal{L})\) and \(U(\mathcal{L}_r)\) respectively. By Theorem 3.1, \(V_{J,r}\) is generated by \((V_{J,r})_2\) when \(d \geq 2\), hence the map (6.6) naturally extends to a VOA homomorphism
\[V_{J,r} \rightarrow \bar{V}_{J,r}. \quad (6.7)\]
The surjectivity of this map follows from (2) of Theorem 1.2 that \(\bar{V}_{J,r}\) is generated by \((\bar{V}_{J,r})_2\) when \(d \geq 2\).

We have the following lemma:

**Lemma 6.2** (\[DLM96\], Theorem 2.4 and Theorem 2.8). If \(V\) is a simple VOA, \(\mathfrak{g}\) is a simple Lie algebra, and \(\mathfrak{g}\) acts on \(V\) semisimply as a derivation, then \(V^\mathfrak{g}\) is simple. Similarly, if \(G\) is a reductive algebraic group, and \(G\) acts algebraically on \(V\) as an automorphism, then the VOA \(V^G\) is also simple.
Let $W$ be an orthosymplectic superspace with $\text{sdim}(W) = (m|0), (0|2n), (1|2n)$, respectively. Using the supersymmetric bilinear form over $W$, it is easy to check that the Fock spaces $\mathcal{H}(\mathfrak{h} \otimes V_m) \ (\mathcal{A}(\mathfrak{h} \otimes W_n), \mathcal{H}(\mathfrak{h}) \otimes \mathcal{A}(\mathfrak{h} \otimes W_n)$, respectively) are simple as $\mathfrak{h} \otimes V_m, \mathfrak{h} \otimes W_n, \mathfrak{h} \otimes W$, respectively)-modules, because the induced invariant bilinear form over the corresponding Fock spaces are non-degenerate (for detailed arguments, see the proof of Proposition 2.2 in [KR87]). This implies that $\mathcal{H}(\mathfrak{h} \otimes V_m) \ (\mathcal{A}(\mathfrak{h} \otimes W_n), \mathcal{H}(\mathfrak{h}) \otimes \mathcal{A}(\mathfrak{h} \otimes W_n)$, respectively) are all simple VO(S)As.

For Case 1 and 2, it is well known that $O(m), Sp(2n)$-actions are semisimple. Therefore by Lemma 6.2, $\overline{V}_{J, r}$ is simple when $r = m \geq 1$ or $r = -2n, n \geq 1$. For Case 3, we first have

$$\overline{V}_{J, -2n+1} = (\mathcal{H}(\mathfrak{h}) \otimes \mathcal{A}(\mathfrak{h} \otimes W_n))^{O(1) \times Sp(2n)}.$$

We note the following lemma:

**Lemma 6.3** (see for example, [Sch79], p.239, Theorem 1). The category of the finite dimensional osp(1|2n)-module is semisimple if and only if $\text{sdim}(W) = (m|0), (0|2n), \text{or} (1|2n)$.

By Lemma 6.3, $M = \mathcal{H}(\mathfrak{h}) \otimes \mathcal{A}(\mathfrak{h} \otimes W_n)$ is decomposed into a direct sum of irreducible osp(1|2n)-modules:

$$M = M^{osp(1|2n)} \bigoplus \bigoplus M^\lambda,$$

where $M^\lambda$ are non-trivial irreducible $osp(1|2n)$-modules. Note that the non-degenerated bilinear form over $M$ is invariant under the $osp(1|2n)$-action, thus $M^{osp(1|2n)}$ is orthogonal to all $M^\lambda$, and we deduce that the invariant bilinear form restricted to $M^{osp(1|2n)}$ is non-degenerate. By similar arguments applied to $G = O(1) \times Sp(2n) \simeq \{\pm 1 \times Sp(2n)\}$ and $V = M^{osp(1|2n)}$, it follows that the invariant bilinear form restricted to $\overline{V}_{J, -2n+1} = V^G$ is also non-degenerated, hence $\overline{V}_{J, -2n+1}$ is a simple VOA [Li94]. Therefore we have:

**Proposition 6.1.** $\overline{V}_{J, r}$ $r \in \mathbb{Z} \neq 0$ are all simple.

Therefore we conclude the proof of (1) in Theorem 1.2 as a corollary of this proposition. We remark that by the second fundamental theorem of invariants for classical groups [GW09] and orthosymplectic supergroup (see [LZ16], [LZ14]), it is easy to see that the kernel of the map (6.6) is non-zero. We can write down some elements in the kernel explicitly, and this explains Proposition 6.1 in [NS10] about the ‘high symmetry of singular vectors’. As another corollary, we also reprove that $\overline{V}_{J, r}$ is reducible when $r \in \mathbb{Z} \neq 0$. 

65
6.4 The Character Formula of the Simple Quotient $\tilde{V}_{J,r}, r = -2n, n \geq 1$

In this section, we prove Theorem 1.3 in the introduction, which gives the character formula for $\tilde{V}_{J,r}$ in Case 2, $r = -2n, n \geq 1$. By Theorem 1.2, $\tilde{V}_{J,-2n} = A(h \otimes W_n)^{Sp(2n)}$.

Because $Sp(2n)$ is simply connected, it is enough to calculate the character of $A(h \otimes W_n)^{sp(2n)}$. We set $g = sp(2n)$, and $V = V_{J,-2n}$ in this section. We also follow the notations mentioned before Theorem 1.3 in Chapter 1.

It is known from Section 6.1 that the Virasoro element $\omega \in V$ given by

$$\omega = \sum_k L_{-2n}^{e_k,e_k},$$

and $L(0) = \omega(1)$ gives the $\mathbb{Z}_{\geq 0}$-grading on $V$:

$$V = \bigoplus_{i \geq 0} V_i, \quad V_i = \{v \in V \mid L(0)v = iv\}.$$

Let $g_0$ be the Cartan subalgebra of $g$. It is easy to check that $g$-action commutes with $L(0)$, hence $A(h \otimes W_n)$ is decomposed into common eigenspaces of $g_0$ and $L(0)$ labeled by a pair $(\alpha, k), \alpha \in (g_0)^*, k \in \mathbb{Z}_{\geq 0}$:

$$A(h \otimes W_n) = \bigoplus_{(\alpha, k)} A(h \otimes W_n)(\alpha, k).$$

Define the $q$-graded formal character $ch_q(A(h \otimes W_n))$:

$$ch_q(A(h \otimes W_n)) \overset{def}{=} \sum_{(\alpha, k)} \dim(A(h \otimes W_n)(\alpha, k)) e^{\alpha} q^k.$$

Note that $A(h \otimes W_n) = \bigwedge (h \otimes W)_-$, therefore

$$ch_q(A(h \otimes W_n)) = \prod_{i=1, \cdots, d} (1 + e^{-\epsilon_i} q^j) d(1 + e^{\epsilon_i} q^j) d. \quad (6.8)$$

In particular, when $d = 1$ we have:

$$ch_q(A(W_n)) = \prod_{i=1, \cdots, d} (1 + e^{-\epsilon_i} q^j)(1 + e^{\epsilon_i} q^j). \quad (6.9)$$
On the other hand, the \(\mathfrak{g}\)-action on the Fock space \(\mathcal{A}(\mathfrak{h} \otimes W_n)\) is semisimple. Because all finite dimensional simple \(\mathfrak{g}\)-modules are isomorphic to \(L(\lambda)\) for some \(\lambda \in \Lambda^0_+\), we have a decomposition:

\[
\mathcal{A}(\mathfrak{h} \otimes W_n) = \bigoplus_{\lambda \in \Lambda^0_+} (L(\lambda) \otimes \mathcal{A}(\mathfrak{h} \otimes W_n)^{L(\lambda)}),
\]

where \(\mathcal{A}(\mathfrak{h} \otimes W_n)^{L(\lambda)}\) is the ‘multiplicity space’ with respect to \(L(\lambda)\) on which \(L(0)\) acts. In particular, \(\mathcal{A}(\mathfrak{h} \otimes W_n)^{0}\) is the multiplicity space with respect to the trivial representation \(\lambda = 0\). Using this decomposition, we have:

\[
\text{ch}_q(\mathcal{A}(\mathfrak{h} \otimes W_n)) \overset{\text{def.}}{=} \sum_{\lambda \in \Lambda^0_+} \text{ch}(L(\lambda)) \text{Tr}|_{\mathcal{A}(\mathfrak{h} \otimes W_n)^{L(\lambda)}} q^{L(0)}. \quad (6.10)
\]

In particular when \(d = 1\) we have:

\[
\text{ch}_q(\mathcal{A}(W_n)) \overset{\text{def.}}{=} \sum_{\lambda \in \Lambda^0_+} \text{ch}(L(\lambda)) \text{Tr}|_{\mathcal{A}(W_n)^{L(\lambda)}} q^{L(0)}.
\]

Following the notation in [CL16], we define the ‘branching functions’ \(B_{\lambda}(q)\):

\[
B_{\lambda}(q) \overset{\text{def.}}{=} \text{Tr}|_{\mathcal{A}(W_n)^{L(\lambda)}} q^{L(0)}.
\]

and we rewrite the above as:

\[
\text{ch}_q(\mathcal{A}(W_n)) \overset{\text{def.}}{=} \sum_{\lambda \in \Lambda^0_+} \text{ch}_q(L(\lambda)) B_{\lambda}(q). \quad (6.11)
\]

We remark that the ‘character’ in [CL16] means \(\text{Tr}_q q^{L(0)} - \frac{1}{2}\), and our definition of the ‘branching functions’ is slightly different here.

The explicit formula for \(B_{\lambda}(q)\) has been obtained by Linshaw and Creutzig in [CL16], as Corollary 5.5. They derive it by applying the Jacobi triple product identity to (6.9) and compare it with (6.11). Introduce an element:

\[
\rho \overset{\text{def.}}{=} \rho_0 - \rho_1.
\]

Let \(W^0\) denote the Weyl group of \(\mathfrak{sp}(2n)\). Then their formula reads:

\[
B_{\lambda}(q) = P(q)^n \sum_{w \in W^0} (-1)^{l(w)} q^{\frac{1}{2}(w(\lambda + \rho_0) - \rho, w(\lambda + \rho_0) - \rho) - \frac{1}{2}(\rho_1, \rho_1)}. \quad (6.12)
\]

Note that the element \(\rho\) is exactly the half sum of positive roots in \(\Phi\), and the Weyl group of \(\mathfrak{so}(2n + 1)\) is isomorphic to \(W^0\), the Weyl group of \(\mathfrak{sp}(2n)\). We can rewrite (6.12) the same as (1.2). Define the ‘specialization of type \(\lambda\’\) \(F_{\lambda}\) on the formal character by:

\[
F_{\lambda}(\epsilon^\mu) = q^{(\lambda, \mu)}.
\]
We have

\[ B_\lambda(q) = P(q)^n q^{\frac{1}{2}(w(\lambda + \rho_0), w(\lambda + \rho_0)) + \frac{1}{2}(\rho, \rho) - \frac{1}{2}(\rho_1, \rho_1)} \sum_{w \in W^0} (-1)^{l(w)} q^{-(\lambda + \rho_0, w(\rho))} \]

\[ = P(q)^n q^{\frac{1}{2}(\lambda + \rho_0, \lambda + \rho_0) + \frac{1}{2}(\rho, \rho) - \frac{1}{2}(\rho_1, \rho_1)} F_{-\lambda - \rho_0}(\sum_{w \in W^0} (-1)^{l(w)} e^{w(\rho)}) \]

\[ = P(q)^n q^{\frac{1}{2}(\lambda + \rho_0, \lambda + \rho_0) + \frac{1}{2}(\rho, \rho) - \frac{1}{2}(\rho_1, \rho_1)} F_{-\lambda - \rho_0}(e^\rho \prod_{\alpha \in \Phi^+} (1 - e^{-\alpha})) \]

\[ = q^{\frac{1}{2}(\lambda + \rho_1, \lambda + \rho_1) - \frac{1}{2}(\rho_1, \rho_1)} P(q)^n \prod_{\alpha \in \Phi^+} (1 - q^{\lambda(\rho_0, \alpha)}). \]

Here we use the denominator identity of \( so(2n + 1) \):

\[ e^\rho \prod_{\alpha \in \Phi^+} (1 - e^{-\alpha}) = \sum_{w \in W^0} (-1)^{l(w)} e^{w(\rho)}, \]

and we note that the inner product \( \langle \cdot, \cdot \rangle \) is invariant under the Weyl group action. The formula (1.2) when \( \lambda = 0 \) is obtained in [CL16], as Corollary 5.6.

Combine (6.8), (6.9), and (6.11), we have:

\[ \text{ch}_q(\mathcal{A}(\mathfrak{h} \otimes W_n)) = (\text{ch}_q(\mathcal{A}(W))^d \]

\[ = (\sum_{\lambda \in \Lambda_0^+} \text{ch}(L(\lambda)) B_\lambda(q))^d \]

\[ = \sum_{\lambda_1, \ldots, \lambda_d \in \Lambda_0^+} \text{ch}(L(\lambda_1)) \cdots \text{ch}(L(\lambda_d)) B_{\lambda_1}(q) \cdots B_{\lambda_d}(q) \]

\[ = \sum_{\mu \in \Lambda_0^n} m^\mu_{\lambda_1, \ldots, \lambda_d} \text{ch}(L(\mu)) B_{\lambda_1}(q) \cdots B_{\lambda_d}(q). \]

Compare this with (6.10), and use the fact that \( \text{ch}(L(\lambda)) \) are linearly independent, we have:

\[ \text{Tr}_{\mathcal{A}(\mathfrak{h} \otimes W_n)^L(\mu)} q^{L(0)} = \sum_{\mu \in \Lambda_0^n} m^\mu_{\lambda_1, \ldots, \lambda_d} B_{\lambda_1}(q) \cdots B_{\lambda_d}(q). \]

Hence Theorem 1.3 is obtained by taking \( \mu = 0 \). We remark that \( m^\nu_{\lambda, \mu} \) are called Clebsch-Gordan coefficients of \( sp(2n) \), and \( m^\mu_{\lambda_1, \ldots, \lambda_d} \) can be expressed by \( m^\nu_{\lambda, \mu} \). It’s an interesting fact that our character formula is related to these Clebsch-Gordan coefficients.
Chapter 7

Construction of the VOA $V_{\mathcal{J},r}$, Where $\mathcal{J}$ is of Hermitian Type

In this chapter, we will construct $V_{\mathcal{J},r}$ where $\mathcal{J}$ is one of the remaining two types of Hermitian Jordan algebras: the type $A$ and type $C$ Jordan algebras. The constructions for these two cases are quite similar to the construction for the type $B$ case given by Ashihara and Miyamoto, which is already described in Section 3.2. We will also give a uniform description of $V_{\mathcal{J},r}$ for all Hermitian Jordan algebras.

We give a brief account of the idea in this chapter. Observe that when $\mathcal{J}$ is a type $B$ Jordan algebra, we construct a corresponding infinite dimensional Lie algebra $\mathcal{L}$ by taking ‘quadratic elements’, together with the center in the enveloping algebra $U(\mathfrak{h})$. By analogy, when $\mathcal{J}$ is a type $C$ Jordan algebra, we take the corresponding symplectic space $W$, and then take the ‘quadratic elements’ together with the center in the enveloping algebra $U(W)$, so that we construct a ‘new’ Lie algebra in the same way. A direct calculation shows that the parallel construction results in a VOA whose Griess algebra is exactly isomorphic to the type $C$ Jordan algebra. The construction for type $A$ case is also similar.

Construction of $V_{\mathcal{J},r}$ when $\mathcal{J}$ is a type $C$ Jordan algebra: Let $W$ be a $2n$-dimensional symplectic space with the symplectic form $\langle \cdot, \cdot \rangle$. Then $W \otimes W$ has an associative algebra structure:

$$(a \otimes b)(u \otimes v) = \langle b, u \rangle a \otimes v,$$

which induces a Jordan algebra structure on $W \otimes W$:

$$x \circ y = \frac{1}{2}(xy + yx), \text{ for all } x, y \in W \otimes W.$$
Although we use the symplectic form instead of the symmetric form, this Jordan algebra is isomorphic to the type $A_n$ Jordan algebra of rank $2n$ introduced before. The type $C_n$ Jordan algebra of rank $2n$ is realized as the Jordan subalgebra of $W \otimes W$ consists of anti-symmetric tensors:

$$J \simeq \wedge^2(W) = \text{span}\{L_{a,b} | a, b \in W\}, \quad L_{a,b} \overset{\text{def.}}{=} a \otimes b - b \otimes a.$$ 

Here we assume that $n \geq 2$. To construct the VOA $V_{J,r}$, we recall the following Lie superalgebra $\hat{W}$ associated to $W$:

$$\hat{W} = W \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}c,$$

with the Lie superbracket:

$$[a(m), b(n)] = m(a, b)\delta_{m+n,0}c, \quad [x, c] = 0, \text{ for all } x \in \hat{W},$$

where $a(m) \overset{\text{def.}}{=} a \otimes t^m$, $a \in W$. We also consider the localized associative algebra $U(\hat{W})[c^{-1}]$, which is obtained by inverting the central element $c$ in $U(\hat{W})$. It is checked that $[U(\hat{W}), U(\hat{W})] \subseteq cU(\hat{W})$, therefore we can define a ‘new’ Lie bracket $[x, y]_{\text{new}}$:

$$[x, y]_{\text{new}} \overset{\text{def.}}{=} \frac{1}{c}[x, y], \text{ for all } x, y \in U(\hat{W}).$$

Because $c^{-1}$ cancels with the $c$ appeared in $[x, y]$, hence the right hand side is still in $U(\hat{W})$. We also check that the subspace $\mathcal{L}$:

$$\mathcal{L} \overset{\text{def.}}{=} \text{span}\{a(m)b(n) | a, b \in W, m, n \in \mathbb{Z}\} \oplus \mathbb{C}c,$$

is a Lie sub-superalgebra. Because $\mathcal{L}$ is even, $\mathcal{L}$ is actually a Lie algebra.

Set

$$L_{a,b}(m, n) \overset{\text{def.}}{=} \frac{1}{2} : a(m)b(n) :.$$

As in Section 3.2, we define the function

$$1_m = \begin{cases} 1, & m \geq 0, \\ 0, & m < 0. \end{cases}$$

We check that

$$\frac{1}{2} a(m)b(n) = L_{a,b}(m, n) + \frac{1}{2} m\delta_{m+n,0}(a, b)1_{m-n}c,$$

and

$$\mathcal{L} = \text{span}\{L_{a,b}(m, n) | a, b \in W, m, n \in \mathbb{Z}\} \oplus \mathbb{C}c.$$

By a direct computation, it is easy to show that for $L_{a,b}(s, t), L_{u,v}(k, l) \in \mathcal{L}$,

$$[L_{a,b}(s, t), L_{u,v}(k, l)]_{\text{new}}$$
\begin{align}
&= \frac{1}{2} t \delta_{t+k,0} \langle b, u \rangle L_{a,v}(s, l) - \frac{1}{2} s \delta_{s+k,0} \langle a, u \rangle L_{b,v}(t, l) \\
&+ \frac{1}{2} t \delta_{t+l,0} \langle b, v \rangle L_{u,a}(k, s) - \frac{1}{2} s \delta_{s+l,0} \langle a, v \rangle L_{u,b}(k, t) \\
&+ \frac{stc}{4} \delta_{t+k,0} \delta_{s+l,0} \langle b, u \rangle 1_s - \frac{stc}{4} \delta_{s+k,0} \delta_{t+l,0} \langle a, u \rangle 1_t \\
&+ \frac{stc}{4} \delta_{s+k,0} \delta_{t+l,0} \langle b, v \rangle 1_{-s} - \frac{stc}{4} \delta_{t+k,0} \delta_{s+l,0} \langle a, v \rangle 1_{-t}. \tag{7.1}
\end{align}

Let

\begin{align*}
\mathfrak{B}_+ & \overset{\text{def.}}{=} \text{span}\{L_{a,b}(m, n)\mid n \geq 0 \text{ or } m \geq 0\}, \\
\mathcal{L}_- & \overset{\text{def.}}{=} \text{span}\{L_{a,b}(m, n)\mid m, n < 0\}, \\
\mathcal{L}_+ & \overset{\text{def.}}{=} \mathfrak{B}_+ \bigoplus \mathbb{C}.
\end{align*}

Then we have a decomposition of \( \mathcal{L} \):

\[ \mathcal{L} = \mathcal{L}_- \bigoplus \mathcal{L}_+ = \mathcal{L}_- \bigoplus \mathfrak{B}_+ \bigoplus \mathbb{C}. \]

Define a 1-dimensional \( \mathcal{L}_+ \)-module \( C1 \):

\[ x1 = 0, \text{ for all } x \in \mathfrak{B}_+, \quad c1 = r1. \]

Then by induction from \( U(\mathcal{L}_+) \) to \( U(\mathcal{L}) \), we have a \( U(\mathcal{L}) \)-module \( M_r \):

\[ M_r \overset{\text{def.}}{=} U(\mathcal{L}) \otimes_{U(\mathcal{L}_+)} \mathbb{C} = U(\mathcal{L}_-) 1 \]

\[ = \text{span}\{L_{a_1,b_1}(-m_1, -n_1) \cdots L_{a_k,b_k}(-m_k, -n_k) \cdot 1 \mid m_i, n_i \in \mathbb{Z}_{\geq 1}, a_i, b_i \in W\}. \tag{7.2} \]

Because \( c \) acts as \( r \) on \( M_r \), we can take \( c = r \) in (7.1).

For \( a, b \in W \), define the operators \( L_{a,b}(l) \) and the fields \( L_{a,b}(z) \):

\[ L_{a,b}(l) \overset{\text{def.}}{=} \sum_{k \in \mathbb{Z}} L_{a,b}(-k + l - 1, k), \quad L_{a,b}(z) \overset{\text{def.}}{=} \sum_{l \in \mathbb{Z}} L_{a,b}(l) z^{-l-1}. \]

Analogous to the proof in [AM09], it can be shown that these fields are mutually local. Therefore by Theorem 2.3, these mutually local fields generate a vertex superalgebra, which is denoted by \( V_{\mathcal{J},r} \):

\[ V_{\mathcal{J},r} \overset{\text{def.}}{=} \text{span}\{L_{a_1,b_1}(m_1) \cdots L_{a_k,b_k}(m_k) \cdot 1 \mid m_i \in \mathbb{Z}, a_i, b_i \in W\}. \]

Because all elements in \( V_{\mathcal{J},r} \) are even, \( V_{\mathcal{J},r} \) is actually a vertex algebra.

By comparing the above construction with the construction for type \( B \) case described in Section 3.2, we see that the Lie algebras \( \mathcal{L} \) play the key roles, and
the constructions in both cases are similar. The common feature of Hermitian Jordan algebras allows us to give a direct and uniform construction of $V_{J,r}$ without using the Lie (super)algebras $\hat{h}$ and $\hat{W}$.

We first gives a uniform construction of $V_{J,r}$ where $J$ is a type $B$ or type $C$ Jordan algebra. Let $U$ be a finite dimensional vector space with a non-degenerated reflexive bilinear form $(\cdot, \cdot)$, then it is known that $(\cdot, \cdot)$ is either symmetric or anti-symmetric (see for example, [Gro02]), which corresponds to type $B$ case or type $C$ case respectively.

Define the following sign function:

$$
\epsilon_U \overset{\text{def.}}{=} \begin{cases} 
1, & (\cdot, \cdot) \text{ is symmetric}, \\
-1, & (\cdot, \cdot) \text{ is anti-symmetric}.
\end{cases}
$$

Then the following space $\mathcal{L}(U)$

$$
\mathcal{L}(U) \overset{\text{def.}}{=} \begin{cases} 
S^2(U \otimes \mathbb{C}[t, t^{-1}]) \bigoplus \mathbb{C}c, & (\cdot, \cdot) \text{ is symmetric}, \\
\wedge^2(U \otimes \mathbb{C}[t, t^{-1}]) \bigoplus \mathbb{C}c, & (\cdot, \cdot) \text{ is anti-symmetric}.
\end{cases}
$$

is a Lie algebra. Define the elements

$$
L_{a,b}(m, n) \overset{\text{def.}}{=} (a \otimes t^m) \otimes (b \otimes t^n) + \epsilon_U (b \otimes t^n) \otimes (a \otimes t^m) \in \mathcal{L}(U), \ a, b \in U.
$$

Then the non-trivial Lie bracket in $\mathcal{L}(U)$ is given by:

$$
\begin{align*}
\left[ L_{a,b}(m, n), L_{u,v}(k, l) \right] &= -\frac{1}{2} n \delta_{n+k,0}(b, u)L_{a,v}(m, l) + \frac{\epsilon_U}{2} m \delta_{m+k,0}(a, u)L_{b,v}(n, l) \\
&\quad + \frac{1}{2} n \delta_{n+t,0}(b, v)L_{a,u}(k, m) + \frac{\epsilon_U}{2} m \delta_{m+t,0}(a, v)L_{b,u}(k, n) \\
&\quad + \frac{mn}{4} \delta_{n+k,0} \delta_{m+t,0} (b, u)(a, v) \mathbb{1}_m + \frac{\epsilon_U m n}{4} \delta_{m+k,0} \delta_{n+t,0} (a, u)(b, v) \mathbb{1}_n \\
&\quad - \frac{\epsilon_U m n}{4} \delta_{m+k,0} \delta_{n+t,0} (b, v)(a, u) \mathbb{1}_m - \frac{m n}{4} \delta_{n+k,0} \delta_{m+t,0} (a, v)(b, u) \mathbb{1}_n,
\end{align*}
$$

and $c$ is the central element in $\mathcal{L}(U)$. The remaining part of the construction is the same as before by assigning $\mathcal{L} = \mathcal{L}(U)$.

**Construction of $V_{J,r}$ when $J$ is a type $A$ Jordan algebra:** Let $U$ be a $n$-dimensional vector space with a non-degenerated symmetric bilinear form $(\cdot, \cdot)$. We now construct $V_{J,r}$, where $J \simeq U \otimes U$ is the type $A$ Jordan algebra in a similar way. Here we also assume that $n \geq 2$. Define the following vector space associated to $U$:

$$
\tilde{\mathcal{L}}(U) \overset{\text{def.}}{=} (U \otimes \mathbb{C}[t, t^{-1}]) \otimes (U \otimes \mathbb{C}[t, t^{-1}]) \bigoplus \mathbb{C}c.
$$

Let

$$
\tilde{L}_{a,b}(m, n) \overset{\text{def.}}{=} (a \otimes t^m) \otimes (b \otimes t^n), \ a, b \in U.
$$
Let $\epsilon = 1$ or $-1$. Then $\tilde{L}(U)$ is a Lie algebra. The non-trivial Lie bracket (associated to $\epsilon$) is given by:

$$[	ilde{L}_{a,b}(m,n),\tilde{L}_{u,v}(k,l)]_\epsilon = \frac{1}{2} n \delta_{n+k,0}(b,u)\tilde{L}_{a,v}(m,l) + \frac{\epsilon}{2} m \delta_{m+l,0}(v,a)\tilde{L}_{u,b}(k,n) + \frac{n mc}{4} \delta_{n+k,0}\delta_{m+l,0}(b,u)(v,a)(1_m - 1_n).$$  \hfill (7.4)

We remark that, although the Lie bracket on $\tilde{L}(U)$ which is given above depends on the number $\epsilon$, the corresponding Lie algebra structures on $\tilde{L}(U)$ are actually isomorphic.

We observe that $\tilde{L}(U)$ can also be realized as a $\mathbb{C}^\times$-fixpoint Lie subalgebra of $L(U \oplus U)$. The $\mathbb{C}^\times$ action is given by:

$$u \cdot (a,b) = (ua,u^{-1}b), \text{ for all } u \in \mathbb{C}^\times, (a,b) \in U \oplus U,$$

and the bilinear form over $U \oplus U$ is given by

$$(a,b)(u,v) = (a,v) + \epsilon (b,u), \text{ for all } (a,b),(u,v) \in U \oplus U, \epsilon = \pm 1.$$

We check that the bilinear form $(\cdot,\cdot)_\epsilon$ is (non-degenerate) symmetric if $\epsilon = 1$, and it is (non-degenerate) anti-symmetric if $\epsilon = -1$.

Let

$$\tilde{\mathfrak{B}}^+_{+}(U) \overset{\text{def}}{=} \text{span}\{\tilde{L}_{a,b}(m,n) | n \geq 0 \text{ or } m \geq 0\},$$

$$\tilde{\mathfrak{L}}_- (U) \overset{\text{def}}{=} \text{span}\{\tilde{L}_{a,b}(m,n) | m,n < 0\}, \quad \tilde{\mathfrak{L}}^+ (U) \overset{\text{def}}{=} \tilde{\mathfrak{B}}^+_{+}(U) \bigoplus \mathbb{C}c.$$

Then we have a decomposition of $L$:

$$\tilde{L}(U) = \tilde{\mathfrak{L}}_- (U) \bigoplus \tilde{\mathfrak{L}}^+ (U) = \tilde{\mathfrak{L}}_- (U) \bigoplus \tilde{\mathfrak{B}}^+_{+}(U) \bigoplus \mathbb{C}c.$$

We note that the above decomposition is compatible with the decomposition of $L = L(U \oplus U)$, if we view $\tilde{L}(U)$ as Lie subalgebra of $L(U \oplus U)$.

Define a 1-dimensional $L_{++}$-module $C1$:

$$x1 = 0, \text{ for all } x \in \tilde{\mathfrak{B}}^+_{+}(U), \quad c1 = r1.$$

Then by induction from $U(\tilde{\mathfrak{L}}^+_{+}(U))$ to $U(\tilde{L}(U))$, we have a $U(\tilde{L}(U))$-module $M_r$:

$$M_r \overset{\text{def}}{=} U(\tilde{L}(U)) \otimes U(\tilde{\mathfrak{L}}^+ (U)) C1 \cong U(\tilde{\mathfrak{L}}_- (U))1 = \text{span}\{\tilde{L}_{a_1,b_1}(-m_1,-n_1) \cdots \tilde{L}_{a_k,b_k}(-m_k,-n_k) \cdot 1 | m_i,n_i \in \mathbb{Z}_{\geq 1}, a_i,b_i \in U\}. \hfill (7.5)$$
For $a, b \in U$, define the operators $\tilde{L}_{a,b}(l)$ and the fields $\tilde{L}_{a,b}(z)$:

$$
\tilde{L}_{a,b}(l) \overset{\text{def.}}{=} \sum_{k \in \mathbb{Z}} \tilde{L}_{a,b}(-k + l - 1, k), \quad \tilde{L}_{a,b}(z) \overset{\text{def.}}{=} \sum_{l \in \mathbb{Z}} \tilde{L}_{a,b}(l) z^{-l-1}.
$$

Analogous to the proof in [AM09], it can be shown that these fields are mutually local.

Hence by Theorem 2.3, these mutually local fields generate a vertex algebra, which is also denoted by $V_{\mathcal{J},r}$:

$$
V_{\mathcal{J},r} \overset{\text{def.}}{=} \text{span}\{\tilde{L}_{a_1, b_1}(m_1) \cdots \tilde{L}_{a_k, b_k}(m_k) \cdot 1 | m_i \in \mathbb{Z}, a_i, b_i \in U\}.
$$

The VOAs $V_{\mathcal{J},r}$ where $\mathcal{J}$ is a Hermitian Jordan algebra has many similar properties. The further discussions and proofs will appear elsewhere.
Bibliography


